Research Article

Common Fixed Point Theorem of
Two Mappings Satisfying a Generalized Weak
Contractive Condition

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Existence of common fixed point for two mappings which satisfy a generalized weak contractive condition is established. As a consequence, a common fixed point result for mappings satisfying a contractive condition of integral type is obtained. Our results generalize, extend, and unify several well-known comparable results in literature.

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1. Introduction and Preliminaries

Let \( X \) be a metric space and \( T : C \rightarrow C \) a mapping. Recall that \( T \) is contraction if \( d(Tx, Ty) \leq kd(x, y) \) for all \( x, y \in X \), where \( 0 \leq k < 1 \). A point \( x \in C \) is a fixed point of \( T \) provided \( Tx = x \). If a map \( T \) satisfies \( F(T) = F(T^n) \) for each \( n \in N \), where \( F(T) \) denotes the set of all fixed points of \( T \), then it is said to have property \( P \). Banach contraction principle which gives an answer on existence and uniqueness of a solution of an operator equation \( Tx = x \) is the most widely used fixed point theorem in all of analysis. Branciari [1] obtained a fixed point theorem for a mapping satisfying an analogue of Banach’s contraction principle for an integral type inequality. Akgun and Rhoades [2] have shown that a map satisfying a Meir–Keeler type contractive condition of integral type has a property \( P \). Rhoades and Abbas [3] extended [4, Theorem 1] for mappings satisfying contractive condition of integral type. They also studied several results for maps which have property \( P \), defined on a metric space satisfying generalized contractive conditions of integral type. Rhoades [5] proved two fixed point theorems involving more general contractive condition of integral type (see, also [6, 7]). If maps \( S \) and \( T \) satisfy \( F(S) \cap F(T) = F(S^n) \cap F(T^n) \) for each \( n \in N \), then they are said to have property \( Q \). Jeong and Rhoades [8] studied the property \( Q \) for pairs of maps satisfying a number of contractive conditions.
Recently Dutta and Choudhury [9] gave a generalization of Banach contraction principle, which in turn generalize [4, Theorem 1] and corresponding result of [10]. Sessa [11] defined the concept of weakly commuting to obtain common fixed point for pairs of maps. Jungck generalized this idea, first to compatible mappings [12] and then to weakly compatible mappings [13]. There are examples that show that each of these generalizations of commutativity is a proper extension of the previous definition. The aim of this paper is to present a common fixed point theorem for weakly compatible maps satisfying a generalized weak contractive condition which is more general than the corresponding contractive condition of integral type. Our results substantially extend, improve, and generalize comparable results in literature [3, 14, 15].

The following definitions and results will be needed in the sequel.

**Definition 1.1.** Let $X$ be a set, and $f, g$ selfmaps of $X$. A point $x$ in $X$ is called a coincidence point of $f$ and $g$ if and only if $fx = gx$. We will call $w = fx = gx$ a point of coincidence of $f$ and $g$.

**Definition 1.2.** Two maps $f$ and $g$ are said to be weakly compatible if they commute at their coincidence points.

**Lemma 1.3** (see [16]). Let $f$ and $g$ be weakly compatible self maps of a set $X$. If $f$ and $g$ have a unique point of coincidence $w$ (say), then $w$ is the unique common fixed point of $f$ and $g$.

### 2. A Common Fixed Point Theorem

Set $f = \{\phi : R^+ \to R^+ : \phi \text{ is a Lebesgue integrable mapping which is summable and nonnegative and satisfies } \int_0^\varepsilon \phi(t) dt > 0, \text{ for each } \varepsilon > 0 \}$ and $G = \{\psi : [0, \infty] \to [0, \infty] : \psi \text{ is continuous and nondecreasing mapping with } \psi(t) = 0 \text{ if and only if } t = 0\}$.

The following is the main result of this paper.

