1. Introduction

We begin with $\mathcal{L}$, a sublattice of a complete (not necessarily atomic) Boolean algebra $\mathcal{B}$. If $\mathcal{L}$ is closed under arbitrary meets, it abstracts the closed sets of a topological space. If not, we introduce a Kurotowski closure operator to define the associated topological lattice. The operators we define generalize complement on a lattice which in turn abstracts the set theoretic operator. Less restricted than those of Banaschewski and Samuel, the operators exhibit some surprising behaviors. We consider properties of such lattices and their interrelations. Many of these properties are abstractions and generalizations of topological spaces. The approach is similar to that of Bachman and Cohen. It is in the spirit of Alexandroff, Frolik, and Nöbeling, although the setting is more general. Proceeding in this manner, we can handle diverse topological theorems systematically before specializing to get as corollaries as the topological results of Alexandroff, Alo and Shapiro, Dykes, Frolik, and Ramsay.

Copyright © 2009 Eva Cogan. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.
theorems systematically before specializing to get as corollaries as the topological results of [5, 8, 12–14].

Section 2 provides some background material and generates a topology on an algebra by means of a sublattice. Section 3 defines operators and topological type properties for a lattice. Section 4 examines filter and measure behavior with respect to the operators. Section 5 looks at covering properties. Section 6 investigates the relationships between two lattices.

2. Background, Terminology, and Notation

We work within a complete Boolean algebra \( B \) with minimal element \( 0 \) and maximal element \( e \). The usual operators are denoted by \( \lor, \land, \lnot \). \( B \) is not necessarily atomic; equivalently, \( B \) is not necessarily completely distributive [15].

(i) \( \mu, \mu_f \) denote finitely additive zero-one measures on \( B \).
(ii) \( \mathcal{L}, \mathcal{L}_1, \) and \( \mathcal{L}_2 \) denote sublattices of \( B \) containing \( 0 \) and \( e \).
(iii) \( \mathcal{A}(\mathcal{L}) \) denotes the algebra generated by \( \mathcal{L} \).
(iv) \( \mathcal{P}(S) \) is the power set of the set \( S \).
(v) The indices \( i, j, k, \) and \( n \) index countable (finite or countably infinite) collections, while \( a, \beta, \) and \( \gamma \) index arbitrary ones.

Definition 2.1. (i) \( F \subseteq \mathcal{L} \) is an \( \mathcal{L} \)-filter if and only if for all \( a, b \in \mathcal{L} \):

(a) \( a, b \in F \Rightarrow a \land b \in F \),
(b) \( a \in F \) and \( a \leq b \Rightarrow b \in F \).

When there is no ambiguity, we simply say that \( F \) is a filter.
(ii) An \( \mathcal{L} \)-filter \( F \) is a prime \( \mathcal{L} \)-filter if and only if for all \( a, b \in \mathcal{L} \), \( a \lor b \in F \Rightarrow \) either \( a \in F \) or \( b \in F \).
(iii) An \( \mathcal{L} \)-filter \( F \) is an \( \mathcal{L} \)-ultrafilter (or ultra) if and only if \( F \) is a maximal \( \mathcal{L} \)-filter.
(iv) A filter \( F \) is fixed if and only if \( \forall F \neq 0. \) Otherwise \( F \) is free.
(v) A filter \( F \) has cmp (countable meet property or countable intersection property) if and only if for any countable subset of \( F, \bigwedge a_i \neq 0 \).

Remark 2.2. It follows that

(i) \( 0 \notin F \) if and only if \( F \neq \mathcal{L} \),
(ii) every ultrafilter is prime.

In this paper, \( 0 \) is not in any filter.

Definition 2.3. A measure \( \mu \) on an algebra \( B \) containing \( \mathcal{L} \) is \( \mathcal{L} \)-regular if and only if for all \( b \in B, \mu(b) = \sup\{\mu(a) : a \in \mathcal{L}, a \leq b\} \).

Lemma 2.4. There exist one-to-one correspondences between

(i) zero-one measures on \( \mathcal{L} \) and prime \( \mathcal{L} \)-filters,
(ii) \( \mathcal{L} \)-regular zero-one measures on \( \mathcal{A}(\mathcal{L}) \), and \( \mathcal{L} \)-ultrafilters [3, 4].
Thus, analogous measure theoretic results may easily be derived from our filter
statements.

We now topologize \( \mathcal{B} \) by means of a sublattice \( \mathcal{L} \). The lattice elements themselves may
not be sufficient to be used as open or closed sets. However, we will generate a topology.

**Definition 2.5.** Let \( \mathcal{B} \) be an algebra, \( \mathcal{L} \) a sublattice of \( \mathcal{B} \), and \( a, b \in \mathcal{B} \).

(i) \( \overline{\mathcal{B}} = \bigwedge \{ a \in \mathcal{L} : a \geq b \} \).

(ii) \( b^0 = \bigvee \{ a' : a \in \mathcal{L} \text{ and } a' \leq b \} \).

(iii) \( b \in \mathcal{B} \) is closed if and only if \( b = \overline{b} \).

(iv) \( \tau(\mathcal{L}) \) denotes the set of closed elements of \( \mathcal{B} \).

**Remark 2.6.** (i) \( b \mapsto \overline{b} \) is a Kurotowski closure operator.

(ii) \( b \mapsto b^0 \) is an interior operator.

(iii) \( \tau(\mathcal{L}) = \{ b \in \mathcal{B} : b = \bigwedge_{a} a, \ a \in \mathcal{L} \} \).

