Research Article

Weak Forms of Continuity and Associated Properties

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We introduce slightly $p$-continuous mapping and almost $p$-open mapping and investigate the relationships between these mappings and related types of mappings, and also study some properties of these mappings.

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1. Introduction and preliminaries

A subset $A$ of a space $X$ is called regular open if $A = \text{Int} \overline{A}$, and regular closed if $X \setminus A$ is regular open, or equivalently, if $A = \text{Int} A$. It is well known that a subset $A$ of a space $X$ is regular open if and only if $A = \text{Int} F$, where $F$ is closed and $A$ is regular closed if and only if $A = \overline{U}$, where $U$ is open. $A$ is called semi-open [1] (resp., preopen [2], semi-preopen [3], $b$-open [4]) if $A \subseteq \text{Int} A$ (resp., $A \subseteq \text{Int} \overline{A}$, $A \subseteq \text{Int} \overline{A}$, $A \subseteq \text{Int} \overline{A} \cup \text{Int} \overline{A}$). It is known that a set $A$ is semi-open if and only if $U \subseteq A \subseteq \overline{U}$ for some open set $U$, and that $A$ is preopen (resp., semi-preopen) if and only if $A = U \cap D$, where $U$ is open (resp., semi-open) and $D$ is dense.

The concept of a preopen set was introduced in [5], where the term locally dense was used and the concept of a semi-preopen set was introduced in [6] under the name $\beta$-open. It was pointed out in [3] that $A$ is semi-preopen if and only if $P \subseteq A \subseteq \overline{P}$ for some preopen set $P$. Clearly, every regular closed set is semi-open, every open set is both semi-open and preopen, semi-open sets as well as preopen sets are $b$-open and $b$-open sets are semi-preopen. It is also known that the closure of every semi-preopen set is regular closed and that the arbitrary union of semi-open (resp., preopen, semi-preopen, $b$-open) sets is semi-open (resp., preopen, semi-preopen, $b$-open). $A$ is called semi-closed (resp., preclosed, semi-preclosed, $b$-closed) if $X \setminus A$ is semi-open (resp., preopen, semi-preopen, $b$-open). It is well known that a subset $A$ is regular closed if and only if $A$ is both closed and semi-open if and only if $A$ is both closed and semi-preopen.
A mapping $f$ from a space $X$ into a space $Y$ is called regular open [7] if it maps regular open subsets onto regular open sets, almost open [8] if $f^{-1}(\overline{U}) \subseteq \overline{f^{-1}(U)}$ whenever $U$ is open in $Y$, slightly continuous [7] if $f(\overline{U}) \subseteq \overline{f(U)}$ whenever $U$ is open in $X$, semi-continuous [1] if the inverse image of each open set is semi-open, $\beta$-continuous [6] if the inverse image of each open set is $\beta$-open, weakly continuous [9] if for each $x \in X$ and for each open set $V$ containing $f(x)$ there exists an open set $U$ containing $x$ such that $f(U) \subseteq \overline{V}$, weakly $\theta$-irresolute [10] if the inverse image of each regular closed set is semi-open, rc-continuous [11] if the inverse image of each regular closed set is regular closed, and wrc-continuous [12] if the inverse image of each regular closed set is semi-open. We will use the term semi-precontinuous to indicate $\beta$-continuous. Clearly, every semi-continuous mapping is semi-precontinuous, every rc-continuous mapping is weakly $\theta$-irresolute, and every weakly $\theta$-irresolute mapping is wrc-continuous. In [7], it is shown that the properties semi-continuous and slightly continuous are independent of each other.

A space $X$ is called a weak $P$-space [13] if for each countable family $\{U_n : n \in N\}$ of open subsets of $X$, $\bigcup U_n = \bigcup \overline{U}_n$. Clearly, $X$ is a weak $P$-space if and only if the countable union of regular closed subsets of $X$ is regular closed (closed).

A space $X$ is called rc-Lindelöf [14] if every regular closed cover of $X$ has a countable subcover, and called almost rc-Lindelöf [15] if every regular closed cover of $X$ has a countable subfamily whose union is dense in $X$.

