Research Article

Subordination and Superordination Results for a Class of Analytic Multivalent Functions

S. P. Goyal,¹ Pranay Goswami,¹ and H. Silverman²

¹ Department of Mathematics, University of Rajasthan, Jaipur 302004, India
² Department of Mathematics, College of Charleston, Charleston, SC 29424, USA

Correspondence should be addressed to H. Silverman, silvermanh@cofc.edu

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We derive subordination and superordination results for a family of normalized analytic functions in the open unit disk defined by integral operators. We apply this to obtain sandwich results and generalizations of some known results.

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1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\Delta := \{z : |z| < 1\}$, and let $\mathcal{H}[a,p]$ be the subclass of $\mathcal{H}$ of the form

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots, \quad p \in \mathbb{N}. \quad (1.1)$$

Let $\mathcal{A}(p)$ be the subclass of $\mathcal{H}$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^{k}, \quad p \in \mathbb{N}. \quad (1.2)$$

If $f$ and $g$ are analytic and there exists a Schwarz function $w(z)$, analytic in $\Delta$ with

$$w(0) = 0, \quad |w(z)| < 1, \quad z \in \Delta, \quad (1.3)$$

such that $f(z) = g(w(z))$, then the function $f$ is called subordinate to $g$ and is denoted by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z), \quad z \in \Delta. \quad (1.4)$$
In particular, if the function $g$ is univalent in $\Delta$, the above subordination is equivalent to
\[
f(0) = g(0), \quad f(\Delta) \subset g(\Delta).
\] (1.5)

Suppose $h$ and $k$ are analytic functions in $\Delta$ and $\phi(r,s,t; z) : C^3 \times \Delta \rightarrow C$. If $h$ and $\phi(h(z),zh'(z),z^2h''(z); z)$ are univalent and if $h$ satisfies the second-order superordination
\[
k(z) < \phi(h(z),zh'(z),z^2h''(z); z),
\] (1.6)
then $h$ is a solution of the differential superordination (1.6). Note that if $f$ is subordinate to $g$, then $g$ is superordinate to $f$. An analytic function $q$ is called subordinant if $q < h$ for all $h$ satisfying (1.6). A univalent subordinant $\tilde{q}$ that satisfies $q < \tilde{q}$ for all subordinants $q$ of (1.6) is said to be the best subordinant. Miller and Mocanu [1] have obtained conditions on $k$, $q$, and $\phi$ for which the following implication holds:
\[
k(z) < \phi(h(z),zh'(z),z^2h''(z); z) \Rightarrow q(z) < h(z).
\] (1.7)

Ali et al. [2] have obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy
\[
q_1(z) < \frac{zf'(z)}{f(z)} < q_2(z),
\] (1.8)
where $q_1$ and $q_2$ are given univalent functions in $\Delta$ with $q_1(0) = 1$ and $q_2(0) = 1$.

Recently, Shanmugam et al. [3, 4] have also obtained sandwich results for certain classes of analytic functions. Further subordination results can be found in [5–8].

2. Definitions and Preliminaries

Definition 2.1. For $f(z) \in \mathcal{A}(p)$, Shams et al. [9] defined the following integral operator:
\[
\mathcal{O}^\sigma f(z) = \frac{(p+1)^\sigma}{2\Gamma(\sigma)} \int_0^z \left( \log \frac{z}{t} \right)^{\sigma-1} f(t) dt
\] (2.1)
\[
= z^p + \sum_{n=p+1}^{\infty} \left( \frac{p+1}{n+1} \right)^{\sigma} a_n z^n, \quad \sigma > 0.
\] (2.2)

For the operator, one easily gets
\[
z[\mathcal{O}^\sigma f(z)]' = (p+1)\mathcal{O}^{\sigma-1} f(z) - \mathcal{O}^\sigma f(z).
\] (2.3)