(iv) \( \mathcal{L} \subseteq \tau(\mathcal{L}) \).

(v) \( \tau(\mathcal{L}) \) is a lattice which is closed under arbitrary meets.

We observe that \( \tau(\mathcal{L}) \) is an obvious abstraction of the closed sets in a topological space.

**Example 2.7.** Let \( \mathcal{L} \) be the lattice of zero sets in a \( T_2 \) completely regular topological space. Then
\( \tau(\mathcal{L}) \) is the lattice of closed sets [11].

**Definition 2.8.** Let \( \mathcal{M} \subseteq \mathcal{B} \) be a meet semilattice (i.e., a subset of \( \mathcal{B} \) closed under finite meets).

Consider the generated lattice \( \mathcal{L}(\mathcal{M}) = \{ \bigvee^n_i a_i : a_i \in \mathcal{M} \} : n \in \mathbb{N} \}. \) All terminology remains the same except that \( F \) is a prime \( \mathcal{M} \)-filter if and only if \( \bigvee^n_i a_i \in F \) (for \( a_i \in \mathcal{M} \)) implies that one of the \( a_i \in F \).

**Lemma 2.9.** (i) There exist one-to-one correspondences between the prime, ultra-, fixed, and free
filters on \( \mathcal{M} \) and those on \( \mathcal{L}(\mathcal{M}) \).

(ii) \( \mathcal{M} \)-filters with cmp correspond with \( \mathcal{L} \)-filters with cmp.

Thus we lose no generality in “treating” \( \mathcal{M} \) like a lattice.

**Example 2.10.** Let \( \mathcal{M} = \{ b \in \mathcal{B} : b = (\overline{\mathcal{B}})^0 \} \). \( \mathcal{M} \) is a meet semi-lattice since
\( \overline{a \wedge b} \leq \overline{a} \wedge \overline{b} \) implies that \( (a \wedge b)^0 \leq (\overline{a} \wedge \overline{b})^0 = (\overline{a})^0 \wedge (\overline{b})^0 = a \wedge b \), for all \( a, b \in \mathcal{M} \). “An open subset \( G \) in
a topological space is regularly open if and only if \( G \) is the interior of its closure” [16]. Thus
the regularly open sets in a topological space form a meet semi-lattice.

Table 1 summarizes the notation used in this paper.

### 3. Lattice Operators and Properties

In this section we define certain operators and lattice properties. These properties reduce to
the conventional topological ones when the operator is taken to be complement.

**Definition 3.1.** Let \( \mathcal{M} \) be a meet semi-lattice containing \( 0 \) and \( e \). We define \( T \) to be a one-to-
one operator on \( \mathcal{M} \) such that \( T(a \vee b) = T(a) \wedge T(b) \) and \( T(e) = 0 \). We define \( a^* = T(a) \) and
\( \mathcal{L}^* = \{ a^* : a \in \mathcal{L} \} \).
Fixed.

When Corollary 3.3.

**Proposition 3.2.** (a) \( a \leq b \iff T(a) \geq T(b) \).
(b) \( \bigvee^n_i T(a_i) \leq T(\bigwedge^n_i a_i) \).
(c) \( \bigvee^n_i T(a_i) = e \Rightarrow \bigwedge^n_i a_i = 0 \).
(d) \( T(\bigwedge^n_i a_i) = 0 \Rightarrow \bigvee^n_i a_i = e \).

**Proof.** (a) \( a \leq b \iff a \lor b = b \iff T(a) \land T(b) = T(b) \iff T(a) \geq T(b) \).
(b) For all \( j \), \( a_j \geq \bigwedge^n_i a_i \Rightarrow \) for all \( j \), \( T(a_j) \leq T(\bigwedge^n_i a_i) \Rightarrow \bigvee^n_j T(a_j) \leq T(\bigwedge^n_i a_i) \).

The proof of (c) and (d) follows readily. \( \square \)

From now on, we assume that \( T \) is defined on a lattice \( \mathcal{L} \). Then \( \mathcal{L}^* \) is a meet semi-lattice. By Lemma 2.9, we may “treat” \( \mathcal{L}^* \) like a lattice.

**Corollary 3.3.** When \( T \) is defined on \( \tau(\mathcal{L}) \), one has the following.

(a) \( \bigvee_\alpha T(a_\alpha) \leq T(\bigwedge_\alpha a_\alpha) \).

(b) \( \bigvee_\alpha T(a_\alpha) = e \Rightarrow \bigwedge_\alpha a_\alpha = 0 \).

**Example 3.4.** Let \( S = \{1, 2, 3, 4\} \) and \( \mathcal{B} = \mathcal{P}(S) \) (the power set of \( S \)) with set union and intersection as the join and meet operations. Let \( \mathcal{L} = \{\emptyset, a, b, S\} \) with \( a = \{1, 2\} \) and \( b = \{3, 4\} \) as in Figure 1. The operator \( T \) is defined by \( T(S) = \emptyset, T(a) = \{4\}, T(b) = \{2\} \), and \( T(\emptyset) = \{2, 4\} \). We have \( T(a \lor b) = T(a) \land T(b) \) and \( T(S) = \emptyset \). In addition \( T(a \land b) = T(a) \lor T(b) \) and \( T(a) \land a = \emptyset \). Note that, unlike in [2], \( T(a) \) is not the maximal element disjoint from \( a \), and although \( \mathcal{L} \) is complemented (that is, \( a \in \mathcal{L} \Rightarrow a' \in \mathcal{L} \), \( T(a) \neq a' \)).

