A Note on Locally Inverse Semigroup Algebras

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Let \( R \) be a commutative ring and \( S \) a finite locally inverse semigroup. It is proved that the semigroup algebra \( \frac{R}{S} \) is isomorphic to the direct product of Munn algebras \( M(R[G_{J}], m_{J}, n_{J}; P_{J}) \) with \( J \in S/J \), where \( m_{J} \) is the number of \( R \)-classes in \( J \), \( n_{J} \) the number of \( L \)-classes in \( J \), and \( G_{J} \) a maximum subgroup of \( J \). As applications, we obtain the sufficient and necessary conditions for the semigroup algebra of a finite locally inverse semigroup to be semisimple.

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1. Main results

A regular semigroup \( S \) is called a locally inverse semigroup if for all idempotent \( e \in S \), the local submonoid \( eSe \) is an inverse semigroup under the multiplication of \( S \). Inverse semigroups are locally inverse semigroups. Inverse semigroup algebras are a class of semigroup algebras which is widely investigated. One of fundamentally important results is that a finite inverse semigroup algebra is the direct product of full matrix algebras over group algebras of the maximum subgroups of this finite inverse semigroup. Consider that all local submonoids of a locally inverse semigroup are inverse semigroups, it is a very natural problem whether a finite locally inverse semigroup algebra has a similar representation to inverse semigroup algebras. This is the main topic of this note.

Let \( \mathcal{A} \) be an \( R \)-algebra. Let \( m \) and \( n \) be positive integers, and let \( P \) be a fixed \( n \times m \) matrix over \( \mathcal{A} \). Let \( \mathcal{M} := M(\mathcal{A}; m, n; P) \) be the vector space of all \( m \times n \) matrices over \( \mathcal{A} \). Define a product \( \circ \) in \( \mathcal{M} \) by

\[
A \circ B = APB \quad (A, B \in \mathcal{M}),
\]

where \( APB \) is the usual matrix product of \( A, P, \) and \( B \). Then \( \mathcal{M} \) is an algebra over \( R \). Following [1], we call \( \mathcal{M} \) the Munn \( m \times n \) matrix algebra over \( \mathcal{A} \) with sandwich matrix \( P \).
By a semisimple semigroup, we mean a semigroup each of whose principal factor is either a completely 0-simple semigroup or a completely simple semigroup. It is well known that a finite regular semigroup is semisimple. The Rees theorem tells us that any completely 0-simple semigroup (completely simple semigroup) is isomorphic to some Rees matrix semigroup \( \mathcal{M}(G, I, \Lambda; P) \) (\( \Lambda(G, I, \Lambda; P) \)), and vice versa (for Rees matrix semigroups, refer to [1]).

In what follows, by the phrase “Let \( S = \bigcup_{J \in S/\mathcal{J}} M^0(G_j; I_j, \Lambda_j; P_j) \) be a finite regular semigroup,” we mean that \( S \) is a finite regular semigroup in which the principal factor of \( S \) determined by the \( \mathcal{J} \)-class \( J \) is isomorphic to the Rees matrix semigroup \( \mathcal{M}(G_j; I_j, \Lambda_j; P_j) \) or \( \Lambda(G_j; I_j, \Lambda_j; P_j) \) for any \( J \in S/\mathcal{J} \).

The following is the main result of this paper.

**Theorem 1.1.** Let \( S = \bigcup_{J \in S/\mathcal{J}} M^0(G_j, I_j, \Lambda_j; P_j) \) be a finite locally inverse semigroup. Then the semigroup algebra \( R[S] \) is isomorphic to the direct product of \( \mathcal{M}(R[G_j]; |I_j|, |\Lambda_j|; P_j) \) with \( J \in S/\mathcal{J} \).

Based on Theorem 1.1 and [1, Lemma 5.17, page 162, and Lemma 5.18, page 163], the following corollary is straightforward.

**Corollary 1.2.** Let \( S = \bigcup_{J \in S/\mathcal{J}} M^0(G_j, I_j, \Lambda_j; P_j) \) be a finite locally inverse semigroup. Then the semigroup algebra \( R[S] \) has an identity if and only if \( |I_j| = |\Lambda_j| \) and \( P_j \) is invertible in the full matrix algebra \( M_{|I_j|}(R[G_j]) \) for all \( J \in S/\mathcal{J} \).