Also for $-1 \leq B < A \leq 1$ and $\lambda \geq 0$, Shams et al. [9] defined a class $\Omega^p_\sigma(A,B; \lambda)$ of functions $f(z) \in \mathcal{A}(p)$, so that
\[
\frac{\lambda}{p} \left( \frac{\mathcal{O}^{\sigma-1} f(z)}{z^p} \right) + \frac{p-\lambda}{p} \left( \frac{\mathcal{O}^\sigma f(z)}{z^p} \right) < \frac{1 + Az}{1 + Bz}.
\] (2.4)
The family $\Omega^p_\sigma(A,B; \lambda)$ is a general family containing various new and known classes of analytic functions (see, e.g., [10, 11]).
Definition 2.2 (see [1]). Denote by $Q$ the set of all functions $f(z)$ that are analytic and injective on $\Delta - E(f)$, where

$$E(f) = \left\{ \zeta \in \partial \Delta : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial \Delta - E(f)$.

We will require certain results due to Miller and Mocanu [1, 12], Bulboacă [13], and Shanmugam et al. [4] contained in the following lemmas.

Lemma 2.3 (see [12]). Let $q(z)$ be univalent in the unit disk $\Delta$, and let $\theta$ and $\phi$ be analytic in the domain $D$ containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that

(i) $Q(z)$ is starlike univalent in $\Delta$;
(ii) $\Re(zh'(z)/Q(z)) > 0$ for $z \in \Delta$.

If $p(z)$ is analytic in $\Delta$, with $p(0) = q(0)$, $p(\Delta) \subseteq D$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) < q(z)$ and $q(z)$ is the best dominant.

Lemma 2.4 (see [4]). Let $q(z)$ be a convex univalent function in $\Delta$ and $\psi, \gamma \in \mathbb{C}$ with $\Re(1 + (zq''(z)/q'(z))) > \max\{0, -\Re(\psi/\gamma)\}$. If $p(z)$ is analytic in $\Delta$ and

$$\psi p(z) + \gamma zp'(z) < \psi q(z) + \gamma zq'(z),$$

then $p(z) < q(z)$ and $q(z)$ is the best dominant.

Lemma 2.5 (see [12]). Let $q(z)$ be univalent in $\Delta$, and let $\phi(z)$ be analytic in a domain containing $q(\Delta)$. If $zq'(z)/\phi(q(z))$ is starlike and

$$zp'(z)\phi(p(z)) < zq'(z)\phi(q(z)),$$

then $p(z) < q(z)$ and $q(z)$ is the best dominant.

Lemma 2.6 (see [13]). Let $q(z)$ be convex univalent in the unit disk $\Delta$, and let $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(\Delta)$. Suppose that

(i) $\Re(\phi'(q(z))/\phi(q(z))) > 0$ for $z \in \Delta$;
(ii) $zq'(z)\phi(q(z))$ is starlike univalent in $z \in \Delta$.

If $p(z) \in \mathcal{K}[q(0), 1] \cap Q$, with $p(\Delta) \subseteq D$, and if $\theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in $\Delta$ and

$$\theta(q(z)) + zq'(z)\phi(q(z)) < \theta(p(z)) + zp'(z)\phi(p(z)),$$

then $q(z) < p(z)$ and $q(z)$ is the best subordinant.

Lemma 2.7 (see [1]). Let $q(z)$ be convex univalent in $\Delta$ and $\gamma \in \mathbb{C}$. Further assume that $\Re(\gamma) > 0$. If $p(z) \in \mathcal{K}[q(0), 1] \cap Q$ and $p(z) + \gamma zp'(z)$ is univalent in $\Delta$, then

$$q(z) + \gamma zq'(z) < p(z) + \gamma zp'(z),$$

which implies that $q(z) < p(z)$ and $q(z)$ is the best subordinant.
3. Subordination for analytic functions

The main object of this paper is to apply a method based on the differential subordination in order to derive several subordination results.

