Research Article

Vanishing Power Values of Commutators with Derivations on Prime Rings

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Let $R$ be a prime ring of char $R \neq 2$, $d$ a nonzero derivation of $R$ and $\rho$ a nonzero right ideal of $R$ such that $[[d(x), x], [y, d(y)]]_n = 0$ for all $x, y \in \rho$, where $n \geq 0$, $m \geq 0$, $t \geq 1$ are fixed integers. If $[\rho, \rho] \rho \neq 0$, then $d(\rho) \rho = 0$.

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1. Introduction

Throughout this paper, unless specifically stated, $R$ always denotes a prime ring with center $Z(R)$ and extended centroid $C, Q$ the Martindale quotients ring. Let $n$ be a positive integer. For given $a, b \in R$, let $[a, b]_0 = a$ and let $[a, b]_1$ be the usual commutator $ab - ba$, and inductively for $n > 1$, $[a, b]_n = [[a, b]_{n-1}, b]$. By $d$ we mean a nonzero derivation in $R$.

A well-known result proven by Posner [1] states that if $[[d(x), x], y] = 0$ for all $x, y \in R$, then $R$ is commutative. In [2], Lanski generalized this result of Posner to the Lie ideal. Lanski proved that if $U$ is a noncommutative Lie ideal of $R$ such that $[[d(x), x], y] = 0$ for all $x \in U, y \in R$, then either $R$ is commutative or char $R = 2$ and $R$ satisfies $S_3$, the standard identity in four variables. Bell and Martindale III [3] studied this identity for a semiprime ring $R$. They proved that if $R$ is a semiprime ring and $[[d(x), x], y] = 0$ for all $x$ in a non-zero left ideal of $R$ and $y \in R$, then $R$ contains a non-zero central ideal. Clearly, this result says that if $R$ is a prime ring, then $R$ must be commutative.

Several authors have studied this kind of Engel type identities with derivation in different ways. In [4], Herstein proved that if char $R \neq 2$ and $[d(x), d(y)] = 0$ for all $x, y \in R$, then $R$ is commutative. In [5], Filippis showed that if $R$ is of characteristic different from 2 and $\rho$ a non-zero right ideal of $R$ such that $[\rho, \rho] \rho \neq 0$ and $[[d(x), x], [d(y), y]] = 0$ for all $x, y \in \rho$, then $d(\rho) \rho = 0$. 
In continuation of these previous results, it is natural to consider the situation when 
\[[d(x), x]_n, [y, d(y)]_m]^t = 0\] for all \(x, y \in \rho, n, m \geq 0, \ t \geq 1\) are fixed integers. We have studied this identity in the present paper.

It is well known that any derivation of a prime ring \(R\) can be uniquely extended to a derivation of \(Q\), and so any derivation of \(R\) can be defined on the whole of \(Q\). Moreover \(Q\) is a prime ring as well as \(R\) and the extended centroid \(C\) of \(R\) coincides with the center of \(Q\). We refer to \([6, 7]\) for more details.

Denote by \(Q \ast C[X, Y]\) the free product of the \(C\)-algebra \(Q\) and \(C[X, Y]\), the free \(C\)-algebra in noncommuting indeterminates \(X, Y\).

2. The Case: \(R\) Prime Ring

We need the following lemma.

**Lemma 2.1.** Let \(\rho\) be a non-zero right ideal of \(R\) and \(d\) a derivation of \(R\). Then the following conditions are equivalent: (i) \(d\) is an inner derivation induced by some \(b \in Q\) such that \(bp = 0\); (ii) \(d(\rho)\rho = 0\) (for its proof refer to [8, Lemma]).

We mention an important result which will be used quite frequently as follows.

**Theorem 2.2** (see Kharchenko [9]). Let \(R\) be a prime ring, \(d\) a derivation on \(R\) and \(I\) a non-zero ideal of \(R\). If \(I\) satisfies the differential identity \(f(r_1, r_2, \ldots, r_n, d(r_1), d(r_2), \ldots, d(r_n)) = 0\) for any \(r_1, r_2, \ldots, r_n \in I\), then either (i) \(I\) satisfies the generalized polynomial identity

\[
f(r_1, r_2, \ldots, r_n, x_1, x_2, \ldots, x_n) = 0,
\]

or (ii) \(d\) is \(Q\)-inner, that is, for some \(q \in Q, d(x) = [q, x]\) and \(I\) satisfies the generalized polynomial identity

\[
f(r_1, r_2, \ldots, r_n, [q, r_1], [q, r_2], \ldots, [q, r_n]) = 0.
\]

**Theorem 2.3.** Let \(R\) be a prime ring of char \(R \neq 2\) and \(d\) a derivation of \(R\) such that 
\[[[d(x), x]_n, [y, d(y)]_m]^t = 0\] for all \(x, y \in R\), where \(n \geq 0, m \geq 0, t \geq 1\) are fixed integers. Then \(R\) is commutative or \(d = 0\).