We now define various properties for \( \mathcal{L} \). They generalize some of the definitions in point set topology, reducing to the conventional properties when \( \mathcal{B} \) is the power set of a set \( X \) and \( T \) is complement.

**Definition 3.5.** (i) \( \mathcal{L} \) is **compact** if and only if every \( \mathcal{L} \)-filter is fixed.
(ii) \( \mathcal{L} \) is **\( \aleph_0 \)-compact** if and only if every \( \mathcal{L} \)-filter has cmp.
(iii) \( \mathcal{L} \) is an **I-lattice (P-lattice)** if and only if every (prime) \( \mathcal{L} \)-filter with cmp is contained in an ultrafilter with cmp.
(iv) \( \mathcal{L} \) is an **R-lattice** if and only if every filter which contains a fixed prime filter is also fixed.
(v) A topological space is an **I-space** if and only if the lattice of closed sets is an **I-lattice** [17].

<table>
<thead>
<tr>
<th>Table 1: Summary of notation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B )</td>
</tr>
<tr>
<td>( \mathcal{L}, \mathcal{L}_n )</td>
</tr>
<tr>
<td>( F, F_n, G, H )</td>
</tr>
<tr>
<td>( \mathcal{A}(\mathcal{L}) )</td>
</tr>
<tr>
<td>( \mu, \mu_f )</td>
</tr>
<tr>
<td>( \tau(\mathcal{L}) )</td>
</tr>
<tr>
<td>( \mathfrak{M} )</td>
</tr>
<tr>
<td>( \mathcal{P}(S) )</td>
</tr>
</tbody>
</table>
Proposition 3.6. As an immediate consequence of Definition 3.5, one has the following.

(i) $\mathcal{L}$ is compact $\Rightarrow$ $\mathcal{L}$ is $\aleph_0$-compact $\Rightarrow$ $\mathcal{L}$ is an I-lattice $\Rightarrow$ $\mathcal{L}$ is a $\mathcal{P}$-lattice.

(ii) The following are equivalent:

(a) $\mathcal{L}$ is compact ($\aleph_0$-compact),
(b) every prime filter is fixed (has cmp),
(c) every ultrafilter is fixed (has cmp).

Definition 3.7. (i) $\mathcal{L}$ is $\aleph_0$-paracompact if and only if whenever there exists $\{a_n\} \subseteq \mathcal{L}$ with $a_n \downarrow 0$, there exists $\{b_n\} \subseteq \mathcal{L}$ such that $a_n \leq b_n^*$ and $b_n^* \downarrow 0$.

(ii) When $\mathcal{L} \subseteq \tau(\mathcal{L}^*)$, we say that $\mathcal{L}$ is perfect.

Proposition 3.8. Every perfect lattice is $\aleph_0$-paracompact.

Proof. Assume $\mathcal{L}$ is perfect and $a_n \downarrow 0$. For each $a_n$, there exists $\{b_{n,k}\} \subseteq \mathcal{L}$ such that $a_n = \bigwedge_k b_{n,k}^*$. Let $c_n^* = \bigwedge\{b_{k,n}^* : k \leq n\}$. Then $c_n^* \geq a_n$ and $c_n^* \downarrow 0$. \qed

Example 3.9. The zero sets in a topological space are perfect (i.e., complement generated in the sense of [11]) and thus $\aleph_0$-paracompact.

Definition 3.10. Let $a, a_i, b \in \mathcal{B}$, and $f, f_i, g, g_i \in \mathcal{L}$. Then the following are given.

(i) $\mathcal{L}$ is $T_1$ if and only if for all $a_1 \not\leq a_2$ there exists $g^* \in \mathcal{L}^*$ such that $a_2 \leq g^*$ but $a_1 \not\leq g^*$.

(ii) $\mathcal{L}$ is Hausdorff if and only if for all $a_1, a_2 \neq 0$, $a_1 \land a_2 = 0$ there exist $g_1^*, g_2^* \in \mathcal{L}^*$ such that $a_1 \land g_1^* \neq 0, a_2 \land g_2^* \neq 0$, and $g_1^* \land g_2^* = 0$.

(iii) $\mathcal{L}$ is regular if and only if for all $b \in \mathcal{B}$, $b \neq 0$, $f \in \mathcal{L}$, $b \land f = 0$ there exist $g_1^*, g_2^* \in \mathcal{L}^*$ such that $b \land g_1^* \neq 0, f \leq g_2^*$, and $g_1^* \land g_2^* = 0$.

The following proposition provides an example.

Proposition 3.11. Let $\mathcal{B}$ be the power set of a topological space $X$, let $\mathcal{L}$ be the lattice of closed sets in $X$, and let $T$ be complement. $X$ is a $T_1$ (Hausdorff, regular) space if and only if $\mathcal{L}$ is $T_1$ (Hausdorff, regular).
Proof. Let $\mathcal{L}$ be the lattice of closed sets in $X$. Suppose $\mathcal{L}$ is a $T_1$ lattice. Let $a_1 \neq a_2$ be atoms in $\mathcal{B}$. Since $a_1 \not\leq a_2$, there exists $g_2' \in \mathcal{L}'$ such that $a_2 < g_2'$ but $a_1 \not\leq g_2'$. By symmetry, there exists $g_1' \in \mathcal{L}'$ such that $a_1 \leq g_1'$ but $a_2 \not\leq g_1'$. Thus $X$ is a $T_1$ space.