**Theorem 3.1.** Let \( q(z) \) be univalent in the unit disk \( \Delta, \lambda \in \mathbb{C}, \) and

\[
\Re \left( 1 + \frac{z q''(z)}{q'(z)} \right) > \max \left\{ 0, -\Re \left( \frac{p(p+1)}{\lambda} \right) \right\}, \quad \lambda \neq 0 \ (p \in \mathbb{N}).
\] (3.1)

If \( f(z) \in \mathcal{A}(p) \) satisfies the subordination

\[
\frac{\lambda}{p} \left( \frac{\mathcal{D}^{p-1} f(z)}{z^p} \right) + \frac{p - \lambda}{p} \left( \frac{\mathcal{D}^p f(z)}{z^p} \right) < q(z) + \frac{\lambda z q'(z)}{p(p+1)},
\] (3.2)

where \( \mathcal{D}^p f(z) \) is defined by (2.1), then

\[
\left( \frac{\mathcal{D}^p f(z)}{z^p} \right) < q(z)
\] (3.3)

and \( q(z) \) is the best dominant.

**Proof.** Consider

\[
h(z) := \left( \frac{\mathcal{D}^p f(z)}{z^p} \right).
\] (3.4)

Differentiating (3.4) with respect to \( z \) logarithmically, we get

\[
zh'(z) = \frac{z [\mathcal{D}^p f(z)]'}{\mathcal{D}^p f(z)} - p.
\] (3.5)

Now, in view of (2.3), we obtain from (3.5) the following subordination:

\[
h(z) + \frac{\lambda z h'(z)}{p(p+1)} < q(z) + \frac{\lambda z q'(z)}{p(p+1)}.
\] (3.6)

An application of Lemma 2.4, with \( \gamma = \lambda / p(p+1) \) and \( \varphi = 1 \), leads to (3.3).

**Corollary 3.2.** Let \(-1 \leq B < A \leq 1 \) and \( \Re((1 - Bz)/(1 + Bz)) > \max \{0, -\Re(p(p + 1)/\lambda)\} (\lambda \neq 0), \ p \in \mathbb{N}. \) If \( f \in \mathcal{A}(p) \) and

\[
\frac{\lambda}{p} \left( \frac{\mathcal{D}^{p-1} f(z)}{z^p} \right) + \frac{p - \lambda}{p} \left( \frac{\mathcal{D}^p f(z)}{z^p} \right) \prec \frac{1 + A z}{1 + B z} + \frac{\lambda (A - B) z}{p(p+1)(1 + B z)^2},
\] (3.7)

then

\[
\mathcal{D}^p f(z) \prec \frac{1 + A z}{1 + B z}
\] (3.8)

and \((1 + A z)/(1 + B z)\) is the best dominant.
Putting $p = 1$ and $q(z) = (1 + z)/(1 - z)$ in Theorem 3.1, we get the following corollary.

**Corollary 3.3.** Let $\text{Re}((1 + z)/(1 - z)) > \max\{0, -\text{Re}(2/\lambda)\}$ and $\lambda \neq 0$. If $f \in \mathcal{A}(1)$ and

$$
\frac{\lambda \mathcal{O}^{\alpha^{-1}} f(z)}{z} + \frac{(1 - \lambda) \mathcal{O}^\beta f(z)}{z} < \frac{1 + z}{1 - z} + \frac{\lambda z}{(1 - z)^2}, \quad (3.9)
$$

then

$$
\mathcal{O}^\beta f(z) < \frac{1 + z}{1 - z},
$$

and $(1 + z)/(1 - z)$ is the best dominant.

**Theorem 3.4.** Let $q(z)$ be univalent in $\Delta$ and $0 \neq \gamma, \mu \in \mathbb{C}$, and $\alpha, \beta \in \mathbb{C}$ such that $\alpha + \beta \neq 0$. Let $f \in \mathcal{A}(p)$ and suppose that $q$ satisfies

$$
\text{Re} \left\{ 1 + \frac{z q''(z)}{q(z)} - \frac{z q'(z)}{q(z)} \right\} > 0. \quad (3.11)
$$

If

$$
1 + \gamma \mu \left[ \frac{\alpha z [\mathcal{O}^{\alpha^{-1}} f(z)]' + \beta z [\mathcal{O}^\beta f(z)]'}{\alpha \mathcal{O}^{\alpha^{-1}} f(z) + \beta \mathcal{O}^\beta f(z)} - p \right] < 1 + \gamma \frac{z q'(z)}{q(z)}, \quad (3.12)
$$

then

$$
\left[ \frac{\alpha \mathcal{O}^{\alpha^{-1}} f(z) + \beta \mathcal{O}^\beta f(z)}{(\alpha + \beta)z^\mu} \right]^\mu < q(z), \quad (3.13)
$$

and $q(z)$ is the best dominant.

