Research Article
Skew Polynomial Extensions over Zip Rings

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In this article, we study the relationship between left (right) zip property of $R$ and skew polynomial extension over $R$, using the skew versions of Armendariz rings.

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1. Introduction

Throughout this paper $R$ denotes an associative ring with identity and $\sigma : R \to R$ an automorphism of $R$, otherwise unless stated. We denote $R[[x;\sigma]]$ ($R[[x,x^{-1};\sigma]]$) the skew series rings (skew Laurent series rings) whose elements are the series $\sum_{i=0}^{\infty} a_i x^i (\sum_{j=0}^{\infty} b_j x^j)$, where the addition is defined as usual and the multiplication is defined by the rule, $xa = \sigma(a)x$ ($x^{-1}a = \sigma^{-1}(a)x$), for any $a \in R$. Note that the skew polynomial rings of automorphism type $R[x;\sigma]$ (skew Laurent of polynomial $R[x,x^{-1};\sigma]$) are subrings of $R[[x;\sigma]]$ ($R[[x,x^{-1};\sigma]]$) whose elements are $\sum_{i=0}^{n} a_i x^i (\sum_{j=0}^{m} b_j x^j)$ where the sum and multiplication are defined as before.

Rege and Chhawchharia in [1] introduced the notion of an Armendariz ring. A ring $R$ is called Armendariz if whenever polynomials $\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{m} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each $0 \leq i \leq n$ and $0 \leq j \leq m$. The name Armendariz ring was chosen because Armendariz [2] had shown that a reduced ring (i.e., ring without nonzero nilpotent elements) satisfies this condition. Some properties of Armendariz rings have been studied by Rege and Chhawchharia [1], Armendariz [2], Anderson and Camillo [3], and Kim and Lee [4].

Faith in [5] called a ring $R$ right zip if the right annihilator $r_R(X)$ of a subset $X$ of $R$ is zero, then $r_R(Y) = 0$ for a finite subset $Y \subseteq X$; equivalently, for a left ideal $L$ of $R$ with $r_R(L) = 0$, there exists a finitely generated left ideal $L_1 \subseteq L$ such that $r_R(L_1) = 0$. $R$ is zip if it is right and left zip. The concept of zip rings was initiated by Zelmanowitz [6] and appeared in various papers [5, 7–12], and references therein. Zelmanowitz stated that any ring satisfying...
the descending chain condition on right annihilators is a right zip ring (although not so-called at that time), but the converse does not hold. Extensions of zip rings were studied by several authors. Beachy and Blair [7] showed that if $R$ is a commutative zip ring, then the polynomial ring $\mathbb{R}[x]$ over $R$ is zip. The authors in [13] proved that $R$ is a right (left) zip ring if and only if $R[x]$ is a right (left) zip ring when $R$ is an Armendariz ring.

In this paper, we study skew polynomial extensions over zip rings by using skew versions of Armendariz rings and we generalized the results of [13]. Our skew versions of Armendariz rings follow the ideas of [14, Definition]. Moreover, we provide some examples to display some of the phenomena of Section 2.

2. Skew polynomial extensions over zip rings

Throughout this paper, $\sigma$ is an automorphism of $R$ unless otherwise stated and $S$ will denote one of the following rings: $R[x; \sigma]$, $R[[x; \sigma]]$, $R[x, x^{-1}; \sigma]$, and $R[[x, x^{-1}; \sigma]]$. A left (right) annihilator of a subset $U$ of $R$ is defined by $l_R(U) = \{a \in R : aU = 0\}$ ($r_R(U) = \{a \in R : Ur = 0\}$). For a ring $R$, put $r\text{Ann}_R(2^R) = \{r_R(U) : U \subseteq R\}$ and $l\text{Ann}_R(2^R) = \{l_R(U) : U \subseteq R\}$.

We begin with the following lemma and use it without further mention.

**Lemma 2.1.** Let $S$ be one of the rings above and $U$ a subset of $R$. The following statements hold:

1. $l_S(U) = S l_R(U)$,
2. $r_S(U) = r_R(U) S$.