Now suppose $X$ is a $T_1$ topological space. Let $b, c \in \mathcal{B}$, $b \not\leq c$. Then there exists an atom $a \leq b$ but $a \not\leq c$. Thus for all atoms $a_a \leq c$, $a \neq a_a$, and there exists $g_a' \in \mathcal{L}'$ such that $a_a \leq g_a'$ but $a \not\leq g_a'$. Let $g' = \bigvee a g_a'$. Then $c \leq g'$, $b \not\leq g'$, and thus $\mathcal{L}$ is $T_1$.

The proofs for Hausdorff and regular are similar. □

**Proposition 3.12.** Suppose $T(a) \land a = 0$ and $\mathcal{L}$ is regular. If $F_1$ is prime and $F_1 \subseteq F_2$, then $\bigwedge F_1 = \bigwedge F_2$.

**Proof.** Suppose there exists $F_1 \subseteq F_2$ such that $\bigwedge F_1 \neq \bigwedge F_2$. Let $a = \bigwedge F_1$. $a \not\subseteq F_2$ implies there exists $f \in F_2$ with $a \not\leq f$. But then $b = a \land f' \neq 0$ and $b \land f = 0$. By regularity, there exist $c_1, c_2 \in \mathcal{L}$ such that $b \land c_1 \neq 0$, $f \leq c_2$, and $c_1 \land c_2 = 0$. Now $f \leq c_2$ implies that $c_2 \not\subseteq F_2 \supseteq F_1$. From $b \land c_1 \neq 0$, it follows that $a \not\leq c_1$ and thus $c_1 \not\subseteq F_1$. But $c_1 \land c_2 = 0$ implies that $c_1 \lor c_2 = e$, and thus $F_1$ is not prime. □

**Example 3.13.** The closed sets in a regular topological space form an $R$-lattice.

**Definition 3.14.** $\mathcal{L}$ is **normal** if and only if for all $f_1, f_2 \in \mathcal{L}$, $f_1 \land f_2 = 0$ there exist $g_1^*, g_2^* \in \mathcal{L}$ such that $f_1 \leq g_1^*$, $f_2 \leq g_2^*$, and $g_1^* \lor g_2^* = 0$.

The following proposition demonstrates an example of an application.

**Proposition 3.15.** Let $X$ be a topological space. Let $T$ be complement and let $\mathcal{L}'$ be the lattice of open sets. Then the following are equivalent.

1. $\mathcal{L}'$ is normal.
2. $a', b' \in \mathcal{L}'$ and $a' \land b' = 0 \Rightarrow \overline{a'} \land \overline{b'} = 0$.
3. $a' \in \mathcal{L}' \Rightarrow \overline{a'} \in \mathcal{L}'$ (i.e., $X$ is extremally disconnected).

**Proof.** Consider the following. (i) (a) implies (b): let $a', b' \in \mathcal{L}'$, $a' \land b' = 0$. Then there exist $c, d \in \mathcal{L}$ such that $a' \leq c$, $b' \leq d$, and $c \land d = 0$ (by normality). But $\overline{c} \leq c$ and $\overline{d} \leq d$ implies that $a' \land b' = 0$.

(ii) (b) implies (c): let $a' \in \mathcal{L}'$. Let $b = \overline{a}$ so that $a' \land b' = 0$. Then $\overline{a'} \land \overline{b'} = 0$ (by hypothesis), so that $a \land \overline{b'} = 0$. Then $b \leq \overline{b'} = b^0$, so that $b = b^0$ and $b \in \mathcal{L}'$. Therefore $\overline{a'} \in \mathcal{L}'$ by definition of $b$.

(iii) (c) implies (b): let $a', b' \in \mathcal{L}'$, $a' \land b' = 0$, so that $\overline{a'} \land \overline{b'} = 0$. Since $\overline{a'} \in \mathcal{L}'$, then $\overline{a'} \land \overline{b'} = 0$.

(iv) (b) implies (a): $\overline{a'}, \overline{b'}$ are elements of $\mathcal{L}$. □

**Proposition 3.16.** Let $T(a) \land a = 0$. If $\mathcal{L}$ is normal and $F$ is a prime $\mathcal{L}$-filter contained in two $\mathcal{L}$-ultrafilters $G$ and $H$, then $G = H$.

**Proof.** Let $G$ and $H$ be two distinct ultrafilters and let $F \subseteq G \cap H$. Then there exist $g \in G$, $h \in H$ with $g \land h = 0$. By normality, there exist $a^*, b^* \in \mathcal{L}$ such that $g \leq a^*$, $h \leq b^*$, and $a^* \land b^* = 0$. But $g \land a = 0$ implies that $a \not\in G$, and $h \land b = 0$ implies that $b \not\in H$, so that $a, b \not\in G \cap H \supseteq F$, that is, $a, b \not\in F$. But $a \land b = e$ implies that $F$ is not prime. □

**Definition 3.17.** $\mathcal{L}$ is $\aleph_0$-**normal** if and only if it is normal and $\aleph_0$-paracompact.
Example 3.18. A normal topological space is countably paracompact if and only if the lattice of closed sets is $\aleph_0$-paracompact [18]. Willard [16] calls such a space binormal.

**Proposition 3.19.** Let $T$ be complement in $\mathcal{B}$. If $\mathcal{L}$ is $\aleph_0$-normal and $F_1$ is a prime filter with cmp, then $F_1 \subseteq F_2$ implies that $F_2$ has cmp.