**Proof.** Let us consider a function $h(z)$ defined by

$$
h(z) := \left[ \frac{\alpha \mathcal{O}^{\alpha^{-1}} f(z) + \beta \mathcal{O}^\beta f(z)}{(\alpha + \beta)z^\mu} \right]^\mu, \quad \mu \neq 0, \quad \alpha + \beta \neq 0. \quad (3.14)
$$

Now, differentiating (3.14) logarithmically, we get

$$
\frac{zh'(z)}{h(z)} = \mu \left[ \frac{\alpha z [\mathcal{O}^{\alpha^{-1}} f(z)]' + \beta z [\mathcal{O}^\beta f(z)]'}{\alpha \mathcal{O}^{\alpha^{-1}} f(z) + \beta \mathcal{O}^\beta f(z)} - p \right]. \quad (3.15)
$$

By setting

$$
\theta(w) = 1, \quad \phi(w) = \frac{Y}{w}, \quad (3.16)
$$

it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$ and that $\phi(w) \neq 0$ is analytic in $\mathbb{C}/\{0\}$. Also, we let

$$
Q(z) = z q'(z) \phi(q(z)) = \gamma \frac{z q'(z)}{q(z)}, \quad (3.17)
$$

$$
p(z) = \theta(q(z)) + Q(z) = 1 + \gamma \frac{z q'(z)}{q(z)}. \quad (3.18)
$$
From (3.11) we see that \( Q(z) \) is starlike univalent in the unit disk \( \Delta \), and from (3.18) we get
\[
\Re\left(\frac{zp'(z)}{Q(z)}\right) = \Re\left(1 + \frac{zq'(z)}{q(z)} - \frac{zq'(z)}{q(z)}\right) > 0. \tag{3.19}
\]
An application of Lemma 2.3 to (3.12) yields the result. \( \square \)

Putting \( \alpha = 0, \beta = 1, \gamma = 1, \) and \( q(z) = (1 + Az)/(1 + Bz) \) in Theorem 3.4, we obtain the following corollary.

**Corollary 3.5.** If \( f(z) \in A(p) \) and for \(-1 \leq A < B \leq 1, \mu \neq 0, \)
\[
1 + \mu \left[ z \left[ \frac{\partial^\mu f(z)}{\partial^\mu f(z)} - p \right] \right] < 1 + \frac{(A - B)z}{(1 + Az)(1 + Bz)}, \tag{3.20}
\]
then
\[
\left[ \frac{\partial^\mu f(z)}{zp} \right]^\mu < 1 + Az \overset{\Delta}{\leq} (1 + Bz) \tag{3.21}
\]
and \((1 + Az)/(1 + Bz)\) is the best dominant.

By setting \( \alpha = 0, \beta = 1, \gamma = 1, \sigma = 0, \) and \( p = 1 \), and \( q(z) = (1 + Bz)_{\mu(A-B)/B} \) in Theorem 3.4, we get the following corollary.

**Corollary 3.6.** Suppose \( f(z) \in A(1) \) and let \(-1 \leq B < A \leq 1 \) and \( B \neq 0 \). If
\[
1 + \mu \left[ \frac{zf'(z)}{f(z)} - 1 \right] < 1 + Az \overset{\Delta}{\leq} (1 + Bz)_{\mu(A-B)/B} \tag{3.22}
\]
then
\[
\left[ \frac{f(z)}{z} \right]^\mu < (1 + Bz)_{\mu(A-B)/B} \tag{3.23}
\]
and \((1 + Bz)_{\mu(A-B)/B}\) is the best dominant.