**Proof.** (i) We only prove for the case $S = R[x; \sigma]$ because the other cases are similar. Let $f(x) = \sum_{i=0}^{n}a_i x^i \in R[x; \sigma]$ such that $f(x) U = 0$. Then $\sigma^{-i}(a_i) U = 0$ for all $0 \leq i \leq n$ and it follows that $\sigma^{-i}(a_i) \in l_R(U)$ for all $0 \leq i \leq n$. Hence $f(x) = \sum_{i=0}^{n}a_i x^i (a_i) \in R[x; \sigma] l_R(U)$. So $l_{R[x; \sigma]}(U) \subseteq R[x; \sigma] l_R(U)$. We clearly have that $R[x; \sigma] l_R(U) \subseteq l_{R[x; \sigma]}(U)$. Therefore, we have $l_{R[x; \sigma]}(U) = R[x; \sigma] l_R(U)$.

(ii) We only prove for the case $S = R[x; \sigma]$ because the other cases are similar. Let $f(x) = \sum_{i=0}^{n}a_i x^i \in R[x; \sigma]$ such that $U f(x) = 0$. Then $U a_i = 0$ for all $0 \leq i \leq n$ and it follows that $a_i \in r_R(U)$ for all $0 \leq i \leq n$. Hence $f(x) = \sum_{i=0}^{n}a_i x^i \in r_R(U) R[x; \sigma]$. So $r_{R[x; \sigma]}(U) \subseteq r_R(U) R[x; \sigma]$. We clearly have that $r_R(U) R[x; \sigma] \subseteq r_{R[x; \sigma]}(U)$. Therefore, we have $r_{R[x; \sigma]}(U) = r_R(U) R[x; \sigma]$.

With the above lemma, we have maps $\phi : r\text{Ann}_R(2^R) \to r\text{Ann}_S(2^S)$ defined by $\phi(I) = IS$ for every $I \in r\text{Ann}_R(2^R)$ and

$$\Psi : l\text{Ann}_R(2^R) \to l\text{Ann}_S(2^S)$$

defined by $\Psi(I) = SI$ for every $I \in l\text{Ann}_R(2^R)$. Moreover, we have maps $\Phi : r\text{Ann}_S(2^S) \to r\text{Ann}_R(2^R)$ defined by $\Phi(J) = J \cap R$ for every $J \in r\text{Ann}_S(2^S)$ and $\Gamma : l\text{Ann}_S(2^S) \to l\text{Ann}_R(2^R)$ defined by $\Gamma(J) = J \cap R$ for every $J \in l\text{Ann}_S(2^S)$. Obviously, $\phi$ is injective and $\Phi$ and $\Gamma$ are surjective. Clearly, $\phi$ is surjective if and only if $\Phi$ is injective, and in this case $\phi$ and $\Phi$ are the inverses of each other. Note that $\Psi$ and $\Gamma$ satisfy the same relations as above. The first item of the definition below appears in [14, Definition].

**Definition 2.2.** (i) Suppose that $\sigma$ is an endomorphism of $R$. A ring $R$ satisfies $S\Lambda 1'$ if for $f(x) = \sum_{i=0}^{m}a_i x^i$ and $g(x) = \sum_{j=0}^{m}b_j x^j \in R[x; \sigma]$, $f(x) g(x) = 0$ implies that $a_i \sigma^j(b_j) = 0$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$. 

(ii) Suppose that $\sigma$ is an endomorphism of $R$. A ring $R$ satisfies SA2' if for $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ in $R[[x;\sigma]]$, $f(x)g(x) = 0$ implies $a_i \sigma^j(b_j) = 0$ for all $i \geq 0$, $j \geq 0$.

(iii) Suppose that $\sigma$ is an automorphism of $R$. A ring $R$ satisfies SA3' if for $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x,x^{-1};\sigma]$, $f(x)g(x) = 0$ implies $a_i \sigma^r(b_j) = 0$ for all $s \leq i \leq q$ and $t \leq j \leq n$.

(iv) Suppose that $\sigma$ is an automorphism of $R$. A ring $R$ satisfies SA4' if for $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[[x,x^{-1};\sigma]]$, $f(x)g(x) = 0$ implies $a_i \sigma^r(b_j) = 0$ for all $i \geq s$ and $j \geq t$.

Note that if $R$ satisfies one of the conditions above, then all subrings $S$ of $R$ such that $\sigma(S) \subseteq S$ satisfies the same property. The following implications are easy to verify: SA4' $\Rightarrow$ SA3' and SA2' $\Rightarrow$ SA1'. Following [15, Example 2.1] when $\sigma = id_R$, the last implication is not reversible.