*Proof.* Let $F_1 \subseteq F_2$ where $F_1$ is a prime $\mathcal{L}$-filter and $F_2$ is an $\mathcal{L}$-filter without cmp. Then there exists $\{f_k\} \subseteq F_2$ such that $\bigwedge_k f_k = 0$. Let $g_n = \bigwedge_i f_k$. Thus there exist $\{b_n\} \subseteq \mathcal{L}$ such that $g_n \leq b_n'$ and $b_n' \downarrow 0$. Now $g_n \wedge b_n = 0$, so there exist two sequences $\{c_n\}, \{d_n\}$ such that $g_n \leq c_n$, $b_n \leq d_n$, $c_n \wedge d_n = 0$, for all $n$. Since $c_n \vee d_n = e$ and $F_1$ is prime, we have $c_n \in F_1$ or $d_n \in F_1$, for all $n$. But $g_n \leq c_n'$ implies that $c_n \notin F_1$, so $d_n \in F_1$, for all $n$. And since $b_n' \geq d_n$ implies that $d_n \downarrow 0$, we have that $F_1$ does not have cmp.

**Remark 3.20.** If $\mathcal{L}$ is a normal lattice that has the stronger property that whenever $\{a_n\} \subseteq \mathcal{L}$ such that $\bigvee a_n a'_n = e$ there exists $\{b_n\} \subseteq \mathcal{L}$ such that $\bigvee b_n = e$ and for all $n$, $a'_n \geq b_n$, then we need only to assume that $T(a) \wedge a = 0$.

**Corollary 3.21.** Let $T$ be complement in $\mathcal{B}$. If $\mathcal{L}$ is $\aleph_0$-normal, then $\mathcal{L}$ is a $P$-lattice.

### 4. Behavior of Filters and Measures Under $T$

In this section we look at the behavior of filters and measures with respect to $T$. It is interesting to see an example where $T$ does not “behave as nicely” as complement.

**Definition 4.1.** Let $D \subseteq \mathcal{L}$. $\tilde{D} = \{a^* \in \mathcal{L}^* : a \notin D\}$.

**Proposition 4.2.** Let $F$ be a prime $\mathcal{L}$-filter. $\tilde{F}$ is a prime $\mathcal{L}^*$-filter.

*Proof.* (a) $0 \notin \tilde{F}$ since $e \in F$.
(b) $a^* \leq b^*, a^* \in \tilde{F} \Rightarrow a \geq b$ and $a \notin F \Rightarrow b \notin F \Rightarrow b^* \in \tilde{F}$.
(c) Let $a^*, b^* \in \tilde{F}$, so that $a, b \notin F$; equivalently, $a \vee b \notin F$. But then $(a \vee b)^* \in \tilde{F}$, and so $a^* \wedge b^* \in \tilde{F}$. Thus (by (a), (b), and (c)), $\tilde{F}$ is a filter.
(d) Let $a^*, b^* \in \mathcal{L}^*$, $a^* \vee b^* \geq c^* \in \tilde{F}$. $(a \wedge b)^* \geq a^* \vee b^* \geq c^* \in \tilde{F}$ \Rightarrow $a \wedge b \leq c \notin F \Rightarrow a \wedge b \notin F$. Thus $\tilde{F}$ is prime.

**Definition 4.3.** A filter $F$ is **coultra** if and only if $\tilde{F}$ is $\mathcal{L}^*$-ultra. $\mu_F$ is **coregular** if and only if $F$ is coultra.

**Proposition 4.4.** $\tilde{F}$ is $\mathcal{L}^*$-ultra if and only if $a \in F$ is equivalent to $a^* \leq (b^*)'$ for some $b^* \in \tilde{F}$ (i.e., $a^* \notin \tilde{F} \Leftrightarrow$ there exists $b^* \in \tilde{F}$ such that $a^* \leq (b^*)'$).

**Remark 4.5.** Proposition 4.4 generalizes a theorem of [12] which we get by taking $T$ to be complement.

As demonstrated by the following example, we can associate measures with the prime filters on $\mathcal{L}$ as usual, but they may lack some of the properties to which we are accustomed.

**Example 4.6.** Let $S = \{1, 2, 3, 4\}$. Let $\mathcal{B} = \mathcal{P}(S)$, with set union and intersection as the join and meet operations. Figure 2 defines the sublattice $\mathcal{L}$.
Define $T$ on $\mathcal{L}$ as follows:

$$
T(e) = 0, \quad T(0) = e, \\
T(a) = g, \quad T(g) = a, \\
T(b) = h, \quad T(h) = b, \\
T(c) = c, \quad T(d) = d, \quad T(f) = f.
$$

Incidentally, $\mathcal{L} = \mathcal{L}^*$ and for all $a \in \mathcal{L}$, $T(T(a)) = a$. However $a \land T(a)$ does not necessarily $= 0$.

Now $T$ can be extended to all of $\mathcal{A}(\mathcal{L}) = \mathcal{B}$ by defining $T(a \lor b) = T(a) \land T(b)$, $T(a \land b) = T(a) \lor T(b)$, and $T(a') = T(a')$.