**Proof.** Let \(R\) be noncommutative. If \(d\) is not \(Q\)-inner, then by Kharchenko’s Theorem [9]

\[
g(x, y, u, v) = [[[u, x]_n, [y, v]_m]^t = 0,
\]

for all \(x, y, u, v \in R\). This is a polynomial identity and hence there exists a field \(F\) such that \(R \subseteq M_k(F)\) with \(k > 1\), and \(R\) and \(M_k(F)\) satisfy the same polynomial identity [10, Lemma 1]. But by choosing \(u = e_{12}, x = e_{11}, v = e_{11}\) and \(y = e_{21}\), we get

\[
0 = [[[u, x]_n, [y, v]_m]^t = (-1)^t e_{11} + (-)^t e_{22},
\]

which is a contradiction.
Now, let $d$ be $Q$-inner derivation, say $d = ad(a)$ for some $a \in Q$, that is, $d(x) = [a, x]$ for all $x \in R$, then we have
\[
[[a, x]_{n+1}, [y, [a, y]]_{m}]^t = 0, \tag{2.5}
\]
for all $x, y \in R$. Since $d \neq 0$, $a \notin C$ and hence $R$ satisfies a nontrivial generalized polynomial identity (GPI). By [11], it follows that $RC$ is a primitive ring with $H = \text{Soc}(RC) \neq 0$, and $eHe$ is finite dimensional over $C$ for any minimal idempotent $e \in RC$. Moreover we may assume that $H$ is noncommutative; otherwise, $R$ must be commutative which is a contradiction.

Notice that $H$ satisfies $[[a, x]_{n+1}, [y, [a, y]]_{m}]^t = 0$ (see [10, Proof of Theorem 1]). For any idempotent $e \in H$ and $x \in H$, we have
\[
0 = [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_{m}]^t. \tag{2.6}
\]
Right multiplying by $e$, we get
\[
0 = [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_{m}]^t e
\]
\[
= [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_{m}]^{t-1}
\]
\[
\cdot \left\{ [a, e]_{n+1} \left( \sum_{j=0}^{m} (-1)^j \binom{m}{j} [a, ex(1-e)]^j [a, ex(1-e)]^{m-j} \right) e
\right\}
\]
\[
- \left( \sum_{j=0}^{m} (-1)^j \binom{m}{j} [a, ex(1-e)]^j [a, ex(1-e)]^{m-j} \right) [a, e]_{n+1} e
\]
\[
= [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_{m}]^{t-1}
\]
\[
\cdot \left\{ - \left( \sum_{j=0}^{m} (-1)^j \binom{m}{j} (-ex(1-e) a)^j [ex(1-e)(aex(1-e))]^{m-j} \right) ae \right\}
\]
\[
= -[[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_{m}]^{t-1} \left( \sum_{j=0}^{m} \binom{m}{j} (ex(1-e)a)^{m+1} \right) e
\]
\[
= -2^m [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_{m}]^{t-1} (ex(1-e)a)^{m+1} e
\]
\[
= (-)^{t} 2^m (ex(1-e)a)^{(m+1)t+1} e.
\]

This implies that $0 = (-)^{t} 2^m ((1-e)aex)^{(m+1)t+1}$. Since char $R \neq 2$, $((1-e)aex)^{(m+1)t+1} = 0$. By Levitzki’s lemma [12, Lemma 1.1], $(1-e)aex = 0$ for all $x \in H$. Since $H$ is prime ring, $(1-e)aex = 0$, that is, $eae = ae$ for any idempotent $e \in H$. Now replacing $e$ with $1-e$, we get that $ea(1-e) = 0$, that is, $eae = ea$. Therefore for any idempotent $e \in H$, we have $[a, e] = 0$. 