Reference [1, Lemma 5.18, page 163] told us that \( \mathcal{M}(R[G_j], m_j, n_j; P_j) \) is isomorphic to the full matrix algebra \( M_{m_j}(R[G_j]) \) if \( \mathcal{M}(R[G_j], m_j, n_j; P_j) \) has an identity. Now, we have the following.

**Corollary 1.3.** Let \( S = \bigcup_{J \in S/\mathcal{J}} M^0(G_j, I_j, \Lambda_j; P_j) \) be a finite locally inverse semigroup. If \( R[S] \) has an identity, then \( R[S] \) is isomorphic to the direct product of the full matrix algebras \( M_{|I_j|}(R[G_j]) \) with \( J \in S/\mathcal{J} \).

The following corollary is a consequence of Corollary 1.3.

**Corollary 1.4.** Let \( S = \bigcup_{J \in S/\mathcal{J}} M^0(G_j, I_j, \Lambda_j; P_j) \) be a finite locally inverse semigroup. Then the semigroup algebra \( R[S] \) is semisimple if and only if for all \( J \in S/\mathcal{J} \),

1. \( |I_j| = |\Lambda_j| \);
2. \( P_j \) is invertible in the full matrix algebra \( M_{|I_j|}(R[G_j]) \);
3. \( R[G_j] \) is semisimple.

### 2. Proof of Theorem 1.1

For our purpose, we have the Möbius inversion theorem [2].

**Lemma 2.1.** Let \((P, \leq)\) be a locally finite partially ordered set (i.e., intervals are finite) in which each principal ideal has a maximum and \( G \) be an Abelian group. Suppose that \( f: P \to G \) is a function and define \( g: P \to G \) by \( g(x) = \sum_{y \leq x} f(y) \). Then \( f(x) = \sum_{y \leq x} g(y) \mu(x, y) \), where \( \mu \) is a Möbius function.

Now assume that \( S \) is a regular semigroup and \( a, b \in S \). Define

\[
a \leq b \iff \text{there exist } e, \ f \in E(S) \text{ such that } a = eb = bf.
\]

(2.1)
Then $\leq$ is a partial order on $S$. Following [3], we call $\leq$ the natural partial order on $S$. Equivalently, $a \leq b$ if and only if for every (for some) $f \in E(R_0)$ ($f \in E(L_k)$), there exists $e \in E(R_a)$ ($e \in E(L_a)$) such that $e \leq f$ and $a = eb$ ($a = be$). Moreover, Nambooripad [3, 4] proved that $S$ is a locally inverse semigroup if and only if the natural partial order $\leq$ is compatible with respect to the multiplication of $S$.

**Lemma 2.2.** Let $S$ be a locally inverse semigroup and $a, b \in S$. Then for any $u \leq ab$, there exist $x \leq a$ and $y \leq b$ such that $u = xy$, $x \in R_a$, and $y \in L_b$.

*Proof.* For any $e \in E(R_a)$, we have $ea = a$ and $eab = ab$. Let $z$ be an inverse of $ab$. Clearly, $abz \in E(R_{ab})$. Note that $eabz = abz$. It is easy to check that $abze \in E(S)$, $abze \leq e$, and $abze \leq e$. Hence $abzeRa$ and there exists $g \in E(S)$ such that $u = gab$ and $g \leq abze \leq e$. Thus $ga \leq a$. On the other hand, since $R$ is a left congruence and since $abzeRa$, we have $u = gabRa = g$; while since $aRa$, we have $gabRg = g$. These imply that $uRa$. Dually, we have $h \in E(S)$ such that $u = abh$, $bh \leq b$ and $uLa$. Since $u = gab = abh = uh = (ga)(bh)$, we know that $ga$ and $bh$ are the required elements $x$ and $y$. 

Define a multiplication $\otimes$ on $S^0 = S \cup \{0\}$ by

$$x \otimes y = \begin{cases} xy & \text{if } x \neq 0, y \neq 0, \text{ and } xy \in J_x; \\ 0 & \text{otherwise}, \end{cases}$$

(2.2)

where $xy$ is the product of $x$ and $y$ in $S$. By the arguments of [4, page 9], $(S^0, \otimes)$ is a semigroup. We denote by $S^0$ the semigroup $(S^0, \otimes)$. For any $J \in S/\mathcal{J}$, we denote $J^0 = J \cup \{0\}$. It is easy to see that $(J^0, \otimes)$ is a subsemigroup of $S^0$, which is isomorphic to the principal factor of $S$ determined by $J$. We will denote the semigroup $(J^0, \otimes)$ by $J^0$. By the definition of $\otimes$, it is easy to see that in the semigroup $S^0$,

(i) $J^0_x \otimes J^0_y \subseteq J^0_x$ for all $x \in S$;

(ii) $J^0_x \otimes J^0_y = 0$ for all $x, y \in S$ such that $x \notin J_y$.

