Research Article

Characterization for the Convergence of Krasnoselskij Iteration for Non-Lipschitzian Operators

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We establish the convergence of Krasnoselskij iteration for various classes of non-Lipschitzian operators.

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1. Introduction

Let $X$ be a real Banach space; $B$ a nonempty, convex subset of $X$; and $T : B \to B$ an operator. Let $x_0 \in B$. The following iteration is known as Krasnoselskij iteration (see [1]):

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n. \quad (1.1)$$

The map $J : X \to 2^X$ given by $Jx := \{ f \in X^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\| \},$ for all $x \in X$, is called the normalized duality mapping. It is easy to see that we have

$$\langle y, j(x) \rangle \leq \|x\||y||, \quad \forall x, y \in X, \forall j(x) \in J(x). \quad (1.2)$$

Denote

$$\Psi := \{ \varphi : [0, +\infty) \to [0, +\infty) \text{ is a strictly increasing map with } \varphi(0) = 0 \}. \quad (1.3)$$

Definition 1.1. Let $X$ be a real Banach space, and let $B$ be a nonempty subset of $X$. A map $T : B \to B$ is called uniformly pseudocontractive if there exists a map $\varphi \in \Psi$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \varphi(\|x - y\|), \quad \forall x, y \in B. \quad (1.4)$$
A map $S : X \to X$ is called uniformly accretive if there exists a map $\psi \in \Psi$ and $j(x - y) \in J(x - y)$ such that

$$\langle Sx - Sy, j(x - y) \rangle \geq \psi(\|x - y\|), \quad \forall x, y \in X.$$  \hspace{1cm} (1.5)

Taking $\psi(a) = \psi(a) \cdot a$, for all $a \in [0, +\infty)$, $(\psi \in \Psi)$, reduces to the usual definitions of $\psi$-strongly pseudocontractive and $\psi$-strongly accretive. Taking $\psi(a) := \gamma \cdot a^2$, $\gamma \in (0, 1)$, for all $a \in [0, +\infty)$, $(\psi \in \Psi)$, we get the usual definitions of strongly pseudocontractive and strongly accretive. Therefore, the class of strongly pseudocontractive maps is included strictly in the class of $\psi$-strongly pseudocontractive maps. The example from [2] shows that this inclusion is proper. Remark, further, that the class of $\psi$-strongly pseudocontractive maps is also included strictly in the class of uniformly pseudocontractive maps (see also [3]).

We will give a characterization for the convergence of (1.1) when applied to uniformly pseudocontractive operators. For this purpose, we need the following lemma similar to [4, Lemma 1]. Next, $\mathbb{N}$ denotes the set of all natural numbers.

**Lemma 1.2.** Let $\{a_n\}$ be a positive bounded sequence and assume that there exists $n_0 \in \mathbb{N}$ such that

$$a_{n+1} \leq (1 - \lambda)a_n + \lambda a_{n+1} - \lambda \frac{\psi(a_{n+1})}{a_{n+1}} + \lambda \epsilon_n, \quad \forall n \geq n_0,$$  \hspace{1cm} (1.6)

where $\lambda \in (0, 1)$, $\epsilon_n \geq 0$, for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \epsilon_n = 0$. Then $\lim_{n \to \infty} a_n = 0$.

**Proof.** There exists an $M > 0$ such that $a_n \leq M$, for all $n \in \mathbb{N}$. Denote $a := \lim \inf a_n$. We will prove that $a = 0$. Suppose on the contrary that $a > 0$. Then there exists an $N_1 \in \mathbb{N}$ such that

$$a_n \geq \frac{a}{2}, \quad \forall n \geq N_1.$$  \hspace{1cm} (1.7)

From $\lim_{n \to \infty} \epsilon_n = 0$, we know that there exists an $N_2 \in \mathbb{N}$ such that

$$\epsilon_n \leq \frac{\psi(a/2)}{2M}, \quad \forall n \geq N_2.$$  \hspace{1cm} (1.8)

Set $N_0 := \max\{N_1, N_2\}$. Using the fact that $-(1/M) \geq -(1/a_{n+1})$, we get the following:

$$a_{n+1} \leq (1 - \lambda)a_n + \lambda a_{n+1} - \lambda \frac{\psi(a_{n+1})}{a_{n+1}} + \lambda \epsilon_n$$

$$\leq (1 - \lambda)a_n + \lambda a_{n+1} - \lambda \frac{\psi(a/2)}{M} + \lambda \frac{\psi(a/2)}{2M}$$

$$\leq (1 - \lambda)a_n + \lambda a_{n+1} - \lambda \frac{\psi(a/2)}{2M},$$  \hspace{1cm} (1.9)

which implies that $(1 - \lambda)a_{n+1} \leq (1 - \lambda)a_n - \lambda (\psi(a/2)/2M)$, or

$$a_{n+1} \leq a_n - \frac{1}{1 - \lambda} \frac{\lambda \psi(a/2)}{2M} \leq a_n - \frac{\lambda \psi(a/2)}{2M},$$  \hspace{1cm} (1.10)
Theorem 2.1. Let $X$ be a real Banach space, $B$ a nonempty, closed, convex, bounded subset of $X$. Let $T: B \to B$ be a uniformly pseudocontractive and uniformly continuous operator with $F(T) \neq \emptyset$. Then for $x_0 \in B$, the Krasnoselskij iteration (1.1) converges to the fixed point of $T$ if and only if $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$.