A subset $A$ of a space $X$ is called an $S$-set in $X$ [16] if every cover of $A$ by regular closed subsets of $X$ has a finite subcover, and called an rc-Lindelöf set in $X$ (resp., an almost rc-Lindelöf set in $X$) [17] if every cover of $A$ by regular closed subsets of $X$ admits a countable subfamily that covers $A$ (resp., the closure of the union of whose members contains $A$). Obviously, every $S$-set is an rc-Lindelöf set and every rc-Lindelöf set is an almost rc-Lindelöf set. It is also clear that a subset $A$ of a weak $P$-space $X$ is rc-Lindelöf in $X$ if and only if it is almost rc-Lindelöf in $X$.

Throughout this paper, $N$ (resp., $Q$, $R$) denotes the set of natural (resp., rational, real) numbers. For the concepts not defined here, we refer the reader to [18].

## 2. Slightly $p$-continuous mappings

This section is mainly devoted to study several properties of slightly $p$-continuous mappings. Now, we begin with the following lemma which was pointed out in [19] without proof. We will, however, state and prove it for its special importance in the material of our paper.

**Lemma 2.1.** (i) Let $f : X \to Y$ be a semi-continuous and almost open mapping. Then $f$ is weakly $\theta$-irresolute.

(ii) Let $f : X \to Y$ be a semi-precontinuous and almost open mapping. Then $f$ is wrc-continuous.

**Proof.** (i) Let $U$ be an open subset of $Y$. Since $f$ is almost open, then $f^{-1}(\overline{U}) \subseteq \overline{f^{-1}(U)}$. Since $f$ is semi-continuous, then $f^{-1}(U)$ is semi-open, hence there exists an open subset $V$ of $X$ such that $V \subseteq f^{-1}(U) \subseteq \overline{V}$, therefore, $V \subseteq f^{-1}(U) \subseteq f^{-1}(\overline{U}) \subseteq \overline{f^{-1}(U)} \subseteq \overline{V}$. Thus $f^{-1}(\overline{U})$ is semi-open, and $f$ is weakly $\theta$-irresolute.

(ii) Let $U$ be an open subset of $Y$. Since $f$ is almost open, then $f^{-1}(\overline{U}) \subseteq \overline{f^{-1}(U)}$. Since $f$ is semi-precontinuous, then $f^{-1}(U)$ is semi-preopen, hence there exists a preopen subset $V$ of $X$ such that $V \subseteq f^{-1}(U) \subseteq \overline{V}$, therefore, $V \subseteq f^{-1}(U) \subseteq f^{-1}(\overline{U}) \subseteq \overline{f^{-1}(U)} \subseteq \overline{V}$. Thus $f^{-1}(\overline{U})$ is semi-preopen, and $f$ is wrc-continuous. $\square$
Corollary 2.2 (see [12]). Let $f : X \to Y$ be a semi-continuous and almost open mapping. Then $f$ is wrc-continuous.

Proposition 2.3. For a mapping $f : X \to Y$, the following are equivalent:

(i) $f$ is slightly continuous;

(ii) $f(\bar{U}) \subset f(\overline{U})$ whenever $U$ is semi-open in $X$.

Proof. Since every open set is semi-open, it suffices to show that (i) $\to$ (ii). Let $U$ be a semi-open subset of $X$. Then there exists an open subset $V$ of $X$ such that $V \subset U \subset V$. Thus by (i), $f(\bar{U}) = f(\bar{V}) \subset f(\overline{U})$.

Proposition 2.4. Let $f : X \to Y$ be a slightly continuous mapping. Then the following are equivalent:

(i) $f$ is weakly $\theta$-irresolute;

(ii) $f$ is rc-continuous.

Proof. Since every regular closed set is semi-open, it suffices to show that (i) $\to$ (ii). Let $A$ be a regular closed subset of $Y$. By (i), $f^{-1}(A)$ is semi-open, but $f$ is slightly continuous, so by Proposition 2.3, $f(f^{-1}(A)) \subset f(f^{-1}(A)) \subset \overline{A} = A$. Thus $f^{-1}(A) \subset f^{-1}(f^{-1}(A)) \subset f^{-1}(A)$, that is, $f^{-1}(A)$ is closed, but $f^{-1}(A)$ is semi-open, so $f^{-1}(A)$ is regular closed. Hence $f$ is rc-continuous.

Corollary 2.5. Let $f : X \to Y$ be a slightly continuous, semi-continuous, and almost open mapping. Then $f$ is rc-continuous.