**Theorem 2.1.** Let $f, g$ be two self maps of a metric space $(X, d)$ satisfying

$$\psi(d(fx, fy)) \leq \psi(d(gx, gy)) - \psi(d(gx, gy)) \quad (2.1)$$

for all $x, y \in X$, where $\psi, \varphi \in G$. If range of $g$ contains the range of $f$ and $g(X)$ is a complete subspace of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover if $f$ and $g$ are weakly compatible, $f$ and $g$ have a unique common fixed point.

**Proof.** Let $x_0$ be an arbitrary point of $X$. Choose a point $x_1$ in $X$ such that $f(x_0) = g(x_1)$. This can be done, since the range of $g$ contains the range of $f$. Continuing this process, having chosen $x_n$ in $X$, we obtain $x_{n+1}$ in $X$ such that $f(x_n) = g(x_{n+1})$, $n = 0, 1, 2, \ldots$. Suppose for any $n, g(x_n) \neq g(x_{n+1})$, since, otherwise, $f$ and $g$ have a point of coincidence. From (2.1), we have

$$\psi(d(gx_{n+1}, gx_n)) = \psi(d(fx_n, fx_{n-1}))$$
$$\leq \psi(d(gx_n, gx_{n-1})) - \psi(d(gx_n, gx_{n-1}))$$
$$< \psi(d(gx_n, gx_{n-1})), \quad (2.2)$$
that is, \( \psi(d(g_{n+1}, g_n)) < \psi(d(g_n, g_{n-1})) \), and hence

\[
d(g_n, g_{n+1}) \leq d(g_n, g_{n-1}).
\] (2.3)

It follows that \( \{d(g_n, g_{n+1})\} \) is monotone decreasing sequence of numbers and consequently there exists \( r \geq 0 \) such that \( d(g_n, g_{n+1}) \to r \) as \( n \to \infty \). Suppose that \( r > 0 \), then

\[
0 < \psi(r) \leq \psi(d(f_n, f_{n-1})) = \psi(d(g_n, g_{n-1})) \leq \psi(d(g_n, g_{n-1})) - \psi(d(g_n, g_{n-1})) = 0,
\] (2.4)

which on taking limit as \( n \to \infty \) yields

\[
\psi(r) \leq \psi(r) - \psi(r) < \psi(r),
\] (2.5)

which is a contradiction. Therefore \( r = 0 \). Now we prove that \( \{g_n\} \) is a Cauchy sequence. If not, then there exist some \( \varepsilon > 0 \) and subsequences \( \{g_{n_k}\} \) and \( \{g_{m_k}\} \) of \( \{g_n\} \) with \( k < n_k < m_k \) such that \( d(g_{n_k}, g_{m_k}) \geq 3\varepsilon \) for each \( k \). As \( d(g_{n_k+1}, g_{m_k}) \to 0 \) as \( k \to \infty \), for large enough \( k \), we have \( d(g_{n_k+1}, g_{n_k}) < \varepsilon \) and \( d(g_{m_k+1}, g_{m_k}) < \varepsilon \). Thus we obtain

\[
d(g_{n_k+1}, g_{m_k}) \geq d(g_{n_k}, g_{m_k}) - d(g_{n_k+1}, g_{n_k}) > \varepsilon,
\]

\[
d(g_{n_k+1}, g_{m_k-1}) \geq d(g_{n_k}, g_{m_k}) - d(g_{m_k-1}, g_{m_k}) - d(g_{n_k+1}, g_{n_k}) > \varepsilon.
\] (2.6)

We may assume that \( n_k \) are even and \( m_k \) are odd and that \( d(g_{n_k}, g_{m_k}) > \varepsilon \) for all \( k \). Put

\[
r_k = \min\{m_k : d(g_{n_k}, g_{m_k}) > \varepsilon\}.
\] (2.7)