**Remark 3.7.** \( q(z) = (1 + Bz)_{\mu(A-B)/B} \) is univalent if and only if \(|(\mu(A-B)/B) - 1| \leq 1 \) or \(|(\mu(A-B)/B) + 1| \leq 1 \) (see [5]).

Again by setting \( \beta = 1, \mu = 1, \alpha = 0, \gamma = 1/b, \) and \( \sigma = 0, \) and by \( q(z) = 1/(1 - z)^{2b} \) \((b \in \mathbb{C} \setminus \{0\})\) in Theorem 3.4, we get the following corollary.

**Corollary 3.8.** Suppose \( f(z) \in A(1) \) and \( b \) is a nonzero complex number for which
\[
1 + \frac{1}{b} \left[ \frac{zf'(z)}{f(z)} - 1 \right] < 1 + z \overset{\Delta}{\leq} \frac{1}{1 - z}. \tag{3.24}
\]
Then,
\[
\frac{f(z)}{z} < \frac{1}{(1 - z)^{2b}} \tag{3.25}
\]
and \( 1/(1 - z)^{2b} \) is the best dominant.
Suppose that

From [3.26] we see that

\[ \psi(z) = \left[ \frac{\alpha \sigma^{-1} f(z) + \beta \sigma f(z)}{(\alpha + \beta)z^p} \right]^\mu \eta + \gamma \mu \left( \frac{az[\sigma^{-1} f(z)]' + \beta z[\sigma f(z)]'}{a\sigma^{-1} f(z) + \beta \sigma f(z)} - p \right) \]  

Let

If

then

\[ \left[ \frac{\alpha \sigma^{-1} f(z) + \beta \sigma f(z)}{(\alpha + \beta)z^p} \right]^\mu < q(z), \quad \alpha + \beta \neq 0, \]  (3.29)

and \( q(z) \) is the best dominant.

**Proof.** Define a function \( h(z) \) by

\[ h(z) := \left[ \frac{\alpha \sigma^{-1} f(z) + \beta \sigma f(z)}{(\alpha + \beta)z^p} \right]^\mu. \]  (3.30)

Then, a computation shows that

\[ \frac{zh'(z)}{h(z)} = \mu \left\{ \frac{az[\sigma^{-1} f(z)]' + \beta z[\sigma f(z)]'}{a\sigma^{-1} f(z) + \beta \sigma f(z)} - p \right\} \]  (3.31)

and hence

\[ zh'(z) = \mu h(z) \left( \frac{z[a\sigma^{-1} f(z)]' + \beta z[\sigma f(z)]'}{a\sigma^{-1} f(z) + \beta \sigma f(z)} - p \right). \]  (3.32)

Set

\[ \theta(w) = \eta w + \delta, \quad \phi(w) = \gamma, \]  (3.33)

and let

\[ Q(z) = zq'(z)\phi(q(z)) = \gamma zq'(z), \]  (3.34)

\[ p(z) = \theta(q(z)) + Q(z) = \eta q(z) + \delta + \gamma zq'(z). \]

From (3.26), we see that \( Q(z) \) is starlike in \( \Delta \) and that

\[ \Re \left\{ \frac{zp'(z)}{Q(z)} \right\} = \Re \left\{ \frac{\eta}{\gamma} + 1 + \frac{zq''(z)}{q'(z)} \right\} > 0, \]  (3.35)

by the hypothesis (3.26) of Theorem 3.9. Thus, applying Lemma 2.3, the proof of Theorem 3.9 is completed.  

\[ \square \]
Corollary 3.10. Let \( f(z) \in A(p) \) and \( \text{Re}(\eta) > 0 \). Suppose that
\[
\text{Re}\left\{ \frac{1 - Bz}{1 + Bz} \right\} > \max\{0, -\text{Re}(\eta)\}. \tag{3.36}
\]
If
\[
\left[ \frac{\partial^\alpha f(z)}{zp} \right]^\mu \left\{ \eta + \mu \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} + \delta \prec \eta \frac{1 + Az}{1 + Bz} + \delta + z \frac{(A - B)}{(1 + Bz)^2}, \tag{3.37}
\]
then
\[
\left[ \frac{\partial^\alpha f(z)}{zp} \right]^\mu < \frac{1 + Az}{1 + Bz} \tag{3.38}
\]
and \( (1 + Az)/(1 + Bz) \) is the best dominant.