**Lemma 2.3.** Let $\sigma$ be an automorphism of $R$. Then

(i) $R$ satisfies SA1' if and only if $R$ satisfies SA3';

(ii) $R$ satisfies SA2' if and only if $R$ satisfies SA4'.

**Proof.** Let $f(x), g(x) \in R[x,x^{-1};\sigma]$ such that $f(x)g(x) = 0$, where $f(x) = \sum_{i=0}^{p} a_i x^i$ and $g(x) = \sum_{j=0}^{q} b_j x^j$. We clearly have $x^q f(x) \in R[x;\sigma]$ and $g(x) x^t \in R[x;\sigma]$, then $x^q f(x)g(x) = 0$. By assumption, $\sigma^q(a_i)\sigma^r(b_j) = 0$ for all $-p \leq i \leq q$ and $-t \leq j \leq s$. Hence $a_i \sigma^r(b_j) = 0$ for all $-p \leq i \leq q$ and $-t \leq j \leq s$. Since $R[x;\sigma] \subseteq R[x,x^{-1};\sigma]$, the converse follows.

The proof of the other statement is similar. \qed

The following definition appears in [16, Definition 2.1].

**Definition 2.4.** Let $R$ be a ring and $\sigma$ an endomorphism of $R$. Then $R$ is said $\sigma$-compatible like right $R$-module, if $ar = 0$ if and only if $a\sigma(r) = 0$ for any $a \in R$ and $r \in R$.

Let $R$ be a ring and $\alpha$ an endomorphism of $R$. Following [17], the endomorphism $\alpha$ is said $\alpha$-rigid if $r\alpha(r) = 0$, then $r = 0$. A ring $R$ is said a rigid ring if it exists a rigid endomorphism $\alpha$ of $R$.

**Proposition 2.5.** Let $\sigma$ be an endomorphism of $R$. If $R$ is a reduced ring and $\sigma$-compatible like right $R$-module, then $R$ is a $\sigma$-rigid ring and hence satisfies SA1' and SA2'.

**Proof.** We only prove the case of SA2' because the other are similar. We claim that $R[[x;\sigma]]$ is a reduced ring. In fact, let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ such that $(f(x))^2 = 0$. We have that $a_i^2 = 0$. Since $R$ is reduced, then $a_0 = 0$. Next, we have $a_1 \sigma(a_1) = 0$, since $R$ is $\sigma$-compatible and reduced, then $a_1 = 0$. By induction, we get $f(x) = 0$. Hence $R[[x;\sigma]]$ is reduced. Using the same ideas of [14, Proposition 3], we have that $R$ is $\sigma$-rigid and using similar ideas of [14, Corollary 4], we obtain that $R$ satisfies SA2'. \qed

Without the assumption that $R$ is $\sigma$-compatible, Proposition 2.5 is not true. In fact, let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\sigma : \mathbb{Z}_2 \to \mathbb{Z}_2$, defined by $\sigma((a,b)) = (b,a)$. By [14, Example 2], $R$ does not satisfy SA2' because $R$ does not satisfy SA1'. Observe that $(1,0)(0,1) = (0,0)$ but $(1,0)\sigma(0,1) \neq (0,0)$ and so $R$ is not $\sigma$-compatible. We have the following natural questions.
Questions

(i) Let \( \sigma \) be an endomorphism of \( R \). Suppose that \( R \) satisfies SA2'. Is \( R\sigma \)-compatible like right \( R \)-module?

(ii) Let \( \sigma \) be an endomorphism of \( R \). Suppose that \( R \) is \( \sigma \)-compatible like right \( R \)-module. Does \( R \) satisfy SA2'?

The question (i) is false. Let \( R_0 \) be any domain and \( R = R_0[x] \). Let \( \sigma : R \rightarrow R \) be defined by \( \sigma(t) = 0 \) and \( \sigma|_{R_0} = \text{id}_{R_0} \). By [16, Example 4.1], \( R \) is not \( \sigma \)-compatible and using the similar ideas of the proof of [14, Proposition 10], we have that \( R \) satisfies SA2' and consequently \( R \) satisfies SA1'.