Let $F = F_f = \{ x \in \mathcal{L} : x \geq f \} = \{ e, b, f \}$. $\bar{F} = \{ a^*, d^*, h^*, c^*, g^*, \emptyset^* \} = \{ g, d, b, c, a, e \} = F_g$. Since $\bar{F}$ is an $\mathcal{L}^*$-ultrafilter, $F$ is a couutra filter. $F$ is a prime filter so we have the associated measure $\mu = \mu_F = \mu_f$, where

$$
\mu(x) = \begin{cases} 
1 & \text{if } x \geq f, \\
0 & \text{otherwise}.
\end{cases}
$$

Now consider that $[3] \in \mathcal{A}(\mathcal{L})$. $\mu([2, 3]) = 1$, $\mu([2]) = 0$, so that $\mu([3]) = \mu([2, 3]) - \mu([2]) = 1$. But $[3] \notin$ any element of $\mathcal{L}^*$ whose measure is one, so $\mu$ is not $\mathcal{L}^*$-regular even though it is $\mathcal{L}$-coregular.

Note that

$$
\mu(a) = 0 \text{ and } \mu(a^*) = \mu(g) = 0, \\
\mu(f) = 1 \text{ and } \mu(f^*) = \mu(f) = 1, \\
\mu(h) = 0 \text{ but } \mu(h^*) = \mu(b) = 1, \\
\mu(b) = 1 \text{ but } \mu(b^*) = \mu(h) = 0.
$$

Thus, $T$ is not necessarily measure inverting $(\mu(T(a)) = 1 - \mu(a) \land \mu)$, preserving, increasing, or decreasing.
Remark 4.7. If \( T(a) \land a = 0 \) and \( T \) is measure inverting, then every \( L \)-coregular measure is \( L^* \)-regular. These concepts are equivalent when \( T \) is complement.

From now on we assume that \( T(a \land b) = T(a) \lor T(b) \). Now \( L^* \) is a lattice.

Example 4.8. See Examples 4.6 and 3.4.

Proposition 4.9. \( F \) is a prime \( L \)-filter if \( \tilde{F} \) is a prime \( L^* \)-filter. (Please see Definition 4.1 and Proposition 4.2 for the converse.)

Proof. (a) \( 0 \notin F \) since \( e \in \tilde{F} \).
(b) \( a, b \in F \Rightarrow a^* \lor b^* \notin \tilde{F} \Rightarrow (a \land b)^* \notin \tilde{F} \Rightarrow a \land b \in F \).
(c) \( a \leq b \) and \( a \in F \Rightarrow b^* \leq a^* \) and \( a^* \notin \tilde{F} \Rightarrow b^* \notin \tilde{F} \Rightarrow b \in F \).
(d) \( a \lor b \in F \Rightarrow (a \lor b)^* \notin \tilde{F} \Rightarrow a^* \land b^* \notin \tilde{F} \Rightarrow a^* \notin \tilde{F} \) or \( b^* \notin \tilde{F} \Rightarrow a \in F \) or \( b \in F \).

Corollary 4.10. (a) \( F \) is a prime \( L \)-filter if and only if \( \tilde{F} \) is a prime \( L^* \)-filter.
(b) \( F \) is a prime filter if \( F \) is a coultra filter.

5. Covering Properties

In this section we define some covering properties for \( L \) and show that they are analogous to the topological ones. In particular, when \( T \) is taken to be complement, we get topological results as corollaries.

Definition 5.1. \( L \) is \textbf{comax compact} if and only if every coultra filter is fixed. \( L \) is \textbf{comax \( \aleph_0 \)-compact} if and only if every coultra filter has cmp.

Proposition 5.2. Every comax compact \( R \)-lattice is compact.

Proof. Let \( F \) be a prime \( L \)-filter. Form \( \tilde{F} \) and extend it to \( \tilde{G} \), an \( L^* \)-ultrafilter. \( G \) is a prime \( L \)-filter and fixed. \( G \subseteq F \) implies that \( F \) is fixed since \( L \) is an \( R \)-lattice. Thus \( L \) is compact by Proposition 3.6.

Corollary 5.3. If \( T(a) \land a = 0 \) and \( L \) is a comax compact regular lattice, then \( L \) is compact.

Definition 5.4. \( L \) is \textbf{(prime, max, comax) complete} if and only if every (prime, ultra-, coultra) filter with cmp is fixed.

We have the implications in Figure 3.

Proposition 5.5. If \( L \) is \( \text{(comax-)} \aleph_0 \)-compact, then it is \( \text{(comax)} \) complete if and only if it is \( \text{(comax)} \) compact.
Proposition 5.6. If \( \mathcal{L} \) is an R-lattice, then \( \mathcal{L} \) is prime complete if and only if it is comax complete.

Proof. In the proof of Proposition 5.2, let \( F \) have cmp. \( \Box \)

Proposition 5.7. If \( \mathcal{L} \) is a max complete I-lattice, then \( \mathcal{L} \) is complete.

Proof. Let \( F \) be a filter with cmp. Since \( \mathcal{L} \) is an I-lattice, \( F \) can be extended to \( G \), an ultrafilter with cmp. \( G \) is fixed because \( \mathcal{L} \) is max complete. Hence \( F \) is fixed and \( \mathcal{L} \) is complete. \( \Box \)

Corollary 5.8. If \( L \) is a max complete P-lattice, then \( \mathcal{L} \) is prime complete.

Proof. In the proof of Proposition 5.7, take \( F \) to be a prime filter. \( \Box \)

Remark 5.9. When \( T \) is defined on \( \tau(\mathcal{L}) \) (as when \( \mathcal{L} = \tau(\mathcal{L}) \)) and \( T(\bigwedge_{a} a_{a}) = \bigvee_{a} T(a_{a}) \forall a_{a} \in \mathcal{L} \), our definitions coincide with the conventional topological ones. (See Examples 3.4 and 4.6.) In particular \( T \) may be taken to be complement.