Again by setting \( \beta = 1, \gamma = 1, \alpha = 0, p = 1, \) and \( \sigma = 0, \) and by \( q(z) = (1 + z)/(1 - z), \) we get the following corollary.

Corollary 3.11. Let \( f(z) \in A(1) \) and
\[
\left[ \frac{f(z)}{z} \right]^\mu \left\{ \eta + \mu \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} + \delta \prec \eta \frac{1 + z}{1 - z} + \delta + \frac{2z}{(1 - z)^2}, \tag{3.39}
\]
then
\[
\left[ \frac{f(z)}{z} \right]^\mu < \frac{1 + z}{1 - z} \tag{3.40}
\]
and \( (1 + z)/(1 - z) \) is the best dominant.

4. Superordination for analytic functions

Theorem 4.1. Let \( q \) be convex univalent in the unit disk \( \Delta, \) and \( \lambda \in \mathbb{C}. \) Suppose \( \lambda \) satisfies \( \text{Re}(\lambda) > 0 \)
and \( \partial^\alpha f(z)/zp \in A(q(0), 1) \cap Q. \) Suppose that
\[
\frac{\lambda}{p} \left( \frac{\partial^{\alpha - 1} f(z)}{zp} \right) + \frac{p - \lambda}{p} \left( \frac{\partial^\alpha f(z)}{zp} \right) \tag{4.1}
\]
is univalent in the unit disk \( \Delta. \) If
\[
q(z) + \frac{\lambda z q'(z)}{p(p + 1)} \prec \frac{\lambda}{p} \left( \frac{\partial^{\alpha - 1} f(z)}{zp} \right) + \frac{p - \lambda}{p} \left( \frac{\partial^\alpha f(z)}{zp} \right), \tag{4.2}
\]
then
\[
q(z) < \frac{\partial^\alpha f(z)}{zp} \tag{4.3}
\]
and \( q(z) \) is the best subordinant.
Let $p(z) = \mathcal{O}^\alpha f(z) / z^p$, $z \neq 0$. Differentiating logarithmically, we get

$$\frac{zp'(z)}{p(z)} = z[\mathcal{O}^\alpha f(z)]'/\mathcal{O}^\alpha f(z) - p.$$  \hspace{1cm} (4.5)

After some computation, we get

$$p(z) + \frac{\lambda z p'(z)}{p(p + 1)} = \frac{1}{p} \left( \frac{\mathcal{O}^{\alpha-1} f(z)}{z^p} \right) + \frac{p - \lambda}{p} \left( \frac{\mathcal{O}^\alpha f(z)}{z^p} \right).$$  \hspace{1cm} (4.6)

Now, using Lemma 2.7, we get the desired result (4.3).

**Corollary 4.2.** Let $q$ be convex univalent in $\Delta$, and $\lambda \in \mathbb{C}$. Suppose $\lambda$ satisfies $\mathbb{R}[\lambda] > 0$ and $\mathcal{O}^\alpha f(z)/z^p \in \mathcal{A}(q(0), 1) \cap Q$. Let

$$\frac{\lambda(A - B)z}{p(p + 1)(1 + Bz)^2} + \frac{1 + Az}{1 + Bz} = \frac{1}{p} \left( \frac{\mathcal{O}^{\alpha-1} f(z)}{z^p} \right) + \frac{p - \lambda}{p} \left( \frac{\mathcal{O}^\alpha f(z)}{z^p} \right),$$

then

$$\frac{1 + Az}{1 + Bz} < \frac{\mathcal{O}^\alpha f(z)}{z^p}$$

and $(1 + Az)/(1 + Bz)$ is the best subordinant.