The question (ii) is false. Let \( R = K[x, y]/(x^2, y^2) \), where \( K \) is a field of characteristic 2, and consider \( T = M_2(R) \). In this case, take \( \sigma = \text{id}_T \). By [18, Example 3.6], \( S \) does not satisfy SA2' because \( T \) does not satisfy SA1'. Moreover, \( T \) is \( \sigma \)-compatible like right \( T \)-module.

In [19] the authors introduced the following version of skew Armendariz rings.

(i) Suppose that \( \sigma \) is an endomorphism of \( R \). Let \( f(x) = \sum_{i=0}^{n} a_i x^i, \ g(x) = \sum_{j=0}^{m} b_j x^j \in R[x; \sigma] \) such that \( f(x)g(x) = 0 \) implies \( a_i b_j = 0 \) for all \( 0 \leq i \leq n \) and \( 0 \leq j \leq m \).

(ii) Suppose that \( \sigma \) is an endomorphism of \( R \). Let \( f(x) = \sum_{i=0}^{n} a_i x^i, \ g(x) = \sum_{j=0}^{m} b_j x^j \in R[[x; \sigma]] \) such that \( f(x)g(x) = 0 \) implies \( a_i b_j = 0 \) for all \( i \geq 0 \) and \( j \geq 0 \).

Note that the item (i) above in [20, Definition 1.1] the authors called it by \( \sigma \)-Armendariz, the item (ii) above is similar with [20, Definition 1.1] and we call it here by \( \sigma \)-power Armendariz.

In the next proposition, we give a relationship between the definition above and the skew versions of Armendariz rings used in this paper. Using [21, Lemma 2.1] and [20, Theorem 1.8], the proof of next proposition is easy to verify.

**Proposition 2.6.** Let \( \sigma \) be an endomorphism of \( R \) and suppose that \( R \) is \( \sigma \)-compatible like right \( R \)-module. Then

(i) \( R \) satisfies SA1' if and only if \( R \) is \( \sigma \)-Armendariz;

(ii) \( R \) satisfies SA2' if and only if \( R \) is \( \sigma \)-power Armendariz.

The proposition above without the compatibility assumption is not true according to [20, Example 1.9] and the authors in [22, Theorem 2.2] obtained an approach of the result above without the compatibility assumption.

The following proposition is a generalization of [18, Proposition 3.4] and partially generalizes [15, Proposition 2.6].

**Lemma 2.7.** Let \( S \) be any of the rings \( R[x; \sigma] \) and \( R[[x; \sigma]] \). The following conditions are equivalent:

(i) \( R \) satisfies SA2' (SA1');

(ii) \( \phi : r\text{Ann}_R(2^R) \rightarrow r\text{Ann}_S(2^S) \) defined by \( \phi(J) = JS \) is bijective;

(iii) \( \Psi : l\text{Ann}_R(2^R) \rightarrow l\text{Ann}_S(2^S) \) defined by \( \Psi(J) = SJ \) is bijective.

**Proof.** We only prove the proposition in the case of SA2' because the equivalence of (i) and (ii) when \( R \) satisfies SA1' was proved in [23, Proposition 3.2]. The equivalence between (i) and (iii) when \( R \) satisfies SA1' has similar proof.

(i) \( \Rightarrow \) (ii). It is only necessary to show that \( \phi \) is surjective. For an element \( f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x; \sigma]] \), define \( C_{f(x)} = \{ \sigma^{-i}(a_i), i \geq 0 \} \), and for a subset \( T \) of \( R[[x; \sigma]] \), we denote the set
\( \bigcup_{f(x) \in T} C_{f(x)} \) by \( C_T \). We show that \( r_{R[[x;\sigma]]}(f(x)) = r_{R[[x;\sigma]]}(C_{f(x)}) \). In fact, given \( g(x) = \sum_{j=0}^{\infty} b_j x^j \) in \( r_{R[[x;\sigma]]}(f(x)) \), we have \( f(x)g(x) = 0 \). Since \( R \) satisfies \( \text{SA}2' \), then \( a_i \sigma^j(b_j) = 0 \) for all \( i \geq 0 \) and \( j \geq 0 \). In particular, \( \sigma^{-i}(a_i)b_j = 0 \) for all \( i \geq 0 \) and \( j \geq 0 \). Hence \( g(x) \in r_{R[[x;\sigma]]}(C_{f(x)}) \).