Now,

\[
\varepsilon < d(g_{n_k}, g_{r_k}) \leq d(g_{n_k}, g_{r_k-2}) + d(g_{r_k-2}, g_{r_k-1}) + d(g_{r_k-1}, g_{n_k})
\] (2.8)

implies that \( d(g_{n_k}, g_{r_k}) \to \varepsilon \) as \( k \to \infty \). Furthermore

\[
d(g_{n_k}, g_{r_k}) - d(g_{n_k}, g_{n_k+1}) - d(g_{r_k}, g_{r_k+1})
\]

\[
\leq d(g_{n_k+1}, g_{n_k+1}) \leq d(g_{n_k}, g_{n_k}) + d(g_{n_k}, g_{n_k+1}) + d(g_{r_k}, g_{r_k+1})
\] (2.9)
gives \( d(gx_{n_k+1}, gx_{n_k}) \to \varepsilon \), as \( k \to \infty \). Therefore

\[
\psi(d(gx_{n_k+1}, gx_{n_k})) = \psi(d(fx_{n_k}, fx_{n_k})) \\
\leq \psi(d(gx_{n_k}, gx_{n_k})) - \psi(d(gx_{n_k}, gx_{n_k})).
\]

(2.10)

Taking limit as \( k \to \infty \) yields

\[
\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon),
\]

(2.11)

which is a contradiction. Hence \( \{gx_n\} \) is a Cauchy sequence. From completeness of \( g(X) \), there exists a point \( q \) in \( g(X) \) such that \( gx_n \to q \) as \( n \to \infty \). Consequently, we can find \( p \) in \( X \) such that \( g(p) = q \). Now

\[
\psi(d(gx_{n+1}, fp)) = \psi(d(fx_n, fp)) \\
\leq \psi(d(gx_n, gp)) - \psi(d(gx_n, gp))
\]

(2.12)

on taking limit as \( n \to \infty \) implies

\[
\psi(d(q, fp)) \leq \psi(0) - \phi(0),
\]

(2.13)

\( \psi(d(q, fp)) = 0 \), and \( f(p) = q \). Hence \( q \) is the point of coincidence of \( f \) and \( g \). Assume that there is another point of coincidence \( r \) in \( X \) such that \( r \neq q \). Then there exists \( s \) in \( X \) such that \( f(s) = g(s) = r \). Using (2.1), we have

\[
\psi(d(gp, gs)) = \psi(d(fp, fs)) \\
\leq \psi(d(gp, gs)) - \psi(d(gp, gs))
\]

(2.14)

which is a contradiction which proves the uniqueness of point of coincidence; the result now follows from Lemma 1.3.

\( \square \)

**Corollary 2.2.** Let \( f, g \) be two self maps of a metric space \( (X, d) \) satisfying

\[
\int_0^{\psi(d(fx, fy))} \phi(t)dt \leq \int_0^{\psi(d(gx, gy))} \phi(t)dt - \int_0^{\psi(d(gx, gy))} \phi(t)dt
\]

(2.15)

for all \( x, y \in X \), where \( \phi \in F \) and \( \psi, \phi \in G \). If range of \( g \) contains the range of \( f \) and \( g(X) \) is a complete subspace of \( X \), then \( f \) and \( g \) have a unique point of coincidence in \( X \). Moreover if \( f \) and \( g \) are weakly compatible, \( f \) and \( g \) have a unique common fixed point.
Proof. Define \( \Phi : \mathbb{R}^+ \to \mathbb{R}^+ \) by \( \Phi(x) = \int_0^x \phi(t) dt \), then \( \Phi \in G \) and (2.15) becomes

\[
\Phi(\psi(d(fx, fy))) \leq \Phi(\psi(d(gx, gy))) - \Phi(\psi(d(gx, gy))),
\]

(2.16)

which further can be written as

\[
\psi_1(d(fx, fy)) \leq \psi_1(d(gx, gy)) - \psi_1(d(gx, gy)),
\]

(2.17)

where \( \psi_1 = \Phi \circ \psi \) and \( \psi_1 = \Phi \circ \varphi \in G \). Clearly \( \psi_1, \varphi_1 \in G \). Hence by Theorem 2.1 \( f \) and \( g \) have unique common fixed point. \( \square \)

Now we present two examples in the support of Theorem 2.1.

