Research Article

Farthest Points and Subdifferential in $p$-Normed Spaces

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We study the farthest point mapping in a $p$-normed space $X$ in virtue of subdifferential of $r(x) = \sup\{\|x - z\|^p : z \in M\}$, where $M$ is a weakly sequentially compact subset of $X$. We show that the set of all points in $X$ which have farthest point in $M$ contains a dense $G_δ$ subset of $X$.

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1. Introduction

Let $X$ be a real linear space. A quasinorm is a real-valued function on $X$ satisfying the following conditions.

(i) $\|x\| \geq 0$ for all $x \in X$, and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(iii) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \| \cdot \|)$ is called a quasinormed space if $\| \cdot \|$ is a quasinorm on $X$. The smallest possible $K$ is called the modules of concavity of $\| \cdot \|$. By a quasi-Banach space we mean a complete quasinormed space, that is, a quasinormed space in which every Cauchy sequence converges in $X$.

This class includes Banach spaces. The most significant class of quasi-Banach spaces, which are not Banach spaces, is $L_p$-spaces for $0 < p < 1$ equipped with the $L_p$-norms $\| \cdot \|_p$. A quasinorm $\| \cdot \|$ is called a $p$-norm ($0 < p < 1$) if $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$. In this case, a quasinormed (quasi-Banach) space is called a $p$-normed ($p$-Banach) space. By the Aoki-Rolewicz theorem [1], each quasinorm is equivalent to some $p$-norm. Since it is much easier to work with $p$-norms than with quasinorms, henceforth we restrict our attention mainly to $p$-norms. See [2–4] for more information.
If \( x^* \) is in \( X^* \), the dual of \( X \), and \( x \in X \) we write \( x^*(x) \) as \( \langle x^*, x \rangle \). We also consider quasinorms with \( K > 1 \). The case where \( K = 1 \) turns out to be the classical normed spaces, so we will not discuss it and refer the interested reader to [5–7] for analogue results concerning normed spaces.

In this paper, using some strategies from [5–7], we study the farthest point mapping in a \( p \)-normed space \( X \) in virtue of subdifferential of \( r(x) = \sup \{ \| x - z \|^p : z \in M \} \), where \( M \) is a weakly sequentially compact subset of \( X \). We show that the set of all points in \( X \) which have farthest point in \( M \) contains a dense \( G_δ \) subset of \( X \).

Let \( X \) be a \( p \)-normed space and let \( M \) be a nonempty bounded subset of \( X \). The mapping \( Q_M : X \to 2^M \) defined by \( Q_M(x) = \{ z \in M : \| x - z \|^p = \sup_{l \in M} \| x - l \|^p \} \) is called the farthest point map of \( X \). We call \( M \) a remotal (uniquely remotal, resp.) set if for each \( x \in X \) the set \( Q_M(x) \) is not empty (is singleton, resp.) [8–10].

## 2. Main results

Let \( X \) be a \( p \)-normed space and let \( M \) be a bounded subset of \( X \). For each \( x \in X \), we define the subdifferential of a function \( f \) at \( x \) by

\[
\partial f(x) = \{ x^* \in X^* : \text{sgn}(x^*, y - x)\langle x^*, y - x \rangle |^p + f(x) \leq f(y) \ \forall y \in X \}.
\]

This set may be empty even if we consider \( X \) to be a Banach space [7, Example 3.8]. In a \( p \)-normed space, it may happen that \( \partial r(x) \neq \emptyset \), although we should note that a \( p \)-normed space may have a trivial dual as well as it might have a nontrivial dual, see [11, Chapter 3], for some examples. To see the nonemptiness, suppose that \( X \) is a \( p \)-normed space, \( x \in X \), and \( M = \{ x \} \).

Thus, \( r(x) = 0 \) and obviously \( 0 \in \partial r(x) \). Also for each \( x^* \in X^* \) with \( \| x^* \| \leq 1 \), we have

\[
|\langle x^*, y - x \rangle|^p \leq \| x^* \|^p \| y - x \|^p \leq \| y - x \|^p \leq r(y) = r(y) - r(x) \quad (y \in X).
\]

It follows that \( \text{sgn}(x^*, y - x)\langle x^*, y - x \rangle |^p + r(x) \leq r(y) \ (y \in X) \), and so \( \partial r(x) \neq \emptyset \). Throughout the rest, we assume \( \partial r(x) \neq \emptyset \) when we deal with this set. For an arbitrary nonempty bounded subset \( M \) of \( X \), finding the set of all \( x \) for which \( \partial r(x) \neq \emptyset \) remains an open question.

### Lemma 2.1

Let \( X \) be a \( p \)-Banach space and let \( M \) be a bounded subset in \( X \). Then for each \( x \in X \), each element of \( \partial r(x) \) has norm less than or equal to 1 and hence \( \partial r(x) \) is \( \omega^* \)-compact.

**Proof.** Let \( x \in X \) and \( x^* \in \partial r(x) \). We have \( \text{sgn}(x^*, y - x)\langle x^*, y - x \rangle |^p + r(x) \leq r(y) \ (y \in X) \). By definition of \( r(x) \) we have \( |r(y) - r(x)| \leq \| x - y \|^p \) for all \( y \in X \) [10, 12].