Proof. Follows from Lemma 2.1(i) and Proposition 2.4.

Proposition 2.6. Let $f : X \to Y$ be a slightly continuous and semi-continuous mapping. Then $f^{-1}(U) \subset f^{-1}(U)$ for every open subset $U$ of $Y$.

Proof. Let $U$ be an open subset of $Y$. Since $f$ is semi-continuous, it follows that $f^{-1}(U)$ is semi-open, but $f$ is slightly continuous, so it follows from Proposition 2.3 that $f(f^{-1}(U)) \subset f^{-1}(U)$ and $f^{-1}(U) \subset f^{-1}(U)$. Thus $f^{-1}(U) \subset f^{-1}(f^{-1}(U)) \subset f^{-1}(U)$.

The following corollary is a slight improvement of Corollary 2.5. This is because the closure of every semi-open set is regular closed.

Corollary 2.7. Let $f : X \to Y$ be a slightly continuous, semi-continuous, and almost open mapping. Then $f^{-1}(U) \subset f^{-1}(U)$ for every open subset $U$ of $Y$.

Proposition 2.8. Let $f : X \to Y$ be an rc-continuous mapping. If $A$ is rc-Lindelöf in $X$, then $f(A)$ is rc-Lindelöf in $Y$.

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of $f(A)$ by regular closed subsets of $Y$. Then $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a cover of $A$ by regular closed subsets of $X$ (as $f$ is rc-continuous). Since $A$ is rc-Lindelöf in $X$, it follows that there exist $\alpha_1, \alpha_2, \ldots \in \Lambda$ such that $A \subset \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$, thus it follows that $f(A) \subset \bigcup_{i=1}^{n} f(f^{-1}(U_{\alpha_i})) \subset \bigcup_{i=1}^{n} U_{\alpha_i}$. Hence $f(A)$ is rc-Lindelöf in $Y$. 

Corollary 2.9 (see [19]). Let \( f : X \to Y \) be a slightly continuous and weakly \( \theta \)-irresolute mapping. If \( A \) is rc-Lindelöf in \( X \), then \( f(A) \) is rc-Lindelöf in \( Y \).

Proof. Follows from Propositions 2.4 and 2.8. \( \square \)

Proposition 2.10 (see [20]). Let \( f : X \to Y \) be a weakly continuous and almost open mapping. Then \( f \) is rc-continuous.

Corollary 2.11. Let \( f : X \to Y \) be a weakly continuous and almost open mapping. If \( A \) is rc-Lindelöf in \( X \), then \( f(A) \) is rc-Lindelöf in \( Y \).

Proof. Follows from Propositions 2.10 and 2.8. \( \square \)

Now, we prove the following known result using a slight modification on the previous proof.

Proposition 2.12 (see [7]). Let \( f : X \to Y \) be a slightly continuous and weakly \( \theta \)-irresolute mapping. If \( A \) is almost rc-Lindelöf in \( X \), then \( f(A) \) is almost rc-Lindelöf in \( Y \).

Proof. Let \( \{U_\alpha : \alpha \in \Lambda\} \) be a cover of \( f(A) \) by regular closed subsets of \( Y \). Since \( f \) is slightly continuous and weakly \( \theta \)-irresolute, it follows from Proposition 2.4 that \( f \) is rc-continuous and thus \( \{f^{-1}(U_\alpha) : \alpha \in \Lambda\} \) is a cover of \( A \) by regular closed subsets of \( X \). Since \( A \) is almost rc-Lindelöf in \( X \), it follows that there exist \( \alpha_1, \alpha_2, \ldots \in \Lambda \) such that \( A \subset \bigcup_{i=1}^{\infty} f^{-1}(U_{\alpha_i}) \). Now, \( f^{-1}(U_\alpha) \) is regular closed and thus semi-open, but the arbitrary union of semi-open sets is semi-open, so \( \bigcup_{i=1}^{\infty} f^{-1}(U_{\alpha_i}) \) is semi-open. Since \( f \) is slightly continuous, it follows from Proposition 2.3 that \( f(A) \subset \bigcup_{i=1}^{\infty} f(f^{-1}(U_{\alpha_i})) \subset \bigcup_{i=1}^{\infty} U_{\alpha_i} \). Hence \( f(A) \) is almost rc-Lindelöf in \( Y \). \( \square \)

Definition 2.13. A mapping \( f \) from a space \( X \) into a space \( Y \) is said to be slightly \( p \)-continuous if \( f(\overline{U}) \subset \overline{f(U)} \) whenever \( U \) is preopen in \( X \).