**Example 2.3.** Let \( X = [0, 1] \cup \{2, 3, 4, \ldots\} \),

\[
d(x, y) = \begin{cases} 
|x - y|, & \text{if } x, y \in [0, 1], \ x \neq y, \\
x + y, & \text{if at least one of } x \text{ or } y \notin [0, 1], \ x \neq y, \\
0, & \text{if } x = y.
\end{cases}
\]

(2.18)

Then \( (X, d) \) is a complete metric space [17]. Consider \( f : X \to X \), and \( \psi, \varphi \in G \) as given in [9]:

\[
f(x) = \begin{cases} 
x - \frac{1}{2} x^2, & \text{if } 0 \leq x \leq 1, \\
x - 1, & \text{if } x > 1,
\end{cases}
\]

(2.19)

\[
\psi(t) = \begin{cases} 
t, & \text{if } 0 \leq t \leq 1, \\
t^2, & \text{if } t \geq 1,
\end{cases}
\]

(2.20)

\[
\varphi(t) = \begin{cases} 
\frac{1}{2} t^2, & \text{if } 0 \leq t \leq 1, \\
\frac{1}{2}, & \text{if } t \geq 1.
\end{cases}
\]

(2.21)

Let \( g : X \to X \) be defined as

\[
g(x) = \begin{cases} 
x, & \text{if } 0 \leq x \leq 1, \\
x + 1, & \text{if } x > 1.
\end{cases}
\]

(2.22)
Assume that $x > y$ and discuss the following cases.

When $x \in [0, 1]$, then

$$\psi(d(fx, fy)) = \left(x - \frac{1}{2} x^2\right) - \left(y - \frac{1}{2} y^2\right)$$

$$\leq (x - y) - \frac{1}{2} (x - y)^2$$

$$= \psi(d(gx, gy)) - \psi(d(gx, gy)).$$

Taking $x$ in $\{3, 4, \ldots\}$, and $y$ in $[0, 1]$, we obtain

$$\psi(d(fx, fy)) = \left(x - 1 + y - \frac{1}{2} y^2\right)^2$$

$$\leq (x + y - 1)^2,$$

$$\psi(d(gx, gy)) = (x + y + 1)^2,$$

$$\psi(d(gx, gy)) = \frac{1}{2}.$$

Hence

$$\psi(d(fx, fy)) \leq \psi(d(gx, gy)) - \psi(d(gx, gy)).$$

Now, when $x \in [3, 4, \ldots]$, and $y \notin [0, 1]$, then

$$\psi(d(fx, fy)) = (x - 1 + y - 1)^2$$

$$< (x + y - 1)^2,$$

$$\psi(d(gx, gy)) = (x + y + 2)^2,$$

$$\psi(d(gx, gy)) = \frac{1}{2}.$$

Obviously (2.31) holds. Finally when $x = 2$, we have $y \in [0, 1]$, $fx = 1$, and

$$d(fx, fy) = 1 - \left(y - \frac{1}{2} y^2\right) \leq 1,$$
so that \( \psi(d(fx, fy)) \leq 1 \), then

\[
\psi(d(gx, gy)) - \psi(d(gx, gy)) = (3 + y)^2 - \frac{1}{2} > 1 \geq \psi(d(fx, fy)).
\] (2.28)

Thus all conditions of Theorem 2.1 are satisfied. Moreover \( f \) and \( g \) have a unique common fixed point.