Hence \( \text{sgn}(x^*, y - x)\langle x^*, y - x \rangle |^p \leq \| x - y \|^p \) and therefore \( \| x^* \| \leq 1 \). \( \square \)

Note that \(-r(x) \leq -\| x - y \|^p \leq -\text{sgn}(x^*, x - y)\langle x^*, x - y \rangle |^p = \text{sgn}(x^*, y - x)\langle x^*, y - x \rangle |^p \), thus \( \inf_{z \in M} \text{sgn}(x^*, z - x)\langle x^*, z - x \rangle |^p \geq -r(x) \).

Now we have the following proposition which is interesting on its own right.

### Proposition 2.2

Let \( X \) be a \( p \)-Banach space and let \( M \) be a bounded subset of \( X \). Then the set \( F = \{ x \in X : \inf_{z \in M} \text{sgn}(x^*, z - x)\langle x^*, z - x \rangle |^p > -r(x) \text{ for some } x^* \in \partial r(x) \} \) is of the first category in \( X \).
Proof. Let
\[ F_n := \{ x \in X : \inf_{z \in M} \text{sgn}(x^*, z - x)|\langle x^*, z - x \rangle| \geq -r(x) + \frac{1}{n} \text{ for some } x^* \in \partial r(x) \}. \] (2.3)

Then \( F = \bigcup_{n=1}^{\infty} F_n \). We will show that for each \( n \),

(i) \( F_n \) is a norm closed subset of \( X \);

(ii) \( F_n \) has empty interior.

To see (i), let \( \{x_m\} \) be a sequence in \( F_n \) which converges to an element \( x \) in \( X \). For each \( m \), choose \( x_m^* \in \partial r(x_m) \) such that
\[
\inf_{z \in M} \text{sgn}(x_m^*, z - x_m)|\langle x_m^*, z - x_m \rangle| \geq -r(x_m) + \frac{1}{n}. \] (2.4)

By Lemma 2.1 \( \|x_m^*\| \leq 1 \) for all \( m \). Without loss of generality, we assume that \( \{x_m^*\} \) converges weak* to \( x^* \). For every \( y \in X \), we have
\[
|\langle x_m^*, y - x_m \rangle - \langle x^*, y - x \rangle| \leq |\langle x_m^*, y - x_m \rangle - \langle x_m^*, y - x \rangle| + |\langle x_m^*, y - x \rangle - \langle x^*, y - x \rangle|
\leq \|x_m - x\| + |\langle x_m^* - x^*, y - x \rangle|.
\] (2.5)

This shows that \( \{\langle x_m^*, y - x_m \rangle\}_{m=1}^{\infty} \) converges to \( \langle x^*, y - x \rangle \). Since \( x_m^* \in \partial r(x_m) \),
\[
\text{sgn}(x_m^*, y - x_m)|\langle x_m^*, y - x_m \rangle| = r(x_m) \leq r(y) \quad (y \in X),
\] (2.6)
or equivalently
\[
\langle x_m^*, y - x_m \rangle|\langle x_m^*, y - x_m \rangle|^{p-1} + r(x_m) \leq r(y) \quad (y \in X).
\] (2.7)

It follows that
\[
\langle x^*, y - x \rangle|\langle x^*, y - x \rangle|^{p-1} + r(x) \leq r(y) \quad (y \in X)
\] (2.8)
and hence
\[
\text{sgn}(x^*, y - x)|\langle x^*, y - x \rangle|^{p} + r(x) \leq r(y) \quad (y \in X).
\] (2.9)

This shows that \( x^* \in \partial r(x) \). It follows from (2.4) that
\[
\text{sgn}(x_m^*, z - x_m)|\langle x_m^*, z - x_m \rangle| \geq -r(x_m) + \frac{1}{n} \quad (z \in M).
\] (2.10)

We use the fact that \( \text{sgn}(x_m^*, y - x_m)|\langle x_m^*, y - x_m \rangle| \geq \langle x_m^*, y - x_m \rangle|\langle x_m^*, y - x_m \rangle|^{p-1} \) once more to obtain the inequality
\[
\text{sgn}(x^*, z - x)|\langle x^*, z - x \rangle|^{p} \geq -r(x) + \frac{1}{n} \quad (z \in M).
\] (2.11)