Suppose that $a$ commutes with all idempotents in $H$. Since $H$ is a simple ring, either $H$ is generated by its idempotents or $H$ does not contain any nontrivial idempotents. The first case gives $a \in C$ contradicting $d \neq 0$. In the last case, $H$ is a finite dimensional division algebra over $C$. This implies that $H = RC = Q$ and $a \in H$. By [10, Lemma 2], there exists a field $F$ such that $H \subseteq M_k(F)$ and $M_k(F)$ satisfies $[[a, x]_{n+1}, [y, [a, y]]_m]^t$. Then by the same argument as earlier, $a$ commutes with all idempotents in $M_k(F)$, again giving the contradiction $a \in C$, that is, $d = 0$. This completes the proof of the theorem.

**Theorem 2.4.** Let $R$ be a prime ring of char $R \neq 2$, $d$ a non-zero derivation of $R$ and $\rho$ a non-zero right ideal of $R$ such that $[[d(x), x]_n, [y, d(y)]_m]^t = 0$ for all $x, y \in \rho$, where $n \geq 0, m \geq 0, t \geq 1$ are fixed integers. If $[\rho, \rho] \neq 0$, then $d(\rho)\rho = 0$.

We begin the proof by proving the following lemma.

**Lemma 2.5.** If $d(\rho)\rho \neq 0$ and $[[d(x), x]_n, [y, d(y)]_m]^t = 0$ for all $x, y \in \rho$, $m, n \geq 0$, $t \geq 1$ are fixed integers, then $R$ satisfies nontrivial generalized polynomial identity (GPI).

**Proof.** Suppose on the contrary that $R$ does not satisfy any nontrivial GPI. We may assume that $R$ is noncommutative; otherwise, $R$ satisfies trivially a nontrivial GPI. We consider two cases.

**Case 1.** Suppose that $d$ is $Q$-inner derivation induced by an element $a \in Q$. Then for any $x \in \rho$,

$$[[a, xX]_{n+1}, [xY, [a, xY]]_m]^t$$

is a GPI for $R$, so it is the zero element in $Q*_{C}C[X, Y]$. Expanding this, we get

$$
\left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} [a, xX]_{n+1} ^j [xY, [a, xY]]_m ^{m-j} - \sum_{j=0}^{m} (-1)^j \binom{m}{j} [a, xX]_{n+1} ^j [xY, [a, xY]]_m ^{m-j} \right) A(X, Y) = 0,
$$

where $A(X, Y) = [[a, xX]_{n+1}, [xY, [a, xY]]_m]^{-1}$. If $ax$ and $x$ are linearly $C$-independent for some $x \in \rho$, then

$$
\left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} (axX)_{n+1} ^j [xY, [a, xY]]_m ^{m-j} - \sum_{j=0}^{m} (-1)^j \binom{m}{j} (axY)_{n+1} ^j [xY, [a, xY]]_m ^{m-j} \right) A(X, Y) = 0.
$$
Again, since \( ax \) and \( x \) are linearly \( C \)-independent, above relation implies that
\[
(-xY[a,y]^m[a,x]l_{n+1})A(X,Y) = 0, \tag{2.11}
\]
and so
\[
(-xY(axY)^m(axX)^{n+1})A(X,Y) = 0. \tag{2.12}
\]
Repeating the same process yields
\[
(-xY(axY)^m(axX)^{n+1})^t = 0 \tag{2.13}
\]
in \( Q\ast_C C[X,Y] \). This implies that \( ax = 0 \), a contradiction. Thus for any \( x \in \rho \), \( ax \) and \( x \) are \( C \)-dependent. Then \( (a - a)\rho = 0 \) for some \( a \in C \). Replacing \( a \) with \( a - a \), we may assume that \( ap = 0 \). Then by Lemma 2.1, \( d(\rho)\rho = 0 \), contradiction.

**Case 2.** Suppose that \( d \) is not \( Q \)-inner derivation. If for all \( x \in \rho \), \( d(x) \in xC \), then \( [d(x), x] = 0 \) which implies that \( R \) is commutative (see [13]). Therefore there exists \( x \in \rho \) such that \( d(x) \notin xC \), that is, \( x \) and \( d(x) \) are linearly \( C \)-independent.

By our assumption, we have that \( R \) satisfies
\[
[[d(x), x]_{n'}, [xY, d(x)y]_{m}]^t = 0. \tag{2.14}
\]
By Kharchenko’s Theorem [9],
\[
[[d(x)X + xr_1, xX]_{n'}, [xY, d(x)Y + xr_2]_{m}]^t = 0, \tag{2.15}
\]
for all \( X, Y, r_1, r_2 \in R \). In particular for \( r_1 = r_2 = 0 \),
\[
[[d(x), x]_{n'}, [xY, d(x)y]_{m}]^t = 0, \tag{2.16}
\]
which is a nontrivial GPI for \( R \), because \( x \) and \( d(x) \) are linearly \( C \)-independent, a contradiction.