On the other hand, let \( h(x) = \sum_{k=0}^{\infty} c_k x^k \) be an element in \( R[[x;\sigma]] \) such that \( C_{f(x)}h(x) = 0 \). It is clear that \( a_i \sigma^j(c_k) = 0 \) for all \( i \geq 0 \) and \( k \geq 0 \). So \( f(x)h(x) = 0 \). Since \( R \) satisfies \( \text{SA}2' \) then \( r_{R[[x;\sigma]]}(T) = r_{R[[x;\sigma]]}(f(x)) \).

Thus
\[
r_{R[[x;\sigma]]}(T) = \bigcap_{f(x) \in T} r_{R[[x;\sigma]]}(f(x)) = \bigcap_{f(x) \in T} r_{R[[x;\sigma]]}(C_{f(x)}) = \left( \bigcap_{f(x) \in T} r_{R}(C_{f(x)}) \right) R[[x;\sigma]] = r_R(C_T)R[[x;\sigma]].
\]

Therefore, \( \phi \) is surjective.

(ii)\( \rightarrow \) (i). Let \( f(x) = \sum_{i=0}^{\infty} a_i x^i \) and \( g(x) = \sum_{j=0}^{\infty} b_j x^j \) be elements in \( R[[x;\sigma]] \) such that \( f(x)g(x) = 0 \). By assumption, \( r_{R[[x;\sigma]]}(f(x)) = BR[[x;\sigma]] \), for some right ideal \( B \) of \( R \). Hence \( g(x) \in BR[[x;\sigma]] \) and we have that \( b_j \in B \subseteq r_{R[[x;\sigma]]}(f(x)) \) for all \( j \geq 0 \). So \( a_i \sigma^j(b_j) = 0 \) for all \( i \geq 0 \) and \( j \geq 0 \).

(iii)\( \rightarrow \) (i). Let \( f(x) = \sum_{i=0}^{\infty} a_i x^i \) and \( g(x) = \sum_{j=0}^{\infty} b_j x^j \) be elements in \( R[[x;\sigma]] \) such that \( f(x)g(x) = 0 \). By assumption, \( l_{R[[x;\sigma]]}(g(x)) = R[[x;\sigma]]B \) for some left ideal \( B \) of \( R \). We can write \( f(x) = \sum_{i=0}^{\infty} a_i x^i \) \( \sigma^{-i}(a_i) \in R[[x;\sigma]]B \). By the equality of the polynomials with the coefficients on the right side, we have that \( \sigma^{-i}(a_i) \in B \subseteq l_{R[[x;\sigma]]}(g(x)) \) for all \( i \geq 0 \). So \( a_i \sigma^j(b_j) = 0 \) for all \( i \geq 0 \) and \( j \geq 0 \).

(i)\( \rightarrow \) (iii). It is only necessary to show that \( \Psi \) is surjective. Let \( f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x;\sigma]] \).

Define \( C_{f(x)} = \{ a_i, i \geq 0 \} \), and for a subset \( T \) of \( R[[x;\sigma]] \), we denote the set \( \bigcup_{f(x) \in T} C_{f(x)} \) by \( C_T \). We show that
\[
l_{R[[x;\sigma]]}(f(x)) = l_{R[[x;\sigma]]}(C_{f(x)}). \tag{2.3}
\]

In fact, given \( g(x) = \sum_{j=0}^{\infty} b_j x^j \in l_{R[[x;\sigma]]}(f(x)) \), we have \( g(x)f(x) = 0 \). Since \( R \) satisfies \( \text{SA}2' \), then \( b_j \sigma^j(a_i) = 0 \) for all \( i \geq 0 \) and \( j \geq 0 \). Hence \( g(x) = \sum_{j=0}^{\infty} x^j \sigma^{-j}(b_j) \in l_{R[[x;\sigma]]}(C_{f(x)}) \).

On the other hand, let \( g(x) \in R[[x;\sigma]] \) such that \( g(x)C_{f(x)} = 0 \). Thus \( g(x)a_i = 0 \) for all \( i \geq 0 \). So \( g(x) \sum_{i=0}^{\infty} a_i x^i = g(x)f(x) = 0 \), and we have that \( g(x) \in l_{R[[x;\sigma]]}(f(x)) \).