Proof. Since $T$ is a self-map of $B$, which is bounded and convex, then, from (1.1), each $x_n \in B$, so $\{x_n\}$ is bounded for each $n \in \mathbb{N}$. Uniqueness of the fixed point follows from (1.4). If $\{x_n\}$ converges to the fixed point of $T$, that is, $\lim_{n \to \infty} x_n = x^*$, then, obviously, $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$. Conversely, we will prove that if $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$, then $\lim_{n \to \infty} x_n = x^*$. Suppose that
\[ x_n = x^* \] for some \( n \in \mathbb{N} \). Then from (1.1), it follows that \( x_m = x^* \) for each \( m > n \), and the theorem is proved. Now suppose that \( x_n \neq x^* \) for each \( n \in \mathbb{N} \). Using (1.1) and (1.2),

\[
\| x_{n+1} - x^* \|^2 = \langle x_{n+1} - x^*, j(x_{n+1} - x^*) \rangle \\
= \langle (1 - \lambda)(x_n - x^*) + \lambda(Tx_n - Tx^*), y(x_{n+1} - x^*) \rangle \\
= \langle (1 - \lambda)(x_n - x^*), y(x_{n+1} - x^*) \rangle + \lambda\langle Tx_n - Tx^*, y(x_{n+1} - x^*) \rangle \\
\leq (1 - \lambda)\| x_n - x^* \| \| x_{n+1} - x^* \| + \lambda \langle Tx_n - Tx^*, j(x_{n+1} - x^*) \rangle + \lambda \langle Tx_n - Tx_{n+1}, j(x_{n+1} - x^*) \rangle \\
\leq (1 - \lambda)\| x_n - x^* \| \| x_{n+1} - x^* \| + \lambda \| x_{n+1} - x^* \|^2 - \lambda \psi(\| x_{n+1} - x^* \|) + \lambda \| Tx_n - Tx_{n+1} \| \| x_{n+1} - x^* \| \\
\leq \| x_{n+1} - x^* \| \left( (1 - \lambda)\| x_n - x^* \| + \lambda \| x_{n+1} - x^* \| - \frac{\lambda}{\| x_{n+1} - x^* \|} \right) + \lambda \| Tx_n - Tx_{n+1} \|. 
\]

(2.1)

Hence

\[
\| x_{n+1} - x^* \| \leq (1 - \lambda)\| x_n - x^* \| + \lambda \| x_{n+1} - x^* \| - \frac{\lambda}{\| x_{n+1} - x^* \|} + \lambda \| Tx_n - Tx_{n+1} \|. 
\]

(2.2)

Since \( \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0 \) and \( T \) is uniformly continuous, it follows that

\[
\lim_{n \to \infty} \| Tx_n - Tx_{n+1} \| = 0. 
\]

(2.3)

Set \( a_n = \| x_n - x^* \|, \varepsilon_n = \| Tx_n - Tx_{n+1} \| \) and use Lemma 1.2 to obtain the conclusion.

Remark 2.2. (1) If \( B \) is not bounded, then Theorem 2.1 holds under the assumption that \( \{ x_n \} \) is bounded.

(2) If \( T(B) \) is bounded, then \( \{ x_n \} \) is bounded.

(3) If \( T \) is strongly pseudocontractive, then automatically \( F(T) \neq \emptyset \).

3. Further results

Let \( I \) denote the identity map. A map \( T : B \to B \) is called pseudocontractive if there exists \( j(x - y) \in J(x - y) \) such that \( \langle Tx - Ty, j(x - y) \rangle \leq \| x - y \|^2 \).

Remark 3.1. The operator \( T \) is a (uniformly, strongly) pseudocontractive map if and only if \( (I - T) \) is a (uniformly, strongly) accretive map.

Remark 3.2. (1) Let \( T, S : X \to X \), and let \( f \in X \) be given. A fixed point for the map \( Tx = f + (I - S)x \), for all \( x \in X \), is a solution for \( Sx = f \).

(2) Let \( f \in X \) be a given point. If \( S \) is an accretive map, then \( T = f - S \) is a strongly pseudocontractive map.
Consider Krasnoselskij iteration with $Tx = f + (I - S)x$,

$$x_{n+1} = (1 - \lambda)x_n + \lambda(f + (I - S)x_n). \quad (3.1)$$

Remarks 3.1 and 3.2 and Theorem 2.1 lead to the following result.

**Corollary 3.3.** Let $X$ be a real Banach space and let $S : X \to X$ be a uniformly accretive and uniformly continuous operator, with $(I - S)(X)$ bounded. Suppose that $Sx = f$ has a solution. Then for any $x_0 \in X$, the Krasnoselskij iteration (3.1) converges to the solution of $Sx = f$ if and only if $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$.

Let $S$ be an accretive operator. The operator $Tx = f - Sx$ is strongly pseudocontractive for a given $f \in X$. A solution for $Tx = x$ becomes a solution for $x + Sx = f$. Consider Krasnoselskij iteration with $Tx := f - Sx$,

$$x_{n+1} = (1 - \lambda)x_n + \lambda(f - Sx_n). \quad (3.2)$$

Again, using Remarks 3.1 and 3.2 and Theorem 2.1, we obtain the following result.

**Corollary 3.4.** Let $X$ be a real Banach space and let $S : X \to X$ be an accretive and uniformly continuous operator, with $(I - S)(X)$ bounded. Suppose that $x + Sx = f$ has a solution. Then for $x_0 \in X$, the Krasnoselskij iteration (3.2) converges to the solution of $x + Sx = f$ if and only if $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$.

**Remark 3.5.** If (1.4) holds for all $x \in B$ and $y := x^* \in F(T)$, then such a map is called uniformly hemicontractive. It is trivial to see that our results hold for the uniformly hemicontractive maps.

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**References**