We are now ready to prove our main theorem.

**Proof of Theorem 2.4.** Suppose that \( d(\rho)\rho \neq 0 \), then we derive a contradiction. By Lemma 2.5, \( R \) is a prime GPI ring, so is also \( Q \) by [14]. Since \( Q \) is centrally closed over \( C \), it follows from [11] that \( Q \) is a primitive ring with \( H = Soc(Q) \neq 0 \).

By our assumption and by [7], we may assume that
\[
[[d(x), x]_{n'}, [y, d(y)]_{m}]^t = 0 \tag{2.17}
\]
is satisfied by $\rho Q$ and hence by $\rho H$. Let $e = e^2 \in \rho H$ and $y \in H$. Then replacing $x$ with $e$ and $y$ with $ey(1 - e)$ in (2.17), then right multiplying it by $e$, we obtain that

$$
0 = \left[ [d(e), e]_{n^t} [ey(1 - e), d(ey(1 - e))]_m \right] e
$$

$$
= \left[ [d(e), e]_{n^t} [ey(1 - e), d(ey(1 - e))]_m \right] e
$$

$$
\cdot \left\{ \left[ d(e), e \right]_{m} \sum_{j=0}^{m} (-1)^j \left( \begin{array}{c} m \\ j \end{array} \right) d(ey(1 - e))^j ey(1 - e)d(ey(1 - e))^{m-j} e \right\}.
$$

(2.18)

$$
- \sum_{j=0}^{m} (-1)^j \left( \begin{array}{c} m \\ j \end{array} \right) d(ey(1 - e))^j ey(1 - e)d(ey(1 - e))^{m-j} [d(e), e]_n e \right\}.
$$

Now we have the fact that for any idempotent $e$, $d(y(1 - e))e = -y(1 - e)d(e), ed(e)e = 0$ and so

$$
0 = \left[ [d(e), e]_{n^t} [ey(1 - e), d(ey(1 - e))]_m \right] e
$$

$$
\cdot \left\{ 0 - \sum_{j=0}^{m} (-1)^j \left( \begin{array}{c} m \\ j \end{array} \right) e(-y(1 - e)d(e))^j y(1 - e)d(ey(1 - e))^{m-j} d(e)e \right\}.
$$

(2.19)

Now since for any idempotent $e$ and for any $y \in R$, $(1 - e)d(ey) = (1 - e)d(e)y$, above relation gives

$$
0 = \left[ [d(e), e]_{n^t} [ey(1 - e), d(ey(1 - e))]_m \right] e
$$

$$
\cdot \left\{ -e \sum_{j=0}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) (y(1 - e)d(e))^j y(1 - e)(d(e)y(1 - e))^{m-j} d(e)e \right\}
$$

$$
= \left[ [d(e), e]_{n^t} [ey(1 - e), d(ey(1 - e))]_m \right] e
$$

$$
\cdot \left\{ -e \sum_{j=0}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) (y(1 - e)d(e))^{m+1} e \right\}
$$

(2.20)

$$
\cdot \left\{ -2^m e(y(1 - e)d(e))^{m+1} e \right\}
$$

This implies that $0 = (-1)^t 2^m ((1 - e)d(e)ey)^{(m+1)t+1}$ for all $y \in H$. Since char $R \neq 2$, we have by Levitzki's lemma [12, Lemma 1.1] that $(1 - e)d(e)ey = 0$ for all $y \in H$. By primeness of $H$, $(1 - e)d(e) = 0$. By [15, Lemma 1], since $H$ is a regular ring, for each $r \in \rho H$, there exists an idempotent $e \in \rho H$ such that $r = er$ and $e \in \rho H$. Hence $(1 - e)d(e)e = 0$ gives $(1 - e)d(e) = (1 - e)d(e^2) = (1 - e)d(e)e = 0$ and so $d(e) = ed(e) \in eH \subseteq \rho H$ and $d(r) = d(er) = d(e)er + ed(er) \in \rho H$. Hence for each $r \in \rho H$, $d(r) \in \rho H$. Thus $d(\rho H) \subseteq \rho H$. Set $J = \rho H.$
Then $\overline{J} = J/(J \cap l_H(J))$, a prime $C$-algebra with the derivation $\overline{d}$ such that $\overline{d}(\overline{x}) = \overline{d(x)}$, for all $x \in J$. By assumption, we have that