We easily have that for each subset \( T \) of \( R[[x;\sigma]] \),
\[
l_{R[[x;\sigma]]}(T) = l_{R[[x;\sigma]]} \left( \bigcup_{f(x) \in T} C_{f(x)} \right). \tag{2.4}
\]

We claim that \( l_{R[[x;\sigma]]}(C_{f(x)}) = R[[x;\sigma]]l_R(C_{f(x)}) \). In fact, let \( g(x) = \sum_{j=0}^{\infty} b_j x^j \) such that \( g(x)C_{f(x)} = 0 \). Then we have that \( 0 = g(x)a_i = \sum_{j=0}^{\infty} b_j x^j a_i = \sum_{j=0}^{\infty} x^j \sigma^{-j}(b_j) a_i \). Thus \( \sigma^{-j}(b_j) \in l_R(C_{f(x)}) \), and it follows that
\[
\sum_{j=0}^{\infty} x^j \sigma^{-j}(b_j) \in R[[x;\sigma]]l_R(C_{f(x)}). \tag{2.5}
\]

The other inclusion is trivial. So
\[
l_{R[[x;\sigma]]}(T) = \bigcap_{f(x) \in T} l_{R[[x;\sigma]]}(C_{f(x)}) = \bigcap_{f(x) \in T} l_{R[[x;\sigma]]}(C_{f(x)}) = R[[x;\sigma]] \left( \bigcap_{f(x) \in T} l_R(C_{f(x)}) \right) = R[[x;\sigma]]l_R(C_T). \tag{2.6}
\]

Therefore, \( \Psi \) is surjective. \( \square \)
Now we are able to prove the main results of this paper.

**Theorem 2.8.** Let \( \sigma \) be an automorphism of \( R \).

(i) Suppose that \( R \) satisfies SAI'. The following conditions are equivalent:

(a) \( R \) is a right (left) zip ring;
(b) \( R[x; \sigma] \) is a right (left) zip ring;
(c) \( R[x, x^{-1}; \sigma] \) is a right (left) zip ring.

(ii) Suppose that \( R \) satisfies SAI". The following conditions are equivalent:

(a) \( R \) is right (left) zip ring;
(b) \( R[[x; \sigma]] \) is right (left) zip ring;
(c) \( R[[x, x^{-1}; \sigma]] \) is right (left) zip ring.

**Proof.** (i) We will show the right case because the left case is similar.

Suppose that \( R[x; \sigma] \) is right zip. Let \( X \) be a subset of \( R \) such that \( r_R(X) = 0 \), and \( f(x) = \sum_{i=0}^{n} a_i x^i \in R[x; \sigma] \) such that \( Xf(x) = 0 \). Thus \( a_i \in r_R(X) = 0 \) and it follows that \( f(x) = 0 \). By assumption, there exists \( X_1 = \{ x_0, \ldots, x_n \} \) such that \( r_R(x_1) = 0 \). Hence \( r_R(X_1) = r_R(x_1) \cap R = (0) \).

Conversely, let \( Y \subseteq R[x; \sigma] \) such that \( r_R(Y) = 0 \). By Lemma 2.7, \( r_R(Y) = r_R(T)R[X; \sigma] \), where \( T = C_Y \cup_{f(x) \in Y} C_f(x) \) such that \( C_f(x) = \{ \sigma^{-i}(a_i) : 0 \leq i \leq n \} \) with \( f(x) = \sum_{i=0}^{n} a_i x^i \in Y \). We have that \( r_R(T) = 0 \) and, by assumption, there exists \( T_1 = \{ \sigma^{-i}(a_i), \ldots, \sigma^{-i}(a_i) \} \) such that \( r_R(T_1) = 0 \). For each \( \sigma^{-i}(a_i) \in T_1 \), there exists \( g_{a_i}(x) \in Y \) such that some of the coefficients of \( g_{a_i}(x) \) are \( a_i \) for each \( 1 \leq j \leq n \). Let \( Y_0 \) be a minimal subset of \( Y \) such that \( g_{a_i}(x) \in Y_0 \) for each \( 1 \leq j \leq n \). Then \( Y_0 \) is nonempty finite subset of \( Y \).

Set \( T_0 = \cup_{f(x) \in Y_0} C_f(x) \) and we have that \( T_1 \subseteq T_0 \). Hence \( r_R(T_0) \subseteq r_R(T_1) = 0 \). By Lemma 2.7, \( r_R(Y_0) = r_R(T_0)R[X; \sigma] \) and it follows that \( r_R(Y_0) = 0 \).