Proposition 5.10. Let \( T \) be defined on \( \tau(\mathcal{L}) \) with \( T(\bigwedge_{a} a_{a}) = \bigvee_{a} T(a_{a}) \) for all \( a_{a} \in \mathcal{L} \).

(a) \( \mathcal{L} \) is compact if and only if \( e = \bigvee_{a} a_{a} \Rightarrow e = \bigvee_{a} a_{a} \).

(b) \( \mathcal{L} \) is complete if and only if \( e = \bigvee_{a} a_{a} \Rightarrow e = \bigvee_{a} a_{a} \).

(c) \( \mathcal{L} \) is \( \aleph_{0} \)-compact if and only if \( e = \bigvee_{a} a_{a} \Rightarrow e = \bigvee_{a} a_{a} \).

Proof. We will prove only part (b). Parts (a) and (c) have similar proofs.

Suppose the condition holds. Let \( F = \{ f_{a} \} \) be a free \( \mathcal{L} \)-filter. Then \( \bigwedge_{a} f_{a} = 0 \) implies that \( \bigvee_{a} f_{a} = e \), so that there exists \( \{ f_{a} \} \subseteq \{ f_{a} \} \) such that \( \bigvee_{a} f_{a} = e \). But then \( \bigwedge_{a} f_{a} = 0 \) and \( F \) does not have cmp, so \( \mathcal{L} \) is complete.

Conversely, suppose that the condition does not hold. Then there exists \( \{ f_{a} \} \) such that \( \bigvee_{a} f_{a} = e \) but \( \bigvee_{a} f_{a} = 0 \) for any countable subset. Then \( 0 \neq \bigwedge_{a} a_{a} \) and \( \{ a_{a} \} \) is a subbase for a filter \( F \). \( F \) has cmp since \( \{ f_{j} \} \subseteq F \) implies that \( \bigwedge_{a} f_{a} \geq \bigwedge_{a} a_{a} > 0 \). \( F \) is free since \( e = \bigvee_{a} a_{a} \) implies that \( \bigwedge_{a} a_{a} = 0 \), and therefore, \( \mathcal{L} \) is not complete. \( \Box \)

Corollary 5.11. Let \( X \) be a topological space. Take \( \mathcal{L} \) to be the closed sets and take \( T \) to be complement. Then \( X \) is compact (Lindelöf, countably compact) if and only if \( \mathcal{L} \) is compact (complete, \( \aleph_{0} \)-compact).

Corollary 5.12. \( X \) is a realcompact I-space if and only if it is a Lindelöf space [14].

Corollary 5.13. If \( X \) is regular, countably paracompact, and almost realcompact, then \( X \) is realcompact [6].

6. Lattice Interrelations

In this section we investigate the implications between the properties of two lattices when one is a sublattice of the other.

Proposition 6.1. Let \( \mathcal{L}_{1} \subseteq \mathcal{L}_{2} \subseteq \tau(\mathcal{L}_{1}) \)

(a) If \( \mathcal{L}_{1} \) is complete, then \( \mathcal{L}_{2} \) is complete.

(b) If \( \mathcal{L}_{1} \) is prime complete, then \( \mathcal{L}_{2} \) is prime complete.

(c) If \( \mathcal{L}_{1} \) is a P-lattice, then that \( \mathcal{L}_{1} \) is max complete implies that \( \mathcal{L}_{2} \) is prime complete.
Proof. (a) Let $F$ be an $L_2$-filter with cmp, $G = F \cap L_1$, and $b \in F$. There exists $\{a_\alpha\} \subseteq L_1$ such that $b = \bigwedge a_\alpha$. As $b \leq a_\alpha$ for all $a_\alpha$, we have $a_\alpha \in F \cap L_1$ (which $G$), for all $a$. Now let $\{b_\beta\} \subseteq F$. $\bigwedge b_\beta = \bigwedge a_\alpha a_\beta \neq 0$, since $G$ is fixed. Thus $F$ is fixed and $L_2$ is complete.

(b) Let $F$ in (a) be prime. Then $G$ is prime.

(c) Since $L_1$ is a P-lattice, $G$ may be extended to an $L_1$-ultrafilter $H$ with cmp. Since $L_1$ is max complete, $H$ is fixed, and hence $G$ and $F$ are fixed.

Corollary 6.2. Let $L_1 \subseteq L_2 \subseteq \tau(L_1)$ and let $L_1$ be a P-lattice. If $L_1$ is max complete, then $L_2$ is max and comax complete.

Corollary 6.3. Let $L_1 \subseteq L_2 \subseteq \tau(L_1)$ and let $L_1$ be $\aleph_0$-normal. If $L_1$ is max complete, then $L_2$ is max and comax complete.

Corollary 6.4. If $X$ is a normal, countably paracompact space, then $Z$-replete implies $F$-replete and realcompact implies $\alpha$-complete [13].

The following proposition generalizes two results of Alexandroff [5].

Proposition 6.5. Let $L_1 \subseteq L_2 \subseteq \tau(L_1)$.

(a) If $L_1$ is compact, then $L_2$ is compact.

(b) If $L_1$ is compact and normal, then $L_2$ is normal.

Proof. (a) Let $F$ be an $L_2$-filter. Let $G = F \cap L_1$. The proof follows as in Proposition 6.1.