\[ \left[ \left[ \overline{d}(\overline{x}), \overline{y} \right]_n, \left[ y, \overline{d}(\overline{y}) \right]_m \right]^t = 0, \tag{2.21} \]

for all $\overline{x}, \overline{y} \in \overline{J}$. By Theorem 2.3, we have either $\overline{d} = 0$ or $\rho_H$ is commutative. Therefore we have that either $d(\rho_H)\rho_H = 0$ or $[\rho_H, \rho_H]\rho_H = 0$. Now $d(\rho_H)\rho_H = 0$ implies that $0 = d(\rho_H)\rho_H = d(\rho)\rho H \rho H$ and so $d(\rho) = 0$. $[\rho_H, \rho_H]\rho_H = 0$ implies that $0 = [\rho_H, \rho_H]\rho_H = [\rho, \rho_H]\rho H \rho H$ and so $[\rho, \rho_H]\rho = 0$, then $0 = [\rho, \rho_H]\rho = [\rho, \rho]\rho H \rho H$ implying that $[\rho, \rho]\rho = 0$. Thus in all the cases we have contradiction. This completes the proof of the theorem. $\square$

3. The Case: $R$ Semiprime Ring

In this section we extend Theorem 2.3 to the semiprime case. Let $R$ be a semiprime ring and $U$ be its right Utumi quotient ring. It is well known that any derivation of a semiprime ring $R$ can be uniquely extended to a derivation of its right Utumi quotient ring $U$ and so any derivation of $R$ can be defined on the whole of $U$ [7, Lemma 2].

By the standard theory of orthogonal completions for semiprime rings, we have the following lemma.

Lemma 3.1 (see [16, Lemma 1 and Theorem 1] or [7, pages 31-32]). Let $R$ be a 2-torsion free semiprime ring and $P$ a maximal ideal of $C$. Then $PU$ is a prime ideal of $U$ invariant under all derivations of $U$. Moreover, $\bigcap \{PU \mid P$ is a maximal ideal of $C$ with $U/PU$ 2-torsion free $\} = 0$.

Theorem 3.2. Let $R$ be a 2-torsion free semiprime ring and $d$ a non-zero derivation of $R$ such that $[[d(x), x]_n, [y, d(y)]_m]^t = 0$ for all $x, y \in R$, $n, m \geq 0, t \geq 1$ fixed are integers. Then $d$ maps $R$ into its center.

Proof. Since any derivation $d$ can be uniquely extended to a derivation in $U$, and $R$ and $U$ satisfy the same differential identities [7, Theorem 3], we have

\[ [[d(x), x]_n, [y, d(y)]_m]^t = 0, \tag{3.1} \]

for all $x, y \in U$. Let $P$ be any maximal ideal of $C$ such that $U/PU$ is 2-torsion free. Then by Lemma 3.1, $PU$ is a prime ideal of $U$ invariant under $d$. Set $\overline{U} = U/PU$. Then derivation $d$ canonically induces a derivation $\overline{d}$ on $\overline{U}$ defined by $\overline{d}(\overline{x}) = \overline{d(x)}$ for all $x \in U$. Therefore,

\[ \left[ \left[ \overline{d}(\overline{x}), \overline{y} \right]_n, \left[ \overline{y}, \overline{d}(\overline{y}) \right]_m \right]^t = 0, \tag{3.2} \]

for all $\overline{x}, \overline{y} \in \overline{U}$. By Theorem 2.3, either $\overline{d} = 0$ or $[\overline{U}, \overline{U}] = 0$, that is, $d(U) \subseteq PU$ or $[U, U] \subseteq PU$. In any case $d(U)[U, U] \subseteq PU$ for any maximal ideal $P$ of $C$. By Lemma 3.1,
\[ \cap \{ PU \mid \text{P is a maximal ideal of} \ C \text{ with } U/PU \text{ 2-torsion free} \} = 0. \text{ Thus } d(U)[U,U] = 0. \]

Without loss of generality, we have \( d(R)[R,R] = 0. \) This implies that

\[
0 = d(R^2)[R,R] = d(R)R[R,R] + Rd(R)[R,R] = d(R)R[R,R]. \quad (3.3)
\]

Therefore \( [R,d(R)]R[R,d(R)] = 0. \) By semiprimeness of \( R, \) we have \( [R,d(R)] = 0, \) that is, \( d(R) \subseteq Z(R). \) This completes the proof of the theorem. \( \square \)

References