The proofs of (a) \( \Leftrightarrow \) (c) and of item (ii) follow similarly. \( \square \)

Let \( \sigma \) be an endomorphism of \( R \) and \( \delta : R \to R \) an additive map of \( R \). The application \( \delta \) is said to be a \( \sigma \)-derivation if \( \delta(ab) = \delta(a)b + \sigma(a)\delta(b) \). The Ore extension \( R[x; \sigma, \delta] \) is the set of polynomials \( \sum_{i=0}^{n} a_i x^i \) with the usual sum, and the multiplication rule is \( xa = \sigma(a)x + \delta(a) \).

Following [16], \( R \) is said to be \( (\sigma, \delta) \)-compatible, where \( \sigma \) is an endomorphism of \( R \) and \( \delta \) is a \( \sigma \)-derivation of \( R \) if \( ab = 0 \Leftrightarrow a\sigma(b) = 0 \) and \( ab = 0 \) implies that \( a\delta(b) = 0 \).

In the next result we obtain a necessary and sufficient condition for \( R[x; \sigma, \delta] \) to be left zip, when \( \sigma \) is an endomorphism of \( R \) using the skew version of Armendariz rings of [19].

**Theorem 2.9.** Let \( \sigma \) be an endomorphism of \( R \) and \( \delta \) a \( \sigma \)-derivation of \( R \). Suppose that if \( f(x)g(x) = 0 \) for \( f(x) = \sum_{i=0}^{n} a_i x^i \) and \( g(x) = \sum_{j=0}^{m} b_i x^j \in R[x; \sigma, \delta] \), then \( a_i b_j = 0 \) for all \( 0 \leq i \leq n \) and \( 0 \leq j \leq m \). Then \( R \) is left zip if and only if \( R[x; \sigma, \delta] \) is left zip.

**Proof.** Let \( X \) be any subset of \( R[x; \sigma, \delta] \) and \( C_X = \cup_{f(x) \in X} C_f(x) \), where \( C_f(x) = \{ a_i, 0 \leq i \leq n \} \) with \( f(x) = \sum_{i=0}^{n} a_i x^i \). Suppose that \( l_R(X) = 0 \). We clearly have \( l_R(C_X) = 0 \). By assumption, there exists \( \{ b_0, \ldots, b_t \} \subseteq C_X \) such that \( l_R(Y) = 0 \). Let \( f_b(x) \in X \) be an element of \( X \) with some of its coefficients are equal to \( b_i \) for all \( 1 \leq i \leq t \). Take \( X_0 \) be a minimal subset of \( X \) with this property. We clearly have that \( X_0 \) is a finite set. We claim that \( l_R(X_0) = 0 \). In fact, we
easily have \( l_R(C_{X_0}) = 0 \), where \( C_{X_0} = \cup_{f(x) \in X_0} C_{f(x)} \) with \( C_{f(x)} \) being defined as before. Next, let 
\[
g(x) = \sum_{i=0}^{m} b_i x^i\]such that \( g(x)X_0 = 0 \). Hence for any \( f(x) = \sum_{i=0}^{m} a_i x^i \in X_0 \), \( g(x)f(x) = 0 \), and we have, by assumption, \( b_i a_i = 0 \) for all \( 0 \leq j \leq m \) and \( 0 \leq i \leq n \). Thus \( b_j C_{X_0} = 0 \) for all \( 0 \leq j \leq m \) and it follows that \( g(x) = 0 \). So \( l_{R[x;\sigma,0]}(X_0) = 0 \).

Using the methods of Theorem 2.8, the converse follows.

\[\square\]

**Remark 2.10.** Let \( R \) be a ring and \( \sigma \) an endomorphism of \( R \). Suppose that \( R \) is \( \sigma \)-power Armendariz and left zip. Using similar methods of [20, Theorem 1.8], \( R \) satisfies SA2' and with similar ideas of Theorem 2.9, we have that \( R \) is a left zip ring if and only if \( R[[x;\sigma]] \) is a left zip ring.

### 3. Examples

In this section, we present some examples of rings that satisfy SA1' and SA2', and they are zip rings. Moreover, an example of a \( \sigma \)-rigid ring that is a zip ring is given.