Therefore \( x \in F_n \). So \( F_n \) is a closed subset of \( X \).
To see (ii), suppose that some $F_k$ has nonempty interior. Then, there exists an open ball $U$ in $X$ of radius $\lambda(2r(y_0))^{1/p}$ for some $\lambda > 0$ and center at $y_0$ such that $U \subseteq F_k$. Let $\alpha = 1/\lambda^p$, $\beta = 1/(\lambda + 1)^p$, and $\epsilon \leq ((1 + \lambda)^p - 1)/k(\alpha + \beta + 1)((1 + \lambda)^p) \min\{r_{1/p}(y_0), 1\}$ and choose $z_0 \in M$ such that

$$r(y_0) - \epsilon < \|y_0 - z_0\|^p \leq r(y_0). \quad (2.12)$$

Let $x_0 = y_0 + \lambda(y_0 - z_0)$ then

$$x_0 - z_0 = (1 + \lambda)(y_0 - z_0). \quad (2.13)$$

Choose $x_1 \in U \subseteq F_k$ such that

$$\|x_1 - x_0\|^p = \epsilon. \quad (2.14)$$

Then there exists $x_1^* \in \partial r(x_1)$ such that

$$\inf_{z \in M} \text{sgn}(x_1^*, z - x_1) \langle x_1^*, z - x_1 \rangle \|z - x_1\|^p \geq -r(x_1) + \frac{1}{k}. \quad (2.15)$$

We will show that

$$\text{sgn}(x_1^*, y_0 - x_1) \langle x_1^*, y_0 - x_1 \rangle \|y_0 - x_1\|^p + r(x_1) > r(y_0). \quad (2.16)$$

This will contradict the fact that $x_1^*$ is a subdifferential of $r$ at $x_1$ and the proof would be completed. To achieve a contradiction, we will consider four cases as follows.

(i) $\text{sgn}(x_1^*, z_0 - x_1) < 0$ and $\text{sgn}(x_1^*, y_0 - x_1) > 0$.
(ii) $\text{sgn}(x_1^*, z_0 - x_1) > 0$ and $\text{sgn}(x_1^*, y_0 - x_1) > 0$.
(iii) $\text{sgn}(x_1^*, z_0 - x_1) > 0$ and $\text{sgn}(x_1^*, y_0 - x_1) < 0$.
(iv) $\text{sgn}(x_1^*, z_0 - x_1) < 0$ and $\text{sgn}(x_1^*, y_0 - x_1) < 0$.

We investigate case (i) in detail. The other cases can be studied similarly. First of all note that

$$z_0 - x_1 = z_0 - y_0 + y_0 - x_1 = \frac{y_0 - x_0}{\alpha} + y_0 - x_1 = \left(1 + \frac{1}{\lambda}\right)(y_0 - x_1) + \frac{1}{\alpha}(x_1 - x_0). \quad (2.17)$$

Now, we have

$$r(y_0) - r(x_1) < \|y_0 - z_0\|^p + \epsilon - r(x_1)$$

$$= \frac{1}{(1 + \lambda)^p} \|x_0 - z_0\|^p + \epsilon - r(x_1) \quad \text{(by (2.13))}$$

$$\leq \frac{1}{(1 + \lambda)^p} \left(\|x_0 - x_1\|^p + \|x_1 - z_0\|^p\right) + \epsilon - r(x_1)$$

$$= \left(1 - \frac{1}{(1 + \lambda)^p}\right)(-r(x_1)) + \left(\frac{1}{(1 + \lambda)^p} + 1\right)\epsilon \quad \text{(by the definition of $r(\cdot)$)}$$

$$\leq \left(1 - \frac{1}{(1 + \lambda)^p}\right) \left[\text{sgn}(x_1^*, z_0 - x_1) \langle x_1^*, z_0 - x_1 \rangle \|z_0 - x_1\|^p - \frac{1}{k}\right] + \left(\frac{1}{(1 + \lambda)^p} + 1\right)\epsilon$$

$$\leq \left(1 - \frac{1}{(1 + \lambda)^p}\right) \left[-\|x_1^*, z_0 - x_1\|^p - \frac{1}{k}\right] + \left(\frac{1}{(1 + \lambda)^p} + 1\right)\epsilon$$
Theorem 2.3. Let $M$ be a weakly sequentially compact subset in a $p$-Banach space $X$. Then the set
\[ \{ x \in X : \| x - z \|^p = r(x) \text{ for some } z \in M \} \]
contains a dense $G_\delta$-set in $X$. In particular, the set of
farthest points of $S$ is nonempty.

Proof. Let $F$ and $F_n$ be defined as in Proposition 2.2 and let $D(M) = X \setminus F$. Then
\[ D(M) = X \setminus \bigcup_{n \in \mathbb{N}} F_n = \bigcap_{n \in \mathbb{N}} (X \setminus F_n), \]
where each $X \setminus F_n$ is an open dense subset of $X$. Hence $D(M)$ is a dense $G_\delta$-set in $X$. For each
$x \in D(M)$ and $x^* \in \partial r(x)$, we have
\[ \inf_{z \in M} \langle x^*, z - x \rangle \| x^*, z - x \|^p = -r(x). \]
By the weak compactness of $M$, there exists a point $z \in M$ with $\langle x^*, z_0 - x \rangle \| x^*, z_0 - x \|^p = -r(x)$. Hence
\[ r(x) \geq \| x - z_0 \|^p \geq \text{sgn}(x^*, x - z_0) \| x^*, x - z_0 \|^p = -\text{sgn}(x^*, z_0 - x) \| x^*, z_0 - x \|^p = r(x). \]
This shows that $D(M) \subseteq \{ x : \| x - z \|^p = r(x) \text{ for some } z \in M \}$. 

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References