Proposition 2.14. For a mapping \( f : X \to Y \), the following are equivalent:

(i) \( f \) is slightly \( p \)-continuous;

(ii) \( f(\overline{U}) \subset \overline{f(U)} \) whenever \( U \) is semi-preopen in \( X \);

(iii) \( f(\overline{U}) \subset \overline{f(U)} \) whenever \( U \) is \( b \)-open in \( X \).

Proof. (i) \( \to \) (ii): Let \( U \) be a semi-preopen subset of \( X \). Then there exists a preopen subset \( V \) of \( X \) such that \( V \subset U \subset \overline{V} \). Thus by (i), \( f(\overline{U}) = f(\overline{V}) \subset \overline{f(V)} \subset \overline{f(U)} \).

(ii) \( \to \) (iii) \( \to \) (i): follow since every preopen set is \( b \)-open and every \( b \)-open set is semi-preopen. \( \square \)

Definition 2.15. A mapping \( f : X \to Y \) is called brc-continuous if \( f^{-1}(A) \) is \( b \)-open for every regular closed subset \( A \) of \( Y \).

Clearly, every weakly \( \theta \)-irresolute mapping is brc-continuous and every brc-continuous mapping is wrc-continuous; the converses are, however, not true as the following two examples tell.
Example 2.16. Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a, b\}\} \), and \( \tau^* = \{X, \phi, \{a\}, \{b, c\}\} \). Then the identity mapping from \( (X, \tau) \) onto \( (X, \tau^*) \) is brc-continuous but not weakly \( \theta \)- irresolute (observe that the regular closed subsets of \( (X, \tau^*) \) are the members of \( \tau^* \), each of which is preopen and thus \( b \)-open in \( (X, \tau) \). However, \( \{a\} \) is not semi-open in \( (X, \tau) \).

Example 2.17. Let \( \tau_a \) be the usual topology on the set of real numbers \( R \) and \( \tau = \{R, \phi, A, R \setminus A\} \), where \( A = [0, 1] \cap Q \). Then the identity mapping from \( (R, \tau_a) \) onto \( (R, \tau) \) is wrc-continuous but not brc-continuous (observe that the regular closed subsets of \( (R, \tau) \) are the members of \( \tau \), each of which is semi-preopen in \( (R, \tau_a) \). However, \( A \) is not \( b \)-open in \( (R, \tau_a) \).

Proposition 2.18. Let \( f : X \to Y \) be a slightly \( p \)- continuous mapping. Then the following are equivalent:

(i) \( f \) is weakly \( \theta \)- irresolute;
(ii) \( f \) is re- continuous;
(iii) \( f \) is wrc- continuous;
(iv) \( f \) is brc- continuous.

Proof. (ii) \( \to \) (i) \( \to \) (iv) \( \to \) (iii): follow since every regular closed set is semi-open, every semi-open set is \( b \)-open and every \( b \)-open set is semi-preopen.

(iii) \( \to \) (ii): let \( A \) be a regular closed subset of \( Y \). By (iii), \( f^{-1}(A) \) is semi-preopen, but \( f \) is slightly \( p \)- continuous, so by Proposition 2.14, \( f(f^{-1}(A)) \subset f^{-1}(A) \subset A \subset X \). Thus \( f^{-1}(A) \subset f^{-1}(f^{-1}(A)) \subset f^{-1}(A) \), that is, \( f^{-1}(A) \) is closed, but \( f^{-1}(A) \) is semi-preopen, so \( f^{-1}(A) \) is regular closed. Hence \( f \) is rc-continuous. \( \square \)

Corollary 2.19. Let \( f : X \to Y \) be a slightly \( p \)-continuous, semi-precontinuous, and almost open mapping. Then \( f \) is rc-continuous.

Proof. Follows from Lemma 2.1(ii) and Proposition 2.18. \( \square \)

Proposition 2.20. Let \( f : X \to Y \) be a slightly \( p \)- continuous and semi-precontinuous mapping. Then \( f^{-1}(U) \subset f^{-1}(U) \) for every open subset \( U \) of \( Y \).