Since the proofs of Theorems 4.3 and 4.4 are similar to the proofs of the previous theorems, we only give statements of these theorems without proofs.

**Theorem 4.3.** Let $q(z)$ be convex univalent in $\Delta$, and $0 \neq \gamma, \mu \in \mathbb{C}$, and $\alpha, \beta \in \mathbb{C}$ such that $\alpha + \beta \neq 0$. Let $f(z) \in \mathcal{A}(p)$. Suppose that $[\mathcal{O}^{\alpha-1} f(z) + \beta \mathcal{O}^\alpha f(z)]/\mathcal{O}^{\alpha-1} f(z) \in \mathcal{A}(q(0), 1) \cap Q$, and

$$1 + \gamma \mu \left[ \frac{az[\mathcal{O}^{\alpha-1} f(z)]'/\mathcal{O}^{\alpha-1} f(z) + \beta z[\mathcal{O}^\alpha f(z)]'}{\alpha \mathcal{O}^{\alpha-1} f(z) + \beta \mathcal{O}^\alpha f(z)} - p \right]$$

is univalent in $\Delta$. If

$$1 + \gamma \frac{z q'(z)}{q(z)} < 1 + \gamma \mu \left[ \frac{az[\mathcal{O}^{\alpha-1} f(z)]'/\mathcal{O}^{\alpha-1} f(z) + \beta z[\mathcal{O}^\alpha f(z)]'}{\alpha \mathcal{O}^{\alpha-1} f(z) + \beta \mathcal{O}^\alpha f(z)} - p \right],$$

then

$$q(z) < \left[ \frac{\alpha \mathcal{O}^{\alpha-1} f(z) + \beta \mathcal{O}^\alpha f(z)}{(\alpha + \beta)z^p} \right]^\mu$$

and $q(z)$ is the best subordinant.
Theorem 4.4. Let \( q \) be convex univalent in the unit disk \( \Delta \), and let \( \gamma \neq 0 \in \mathbb{C} \). Suppose \( q \) satisfies (3.1). If \( \mathcal{O} f(z)/z^p \in \mathcal{K}(q(0), 1) \cap \mathcal{Q} \), and
\[
\text{Re}\left\{ \frac{\eta q(z)}{\gamma} \right\} > 0.
\]
(4.13)

If
\[
\eta q(z) + \delta + \gamma z q'(z) < \left[ \frac{\alpha \mathcal{O}^{-1} f(z) + \beta \mathcal{O} f(z)}{(\alpha + \beta)z^p} \right] \left\{ \eta + \gamma \mu \left( \frac{za[\mathcal{O}^{-1} f(z)'] + z\beta[\mathcal{O} f(z)']}{\alpha \mathcal{O}^{-1} f(z) + \beta \mathcal{O} f(z)} - p \right) \right\} + \delta,
\]
then
\[
q(z) < \left[ \frac{\alpha \mathcal{O}^{-1} f(z) + \beta \mathcal{O} f(z)}{(\alpha + \beta)z^p} \right] ^\mu
\]
(4.14)
and \( q(z) \) is the best subordinator.

5. Sandwich results

Combining results of differential subordinations and superordinations, we arrive at the following “sandwich results.”

Theorem 5.1. Let \( q_1(z) \) be convex univalent, and let \( q_2(z) \) be univalent in \( \Delta \), and \( \lambda \in \mathbb{C} \). Suppose \( q_1 \) satisfies \( \text{Re}\{\lambda\} > 0 \) and \( q_2 \) satisfies (3.1). If \( \mathcal{O} f(z)/z^p \in \mathcal{K}(q(0), 1) \cap \mathcal{Q} \) and
\[
\left[ \frac{\alpha \mathcal{O}^{-1} f(z) + \beta \mathcal{O} f(z)}{(\alpha + \beta)z^p} \right] ^\mu, \quad \alpha + \beta \neq 0,
\]
is univalent in \( \Delta \), and if
\[
q_1(z) + \frac{\lambda z q_1(z)}{p(p + 1)} < \frac{\mathcal{O}^{-1} f(z)}{z^p} + \frac{(p - \lambda) \mathcal{O} f(z)}{z^p} < q_2(z) + \frac{\lambda z q_2(z)}{p(p + 1)},
\]
then
\[
q_1(z) < \left( \frac{\mathcal{O} f(z)}{z^p} \right) < q_2(z)
\]
(5.3)
and \( q_1(z) \) and \( q_2(z) \) are, respectively, the best subordinant and the best dominant.