**Example 2.4.** Let \( X = [0, 1] \) and \( f, g : X \rightarrow X \) be given as

\[
f(x) = \frac{2}{5}x^2 + \frac{3}{5}, \quad g(x) = \frac{2}{3}x^2 + \frac{1}{3}.
\] (2.29)

Consider \( \psi, \varphi \in G \) as \( \psi(t) = (1/2)t \) and \( \varphi(t) = (1/10)t \). Then we have

\[
\psi(d(fx, fy)) = \frac{2}{10}(x^2 - y^2) \leq \frac{1}{2} \frac{2}{3}(x^2 - y^2) - \frac{1}{10} \frac{2}{3}(x^2 - y^2) = \psi(d(gx, gy)) - \psi(d(gx, gy)).
\] (2.30)

Note that \( x = 1 \) is the unique coincidence point of \( f \) and \( g \), and \( f \) and \( g \) are commuting at \( x = 1 \). Hence all conditions of Theorem 2.1 are satisfied. Moreover, \( x = 1 \) is the unique common fixed point of \( f \) and \( g \).

Following theorem can be viewed as generalization and extension of [3, Theorem 3].

**Theorem 2.5.** Let \( f \) be a self map of a complete metric space \( (X, d) \) satisfying

\[
\int_0^{\psi(d(fx, fy))} \phi(t)dt \leq \int_0^{\psi(d(x, y))} \phi(t)dt - \int_0^{\psi(d(x, y))} \phi(t)dt
\] (2.31)

for all \( x, y \in X \), where \( \phi \in \Gamma \) and \( \psi, \varphi \in G \). Then \( f \) has a unique fixed point. Moreover \( f \) has property P.
Proof. Existence and uniqueness of fixed point of $f$ follows from Corollary 2.2. Now we prove that $f$ has property $P$. Let $u \in F(f^n)$. We shall always assume that $n > 1$, since the statement for $n = 1$ is trivial. We claim that $fu = u$. If not, then, by (2.31),

$$
\int_0^\phi \frac{(d(u, fu))}{\phi(t)dt} = \int_0^\phi \frac{(d(f^n u, f^n u))}{\phi(t)dt} = \int_0^\phi \frac{(d(f^{n-1} u, f^{n-1} u))}{\phi(t)dt} \\
\leq \int_0^\phi \frac{(d(f^{n-1} u, f^{n-1} u))}{\phi(t)dt} - \int_0^\phi \frac{(d(f^{n-2} u, f^{n-2} u))}{\phi(t)dt} \\
\leq \int_0^\phi \frac{(d(f^{n-2} u, f^{n-2} u))}{\phi(t)dt} - \int_0^\phi \frac{(d(f^{n-3} u, f^{n-3} u))}{\phi(t)dt} \\
\leq \int_0^\phi \frac{(d(f^{n-3} u, f^{n-3} u))}{\phi(t)dt} \\
\leq \int_0^\phi \frac{(d(f^{n-4} u, f^{n-4} u))}{\phi(t)dt} \\
\leq \int_0^\phi \frac{(d(f^{n-5} u, f^{n-5} u))}{\phi(t)dt} \\
\leq \int_0^\phi \frac{(d(f^{n-k} u, f^{n-k} u))}{\phi(t)dt} \\
\leq \int_0^\phi \frac{(d(f^{n-k+1} u, f^{n-k+1} u))}{\phi(t)dt}
$$

(2.32)

Continuing this process we arrive at

$$
\int_0^\phi \frac{(d(u, fu))}{\phi(t)dt} \leq \int_0^\phi \frac{(d(f^{n-k} u, f^{n-k} u))}{\phi(t)dt} - \int_0^\phi \frac{(d(f^{n-k+1} u, f^{n-k+1} u))}{\phi(t)dt} \\
< \int_0^\phi \frac{(d(f^{n-k} u, f^{n-k} u))}{\phi(t)dt},
$$

(2.33)

which is a contradiction. Hence the result follows.

Remarks 2.6. Existence and uniqueness of fixed point of $f$ in above theorem also follows from [9, Theorem 1].

Remarks 2.7. (a) It is noted that if maps $f$ and $g$ involved in Theorem 2.1 are commuting, then they have property $Q$.

(b) Suzuki [18] observed that Branciari [1, Theorem 1] is a particular case of Meir-Keeler fixed point theorem [19]. We pose an open problem to see if a link exists between the contractive conditions (2.15) and the Meir-Keeler condition.

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