(b) Let $a_1, a_2 \in L_2$, $a_1 \wedge a_2 = 0$, where $a_1 = \bigwedge b_\beta, a_2 = \bigwedge c_\gamma$, and of course $b_\beta, c_\gamma \in L_1$. $a_1 \leq b_\beta$ and $a_2 \leq c_\gamma$, for all $\beta, \gamma$. Now there must exist $b_{\beta_0}, c_{\gamma_0}$ such that $b_{\beta_0} \wedge c_{\gamma_0} = 0$. (If not, $\{b_\beta, c_\gamma\}$ would form a subbase for a free filter in a compact space.) By normality of $L_1$, there exist $d_1^*, d_2^* \in L_1^*$ such that $d_1^* \geq b_{\beta_0}$, $d_2^* \geq c_{\gamma_0}$, and $d_1^* \wedge d_2^* = 0$. But then $d_1^* \geq a_1$ and $d_2^* \geq a_2$ and so $L_2$ is normal.

Proposition 6.6. Let $\delta(\mathcal{L})$ denote the smallest set containing $\mathcal{L}$ and closed under countable meets. Let $L_1 \subseteq L_2 \subseteq \delta(\mathcal{L}_1)$. If $L_1$ is $\aleph_0$-compact, then $L_2$ is $\aleph_0$-compact.

Proposition 6.7. Let $L_1 \subseteq L_2$. If $L_2$ is complete, then $L_1$ is complete.

Proof. Let $F$ be an $L_1$-filter with cmp. Let $G = \{a \in L_2 : a \geq f \text{ for some } f \in F\}$. $G$ has cmp since $\bigwedge g_i \geq \bigwedge f_i > 0$, $g_i \in G$, and $f_i \in F$. $L_2$ is complete so $G$ is fixed. Thus $F$ is fixed and $L_1$ is complete.

Corollary 6.8. Let $L_1 \subseteq L_2$ where $L_2$ is an I-lattice.

(a) If $L_2$ is max complete, then $L_1$ is complete.

(b) If $L_2$ is a comax complete R-lattice, then $L_1$ is complete.

Proof. (a) Since $L_2$ is an I-lattice, $L_2$ that is max complete implies that $L_2$ is complete, which implies by Proposition 6.7 that $L_1$ is complete.

(b) Since $L_2$ is an R-lattice, its being comax complete implies that it is prime complete. Since $L_2$ is an I-lattice, $L_2$ is complete and thus $L_1$ is complete by Proposition 6.7.

Definition 6.9. $L_2$ is an $L_1$-$P$-lattice if and only if every $L_1$ prime filter with cmp is contained in an $L_2$ ultrafilter with cmp.
Definition 6.10. (i) $L_2$ is $L_1$-normal if and only if for all $f_1, f_2 \in L_2$ with $f_1 \wedge f_2 = 0$ there exist $g_1^*, g_2^* \in L_1^*$ such that $f_1 \leq g_1^*$, $f_2 \leq g_2^*$, and $g_1^* \wedge g_2^* = 0$.

(ii) $L_2$ is $L_1$-$\aleph_0$-paracompact if and only if for each $\{a_n\} \subseteq L_2$ such that $a_n \downarrow 0$ there exists a sequence $\{b_n\} \subseteq L_1$ such that $a_n \leq b_n^*$ and $b_n^* \downarrow 0$.

(iii) $L_2$ is $L_1$-$\aleph_0$-normal if and only if $L_2$ is $L_1$-normal and $L_1$-$\aleph_0$-paracompact.

Proposition 6.11. Let $T$ be complement. If $L_2$ is $L_1$-$\aleph_0$-normal and $F_1$ is a prime $L_1$-filter with cmp contained in an $L_2$-filter $F_2$, then $F_2$ has cmp.

Proof. Similar to the proof of Proposition 3.19.

Corollary 6.12. Let $T$ be complement.

(a) That $L_2$ is $L_1$-$\aleph_0$-normal implies that $L_2$ is an $L_1$-P-lattice.

(b) If $L_2$ is $\aleph_0$-paracompact and $L_1$ separates $L_2$, then $L_2$ is $L_1$-$\aleph_0$-paracompact.

Corollary 6.13. (a) Let $L_2$ be an $L_1$-P-lattice. That $L_2$ is max complete implies that $L_1$ is prime complete.

(b) If in addition $L_2$ is an R-lattice, then that $L_2$ is comax complete implies that $L_1$ is prime complete.

We get Frolik’s [8] theorems as our final corollary.

Corollary 6.14. (a) If $L_1 \subseteq L_2$ and $L_2$ is $L_1$-$\aleph_0$-normal and prime complete, then $L_1$ is prime complete.

(b) Let $X$ be a normal $T_2$ space. If $X$ is almost realcompact and countably paracompact, then $X$ is realcompact.

Acknowledgments

This paper is based on the Ph.D. thesis the author wrote under the supervision of the late George Bachman at The Polytechnic Institute of Brooklyn, now Polytechnic University of NYU. The author is grateful for Professor Bachman’s patient guidance and encouragement. This material was presented at The Second Annual Dr. George Bachman Memorial Conference at St. John’s University Manhattan Campus on June 7, 2009. The author appreciates the comments and encouragement provided by Professors Keith Harrow, Pao-Sheng Hsu, and Noson Yanofsky and the questions raised by the anonymous reviewers. This paper is dedicated to the memory of Professor George Bachman.

References