Proof. Let \( U \) be an open subset of \( Y \). Since \( f \) is semi-precontinuous, it follows that \( f^{-1}(U) \) is semi-preopen, but \( f \) is slightly \( p \)-continuous, so it follows from Proposition 2.14 that \( f(f^{-1}(U)) \subset f^{-1}(U) \subset U \). Thus \( f^{-1}(U) \subset f^{-1}(f^{-1}(U)) \subset f^{-1}(U) \). \( \square \)

Observing that the closure of every semi-preopen set is regular closed, the following corollary seems a slight improvement of Corollary 2.19.

Corollary 2.21. Let \( f : X \to Y \) be a slightly \( p \)- continuous, semi-precontinuous, and almost open mapping. Then \( f^{-1}(U) = f^{-1}(U) \) for every open subset \( U \) of \( Y \).

Obviously, every continuous mapping is both semi-continuous and slightly \( p \)- continuous and every slightly \( p \)- continuous mapping is slightly continuous, the converses are, however, not true as the following two examples tell.

Example 2.22. Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a, b\}\} \), and \( \tau^* = \{X, \phi, \{a\}, \{a, b\}\} \). Then the identity mapping from \( (X, \tau) \) onto \( (X, \tau^*) \) is slightly continuous and weakly \( \theta \)- irresolute (observe that the regular closed subsets of \( (X, \tau^*) \) are \( X \) and \( \phi \)). However, it is not slightly...
$p$-continuous (consider the preopen subset $\{b,c\}$ of $(X,\tau)$). We observe also that this is an example of a mapping that is both slightly continuous and semi-precontinuous but neither slightly $p$-continuous nor semi-continuous (observe that $\{a\}$, $\{a,b\}$ are both dense and thus preopen in $(X,\tau)$. However, $\{a\}$ is not semi-open in $(X,\tau)$). This example also shows that the converses of Propositions 2.6 and 2.20 are not true.

**Example 2.23.** Let $X = \{a,b,c\}$, $\tau = \{X,\phi,\{a\}\}$, and $\tau^* = \{X,\phi,\{a,b\}\}$. Then the identity mapping from $(X,\tau)$ onto $(X,\tau^*)$ is slightly $p$-continuous (observe that the nonempty preopen subsets of $(X,\tau)$ are the supersets of $\{a\}$); it is, moreover, semi-continuous and almost open. However, it is not continuous.

**Corollary 2.24.** Let $f : X \rightarrow Y$ be a slightly $p$-continuous and wrc-continuous mapping. If $A$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$, then $f(A)$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $Y$.

**Proof.** We observe from Proposition 2.18 that a mapping that is both slightly $p$-continuous and wrc-continuous is both slightly continuous and weakly $\theta$-irresolute (the converse is not true as Example 2.22 tells). Thus the result follows from Corollary 2.9 and Proposition 2.12. □

**Corollary 2.25.** Let $f : X \rightarrow Y$ be a slightly $p$-continuous, semi-precontinuous, and almost open mapping. If $A$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$, then $f(A)$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $Y$.

**Proof.** Follows from Lemma 2.1(ii) and Corollary 2.24. □

**Remark 2.26.** Since every dense set is preopen, one easily observes that if $f$ is a slightly $p$-continuous mapping from a space $X$ onto a space $Y$, then $f$ maps dense subsets of $X$ onto dense subsets of $Y$.

Recall that a space $X$ is called submaximal (resp., strongly irresolvable) if every dense subset of $X$ is open (resp., semi-open), or equivalently if, every preopen subset of $X$ is open (resp., semi-open).

The following proposition is a direct consequence of Proposition 2.3.

**Proposition 2.27.** Let $f : X \rightarrow Y$ be a mapping from a strongly irresolvable space $X$ into a space $Y$. Then the following are equivalent:

(i) $f$ is slightly $p$-continuous;

(ii) $f$ is slightly continuous.

### 3. Almost $p$-open mappings

**Definition 3.1.** A mapping $f$ from a space $X$ into a space $Y$ is said to be semi-regular open (resp., semi-$p$-regular open) if it maps regular open subsets onto semi-closed (resp., semi-preclosed) subsets.