Theorem 5.2. Let \( q_1(z) \) be convex univalent, and let \( q_2(z) \) be univalent in \( \Delta \), and \( \lambda \in \mathbb{C} \). Suppose that \( q_2 \) satisfies (3.11). Further suppose that \( \left[ \frac{(\alpha \mathcal{O}^{-1} f(z) + \beta \mathcal{O} f(z))}{(\alpha + \beta)z^p} \right] ^\mu \in \mathcal{K}(q(0), 1) \cap \mathcal{Q} \) and
\[
1 + \gamma \mu \left[ (az[\mathcal{O}^{-1} f(z)]') + z\beta[\mathcal{O} f(z)]' \right] / (\alpha \mathcal{O}^{-1} f(z) + \beta \mathcal{O} f(z) - p) \text{ is univalent in } \Delta.
\]
If
\[
1 + \gamma \frac{z q_1'(z)}{q_1(z)} < 1 + \gamma \mu \left[ az[\mathcal{O}^{-1} f(z)]' + \beta z[\mathcal{O} f(z)]' \right] / (\alpha \mathcal{O}^{-1} f(z) + \beta \mathcal{O} f(z) - p) < 1 + \gamma \frac{z q_2'(z)}{q_2(z)},
\]
then
\[
q_1(z) < \left[ \frac{\alpha \mathcal{O}^{-1} f(z) + \beta \mathcal{O} f(z)}{(\alpha + \beta)z^p} \right] ^\mu < q_2(z), \quad \alpha + \beta \neq 0,
\]
(5.4)
and \( q_1(z) \) and \( q_2(z) \) are, respectively, the best subordinant and the best dominant.
Theorem 5.3. Let $q_1(z)$ be convex univalent, and let $q_2(z)$ be univalent in $\Delta$, and $\lambda \in \mathbb{C}$. Suppose that $q_1(z)$ satisfies (4.13) and $q_2(z)$ satisfies (3.28). Further suppose that \[ \left( \frac{a\sigma^{-1}f(z) + \beta\sigma f(z)}{(a + \beta)z^\mu} \right) \in \mathcal{K}(q(0), 1) \cap Q \text{ with } a + \beta \neq 0, \text{ and that} \]
\[ \left( \frac{a\sigma^{-1}f(z) + \beta\sigma f(z)}{(a + \beta)z^\mu} \right) \leq \eta + \gamma \mu \left( \frac{za[\sigma^{-1}f(z)]' + z\beta[\sigma f(z)]'}{a\sigma^{-1}f(z) + \beta\sigma f(z)} - p \right) + \delta \] (5.6)
is univalent in $\Delta$. If
\[ \eta q_1(z) + \delta + \gamma zq_1'(z) < \left( \frac{a\sigma^{-1}f(z) + \beta\sigma f(z)}{(a + \beta)z^\mu} \right)^\mu \left\{ \eta + \gamma \mu \left( \frac{za[\sigma^{-1}f(z)]' + z\beta[\sigma f(z)]'}{a\sigma^{-1}f(z) + \beta\sigma f(z)} - p \right) \right\} + \delta \]
< $\eta q_2(z) + \delta + \gamma zq_2'(z),$ (5.7)
then
\[ q_1(z) < \left( \frac{a\sigma^{-1}f(z) + \beta\sigma f(z)}{(a + \beta)z^\mu} \right)^\mu < q_2(z) \] (5.8)
and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinator and the best dominant.

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References