**Example 3.1.** Let \( F \) be any field and \( \sigma : F \rightarrow F \) any automorphism of \( F \). Following [14, page 113], we consider the ring \( T(F,F) \) with automorphism \( \overline{\sigma}(a,b) = (\sigma(a),\sigma(b)) \) and we denote it by \( \sigma \). Note that
\[
T(F,F) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a,b \in F \right\}.
\] (3.1)

By [14, Proposition 15], \( T(F,F) \) satisfies SA1', and using similar methods, we can prove that \( T(F,F) \) satisfies SA2'. We claim that \( T(F,F) \) is a zip ring. In fact, the unique one-sided ideals of \( T(F,F) \) are \( \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \),
\[
I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in F \right\},
\] (3.2)

and \( T(F,F) \). Note that \( r_{T(F,F)}(I) \neq 0 \) and \( l_{T(F,F)}(I) \neq 0 \). So we easily have that \( T(F,F) \) is a zip ring.

**Example 3.2.** Let \( F \) be any field and \( \sigma \) a monomorphism of \( F \), and let
\[
R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a,b,c \in F \right\}
\] (3.3)

with usual addition and multiplication of matrix. Note that the monomorphism \( \sigma \) is naturally extended to \( R \), and \( R \) has the following one-sided ideals:
\[
I_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} : a \in F \right\}, \quad I_2 = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : c \in F \right\},
\] (3.4)

\( R \) and the zero ideal. We easily have \( r_R(I_2) \neq 0 \), \( l_R(I_2) \neq 0 \), \( r_R(I_1) \neq 0 \), and \( l_R(I_1) \neq 0 \). Now we clearly have that \( R \) is a zip ring and by [14, Proposition 17], \( R \) satisfies SA1', and with similar methods of [14, Proposition 17], we can prove that \( R \) satisfies SA2'.
Example 3.3. Let $D$ be any domain with identity, $R = D[x]$, $\sigma$ an endomorphism of $R$ defined by $\sigma(f(x)) = f(0)$. Since $R$ is a domain, then $R$ is right and left zip. Moreover, using similar methods of [14, Example 5], we have that $R$ satisfies $SA1'$ and $SA2'$.

Example 3.4. Let $D$ and $D_1$ be any domains, $\sigma$ an monomorphism of $D$, and $\tau$ an monomorphism of $D_1$. Set $R = D \times D_1$ with usual addition and multiplication, and we define an endomorphism $\gamma$ of $R$ by $\gamma(a, b) = (\sigma(a), \tau(b))$. We easily have that $\gamma$ is a monomorphism of $R$. Since $D$ is $\sigma$-rigid and $D_1$ is $\tau$-rigid, we easily obtain that $R$ is $\gamma$-rigid. We claim that $R$ is left and right zip. In fact, let $I$ be any left ideal of $R$. It is well known that $I = A \times B$, where $A$ is a left ideal of $D$ and $B$ is a left ideal of $D_1$. Suppose that $r_R(I) = 0$. Then $A \neq 0$ and $B \neq 0$. It is not difficult to show that $r_D(A) = 0$ and $r_{D_1}(B) = 0$. Since $D$ and $D_1$ are left zip, then there exists a left finitely generated ideal $L$ of $D$ contained in $A$ such that $r_D(L) = 0$ and a left finitely generated ideal $L_1$ of $D_1$ contained in $B$ such that $r_{D_1}(L_1) = 0$. Thus $r_R(L \times L_1) = 0$ and $L \times L_1$ is a left finitely generated ideal of $R$ contained in $A \times B$. Hence $R$ is left zip. Using similar methods, we have that $R$ is right zip.

Example 3.5. Let $F$ be a field, $\sigma$ an automorphism of $F$,

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c \in F \right\},$$

and $D$ a domain with automorphism $\tau$. Set $T = R \times D$ and we define an endomorphism $\gamma$ of $T$ by $\gamma(a, b) = (\sigma(a), \tau(t))$. It is clear that $\gamma$ is an automorphism of $T$ and it is not difficult to show that $T$ satisfies $SA1'$ and $SA2'$ because $R$ and $D$ satisfy $SA1'$ by [14, Proposition 17] and [14, Proposition 10], respectively, and using similar methods of [14, Proposition 17] and [14, Proposition 10], $R$ and $D$ satisfy $SA2'$, respectively.

Using similar methods of Example 3.4, we have that $T$ is right and left zip and note that $T$ is not $\gamma$-rigid, since $T$ is not a reduced ring.

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