Thus $R_0[S^0]$ is the direct sum of the contracted semigroup algebras $R_0[J^0]$ with $J \in S/\mathcal{J}$. Note that $J^0$ is isomorphic to some principal factor of $S$. We observe that $J^0$ is a completely 0-simple semigroup since $S$ is a semisimple semigroup, and thus $J^0$ is isomorphic to some Rees matrix semigroup $\mathcal{M}^0(G, I_f, \Lambda_f; P_f)$. By a result of [1], $R_0[\mathcal{M}(G, I_f, \Lambda_f; P_f)]$ is isomorphic to $\mathcal{M}(R[G], |I_f|, |\Lambda_f|; P_f)$. Consequently, to verify Theorem 1.1, we need only to prove that $R[S]$ is isomorphic to $R_0[S^0]$.

For the convenience of description, we introduce the semigroup $\overline{S}$. Put $\overline{S} = \{x \mid x \in S\} \cup \{0\}$. Define a multiplication on $\overline{S}$ as follows:

$$\overline{x} \ast \overline{y} = \overline{xy},$$

(2.3)

where we will identify $\overline{0}$ with $0$. It is easy to see that $\overline{S}$ is isomorphic to $S^0$. Hence the contracted semigroup algebra $R_0[\overline{S}]$ is isomorphic to the contracted semigroup algebra $R_0[S^0]$. For $J \in S/\mathcal{J}$, we denote $\overline{J} = \{x \mid x \in J\} \cup \{0\}$. It is easy to check that $(\overline{J}, \ast)$ is a subsemigroup of $\overline{S}$ isomorphic to the semigroup $J^0$. So, for any $J, K \in S/\mathcal{J}$, we have

$$\overline{J} \ast \overline{K} = \begin{cases} \overline{J} & \text{if } K = J, \\ 0 & \text{otherwise}. \end{cases}$$

(2.4)
Lemma 2.3. $R[S] \cong R_0[\overline{S}]$.

Proof. We consider the mapping $\varphi : R[S] \to R_0[\overline{S}]$ given on the basis by $\varphi(s) = \sum_{t \leq s} t (s \in S)$. Clearly, $\varphi$ is well defined. Of course, $\varphi$ and $\tilde{\varphi}$ may be regarded as the mappings of the ordered set $(S, \leq)$ into the additive group of $R_0[\overline{S}]$. Now, by applying the Möbius inversion theorem to the mappings $\varphi$ and $\tilde{\varphi}$, we have

$$\overline{s} = \sum_{t \leq s} \varphi(t) \mu(t, s) = \varphi \left( \sum_{t \in S} t \mu(t, s) \right),$$

where $\mu$ is the Möbius function for $(S, \leq)$. Hence $\varphi$ is surjective.

We will prove that $\varphi$ is injective. For $a_0 = \sum_{x \in S} p_x^0 x \in R[S]$, we denote by $\text{supp}(a_0)$ the set $\{x \in S \mid p_x^0 \neq 0\}$ and by $M(a_0)$ the set of maximal elements in the set $\text{supp}(a_0)$ with respect to the partial order $\leq$. In recurrence, we define $a_n = a_{n-1} - \sum_{x \in M(a_n)} p_x^{n-1} x$, where $a_n = \sum_{x \in \text{supp}(a)} p_x^n x$. Let $\beta_n = \sum_{x \in \text{supp}(\beta)} q_x^n x$ with $n = 0, 1, 2, \ldots$. If $\varphi(a_n) = \varphi(\beta_n)$, then by the definition of $\varphi$, $\sum_{x \in M(\alpha)} p_x^\alpha x + \Gamma = \sum_{y \in M(\beta)} q_y^\beta y + \Gamma$, where $\Gamma = \sum_{x \in M(\alpha)} \sum_{y \in S \setminus M(\beta)} q_x^\alpha q_y^\beta$ and $\Gamma = \sum_{x \in M(\beta)} \sum_{y \in S \setminus M(\beta)} q_x^\alpha q_y^\beta$, and hence $\sum_{x \in M(\alpha)} p_x^\alpha x = \sum_{x \in M(\beta)} q_x^\alpha x$, thus $M(\alpha) = M(\beta)$ and $a_n = \beta_n$ for any $x \in M(\alpha)$. This can imply the following.