**Remark 3.2.** Since every regular open set is semi-closed and every semi-closed set is semi-preclosed, it is obvious that every regular open mapping is semi-regular open and every semi-regular open mapping is semi-$p$-regular open. The converses are, however, not true as the following examples show.
Example 3.3. Let \( X = \{a, b, c\}, \tau = \{X, \phi, \{a, c\}, \{b\}\} \), and \( \tau^* = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \). Then the identity mapping \( f \) from \( (X, \tau) \) onto \( (X, \tau^*) \) is semi-regular open (observe that the regular open subsets of \( (X, \tau) \) are the members of \( \tau \), each of which is semi-closed in \( (X, \tau^*) \)); it is, however, not regular open since \( \{a, c\} \) is not regular open in \( (X, \tau^*) \).

Example 3.4. Let \( X = \{a, b, c\}, \tau = \{X, \phi, \{a, c\}, \{b\}\} \), and \( \tau^* = \{X, \phi, \{a, b\}, \{c\}\} \). Then the identity mapping \( f \) from \( (X, \tau) \) onto \( (X, \tau^*) \) is semi-\( p \)-regular open (observe that \( \{a, c\} \) and \( \{b\} \) are preopen and thus semi-preopen in \( (X, \tau^*) \)); it is, however, not semi-regular open since \( \{a, c\} \) is not semi-closed in \( (X, \tau^*) \).

Definition 3.5. A mapping \( f \) from a space \( X \) into a space \( Y \) is said to be almost \( p \)-open if \( f^{-1}(U) \cap f^{-1}(V) \) whenever \( U \) is preopen in \( Y \).

Proposition 3.6. For a mapping \( f : X \rightarrow Y \), the following are equivalent:

(i) \( f \) is almost \( p \)-open;

(ii) \( f^{-1}(U) \cap f^{-1}(V) \) whenever \( U \) is semi-open in \( Y \).

Proof. Since every open set is semi-open, it suffices to show that (i) \( \rightarrow \) (ii). Let \( U \) be a semi-open subset of \( Y \). Then there exists an open subset \( V \) of \( Y \) such that \( V \subseteq U \subseteq V \). Thus by (i),

\[ f^{-1}(U) \cap f^{-1}(V) = f^{-1}(V) \subseteq f^{-1}(U) \cap f^{-1}(V) \subseteq f^{-1}(U). \]

\( \square \)

Proposition 3.7. For a mapping \( f : X \rightarrow Y \), the following are equivalent:

(i) \( f \) is almost \( p \)-open;

(ii) \( f^{-1}(U) \cap f^{-1}(V) \) whenever \( U \) is semi-preopen in \( Y \);

(iii) \( f^{-1}(U) \cap f^{-1}(V) \) whenever \( U \) is \( b \)-open in \( Y \).

Proof. (i) \( \rightarrow \) (ii): Let \( U \) be a semi-preopen subset of \( Y \). Then there exists a preopen subset \( V \) of \( Y \) such that \( V \subseteq U \subseteq V \). Thus by (i),

\[ f^{-1}(U) \cap f^{-1}(V) = f^{-1}(V) \subseteq f^{-1}(V) \subseteq f^{-1}(U). \]

(ii) \( \rightarrow \) (iii) \( \rightarrow \) (i): follow since every preopen set is \( b \)-open and every \( b \)-open set is semi-preopen.

\( \square \)

Remark 3.8. Since every open set is preopen, it is obvious that every almost \( p \)-open mapping is almost open. However, the converse is not true as the following example tells.

Example 3.9. Let \( X = \{a, b, c\}, \tau = \{X, \phi, \{a\}\} \), and \( \tau^* = \{X, \phi, \{a, b\}, \{c\}\} \). Then the identity mapping \( f \) from \( (X, \tau) \) onto \( (X, \tau^*) \) is almost open and even regular open (observe that the regular open subsets of \( (X, \tau) \) are \( X \) and \( \phi \)); it is, however, not almost \( p \)-open since \( \{b, c\} \) is dense and thus preopen in \( (X, \tau^*) \) but not dense in \( (X, \tau) \).