Fact 2.4. If $\varphi(a_n) = \varphi(\beta_n)$, then $M(\alpha_n) = M(\beta_n)$ and by the definition of $\varphi$, $\varphi(a_{n+1}) = \varphi(\beta_{n+1})$.

By the definition of $\varphi$, the following facts are immediate.

Fact 2.5. $a_n = \beta_n$ if and only if $M(a_n) = M(\beta_n)$ and $a_{n+1} = \beta_{n+1}$.

Fact 2.6. If $\varphi(a_n) = \varphi(\beta_n)$ and $M(a_n) = M(\beta_n)$, $M(\alpha_n) = M(\beta_n)$, then $a_n = \beta_n$.

Note that $|\text{supp}(a_0)| < \infty$ and $\text{supp}(a_{n+1}) \subseteq \text{supp}(a_n)$. We thus have a smallest integer $k$ such that $M(a_k) = \text{supp}(a_k)$. Clearly, $a_k = 0$. This means that $k$ is the smallest integer $t$ such that $a_{t+1} = 0$. Similarly, there exists the smallest integer $l$ such that $\beta_{l+1} = 0$ and $M(\beta_l) = \text{supp}(\beta_l)$. Now, assume $\varphi(a_0) = \varphi(\beta_0)$. By using Fact 2.4 repeatedly,

$$\varphi(a_1) = \varphi(\beta_1), \quad \varphi(a_2) = \varphi(\beta_2), \ldots, \quad \varphi(a_{k+1}) = \varphi(\beta_{k+1}).$$

But $\varphi(a_{k+1}) = 0$, we have $\varphi(\beta_{k+1}) = 0$ and by the definition of $\varphi$, $\beta_{k+1} = 0$. Thus $k + 1 \geq l + 1$ by the minimality of $l$, and $k \geq l$. Therefore $k = l$. Since $\varphi(a_k) = \varphi(\beta_k)$, by Fact 2.6, we have $a_k = \beta_k$ since $M(a_k) = \text{supp}(a_k)$ and $M(\beta_k) = \text{supp}(\beta_k)$. Again by the hypothesis $\varphi(a_0) = \varphi(\beta_0)$, and by Fact 2.4, $M(a_0) = M(\beta_0)$; and by (2.6), $M(a_1) = M(\beta_1), M(a_2) = M(\beta_2), \ldots, M(a_k) = M(\beta_k)$. By Fact 2.5, $M(a_{k-1}) = M(\beta_{k-1})$, and $a_k = \beta_k$ imply $a_{k-1} = \beta_{k-1}$; moreover, by using Fact 2.5 repeatedly, $a_{k-2} = \beta_{k-2}, \ldots, a_1 = \beta_1$ and $a_0 = \beta_0$. We have now proved that $\varphi$ is injective.

Finally, for any $s, t \in S$, by (2.4), we have

$$\overline{s \ast t} = \begin{cases} st & \text{if } s, t \in J_{st}, \\ 0 & \text{otherwise}, \end{cases}$$

for Theorem 1.1, it remains to prove the following lemma.

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and by Lemma 2.2,

\[ \phi(s) * \phi(t) = \left( \sum_{x \leq s} \bar{x} \right) \ast \left( \sum_{y \leq t} \bar{y} \right) \]
\[ = \sum_{x \in J_{st}} \sum_{y \in J_{st}} \bar{x} \cdot \bar{y} \]
\[ = \sum_{x \leq s, x \in J} \sum_{y \leq t, y \in J} \bar{x} \cdot \bar{y}. \]  \hspace{1cm} (2.8)

Moreover, by Lemma 2.2, we have

\[ \phi(st) = \sum_{u \leq st} \bar{u} = \sum_{x \in J_{st}, x \leq s} \sum_{y \in J_{st}, y \leq t} \bar{x} \cdot \bar{y} \]
\[ = \sum_{x \leq s, x \in J} \sum_{y \leq t, y \in J} \bar{x} \cdot \bar{y} = \phi(s) * \phi(t). \]  \hspace{1cm} (2.9)

Thus \( \phi \) is a homomorphism of \( R[S] \) into \( R_0[S] \). Consequently, \( \phi \) is an isomorphism of \( R[S] \) onto \( R_0[S] \).  \( \square \)

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**References**