Proposition 3.10. For an almost \( p \)-open mapping \( f : X \rightarrow Y \), the following are equivalent:

(i) \( f \) is semi-\( p \)-regular open;

(ii) semi-regular open;

(iii) regular open.
Remark 3.14. Since every dense set is preopen, one easily observes that if \( f \) is an almost p-open mapping from a space \( X \) into a space \( Y \), then the inverse image of a dense subset of \( Y \) is a dense subset of \( X \).

The following proposition is a direct consequence of Proposition 3.6.

Proposition 3.15. Let \( f : X \to Y \) be a mapping from a space \( X \) into a strongly irresolvable space \( Y \). Then the following are equivalent:

(i) \( f \) is almost p-open;
(ii) \( f \) is almost open.

Proof. (i) \( \implies \) (iii): Let \( A \) be a regular open subset of \( X \). By assumption, \( f(A) \) is semi-preclosed, that is, \( Y \setminus f(A) \) is semi-preopen. By Proposition 3.7, \( f^{-1}(Y \setminus f(A)) \subset f^{-1}(Y \setminus f(A)) \subset X \setminus A = X \setminus A \). Thus \( f^{-1}(Y \setminus f(A)) \cap A = \emptyset \) and, therefore, \( (Y \setminus f(A)) \cap A = \emptyset \), that is, \( f(A) \subset \text{Int} f(A) \), that is, \( f(A) \) is open, but \( f(A) \) is semi-preclosed, so \( f(A) \) is regular open.

(iii) \( \implies \) (ii) \( \implies \) (i): follow since every regular open mapping is semi-regular open and every semi-regular open mapping is semi-p-regular open.

Proposition 3.11. For an almost open mapping \( f : X \to Y \), the following are equivalent:

(i) semi-regular open;
(ii) regular open.

Proof. Since every regular open mapping is semi-regular open, it suffices to show that (i) \( \implies \) (ii). Let \( A \) be a regular open subset of \( X \). By assumption, \( f(A) \) is semi-closed, that is, \( Y \setminus f(A) \) is semi-open. By Proposition 3.6, \( f^{-1}(Y \setminus f(A)) \subset f^{-1}(Y \setminus f(A)) \subset X \setminus A = X \setminus A \). Thus \( f^{-1}(Y \setminus f(A)) \cap A = \emptyset \) and, therefore, \( (Y \setminus f(A)) \cap A = \emptyset \), that is, \( f(A) \subset \text{Int} f(A) \), that is, \( f(A) \) is open, but \( f(A) \) is semi-closed, so \( f(A) \) is regular open.

Proposition 3.12 (see [19]). Let \( f \) be an almost open and regular open mapping from a space \( X \) onto a space \( Y \). Then the following hold.

(i) If for each \( y \in Y \), \( f^{-1}(y) \) is an \( S \)-set in \( X \), then \( f^{-1}(A) \) is almost rc-Lindelöf in \( X \) whenever \( A \) is almost rc-Lindelöf in \( Y \).

(ii) If for each \( y \in Y \), \( f^{-1}(y) \) is rc-Lindelöf in \( X \), then \( f^{-1}(A) \) is rc-Lindelöf in \( X \) whenever \( A \) is almost rc-Lindelöf in \( Y \) provided that \( X \) is a weak \( P \)-space.

Corollary 3.13. Let \( f \) be an almost p-open and semi-p-regular open mapping from a space \( X \) onto a space \( Y \). Then the following hold.

(i) If for each \( y \in Y \), \( f^{-1}(y) \) is an \( S \)-set in \( X \), then \( f^{-1}(A) \) is almost rc-Lindelöf in \( X \) whenever \( A \) is almost rc-Lindelöf in \( Y \).

(ii) If for each \( y \in Y \), \( f^{-1}(y) \) is rc-Lindelöf in \( X \), then \( f^{-1}(A) \) is rc-Lindelöf in \( X \) whenever \( A \) is almost rc-Lindelöf in \( Y \) provided that \( X \) is a weak \( P \)-space.

Proof. We observe from Proposition 3.10 that a mapping that is both almost p-open and semi-p-regular open is both almost open and regular open (the converse is not true as Example 3.9 tells). Thus the result follows from Proposition 3.12.

Remark 3.14. Since every dense set is preopen, one easily observes that if \( f \) is an almost p-open mapping from a space \( X \) into a space \( Y \), then the inverse image of a dense subset of \( Y \) is a dense subset of \( X \).
References


