Using the method of Jakimovski and Leviatan from their work in 1969, we construct a general class of linear positive operators. We study the convergence, the evaluation for the rate of convergence in terms of the first modulus of smoothness and we give a Voronovskaja-type theorem for these operators.

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1. Introduction

The aim of this paper is to construct a class of linear operators in more general conditions. The method was inspired by Jakimovski and Leviatan (see [1]). We do not study the convergence of these operators with the well-known theorem of Bohman-Korovkin. The evaluation theorems for the rate of convergence are different from the well-known theorem of Shisha-Mond. We prove the Voronovskaja-type theorem for these operators. In the end, we give particularizations of these operators.

We recall some notions and results which we will use in this paper.

Let \( \mathbb{N} \) be the set of positive integer numbers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). For a given interval \( I \), we will use the following function sets: \( B(I) = \{ f \mid f : I \to \mathbb{R}, f \text{ bounded on } I \} \), \( C(I) = \{ f \mid f : I \to \mathbb{R}, f \text{ continuous on } I \} \), and \( C_B(I) = B(I) \cap C(I) \).

For any \( x \in I \), consider the functions \( \psi_x : I \to \mathbb{R} \) defined by \( \psi_x(t) = t - x \) and \( e_i : I \to \mathbb{R}, e_i(t) = t^i \) for any \( t \in I, i \in \{0, 1, 2, 3, 4\} \).

For \( f \in C_B(I) \), by the first-order modulus of smoothness of \( f \) is meant the function \( \omega(f; \cdot) : [0, \infty) \to \mathbb{R} \) defined for any \( \delta \geq 0 \) by

\[
\omega(f; \delta) = \sup \{ | f(x') - f(x'') | : x', x'' \in I, |x' - x''| \leq \delta \}. 
\]
In the following, we take into account the properties of the first-order modulus of smoothness and the properties of the linear positive functional.

**Lemma 1.1.** Let \( f \in C_B(I) \). Then, \( \omega(f; \cdot) \) has the following properties:

(a) \( \omega(f; 0) = 0 \),
(b) \( \omega(f; \cdot) \) is an increasing function,
(c) \( \omega(f; \cdot) \) is a uniform continuous function on \( I \),
(d) for any \( \delta > 0, x, t \in I \), one has
\[
|f(t) - f(x)| \leq [1 + \delta^{-2}\psi^2_x(t)]\omega(f; \delta).
\]

**Lemma 1.2.** Let \( A : E(I) \to \mathbb{R} \) be a linear positive functional. Then,

(a) for \( f, g \in E(I) \) with \( f(x) \leq g(x) \) for any \( x \in I \), one has
\[
A(f) \leq A(g); \quad \text{(1.2)}
\]
(b) \( |A(f)| \leq A(|f|) \) for any \( f \in E(I) \), where \( E(I) \) is a subset of the set of real functions defined on \( I \).

In [2] we have demonstrated the following theorem.

**Theorem 1.3.** Let \( I \) be an interval \( x \in I \), and let the function \( f : I \to \mathbb{R} \) be \( s \) times differentiable in \( x \). According to the Taylor Expansion Theorem, one has

\[
f(t) = \sum_{i=0}^{s} \frac{(t-x)^i}{i!} f^{(i)}(x) + (t-x)^s \mu(t-x), \quad \text{(1.3)}
\]

where \( \mu \) is a bounded function and \( \lim_{t \to x^+} \mu(t-x) = 0 \). If \( f^{(s)} \) is a continuous function on \( I \), then for any \( \delta > 0 \) and \( x \in I \) one has

\[
|\mu(t-x)| \leq \frac{1}{s!} [1 + \delta^{-2}\psi^2_x(t)]\omega(f^{(s)}; \delta). \quad \text{(1.4)}
\]

**2. Preliminaries**

In this section, we construct a general class of linear and positive operators and we demonstrate for these operators an approximation theorem and a Voronovskaja-type theorem.

Let \( I, J \) be intervals and \( I \cap J \) is a nonempty interval. For any \( m \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \), consider the function \( \varphi_{m,k} : J \to \mathbb{R} \) with the property \( \varphi_{m,k}(x) \geq 0 \) for any \( x \in J \) and the linear and positive functional \( A_{m,k} : E(I) \to \mathbb{R} \).
In the following, let $E(I)$ and $F(J)$ be subsets of the set of real functions defined on $I, J$ respectively, such that the series $\sum_{k=0}^{\infty} \varphi_{m,k}(x)A_{m,k}(f)$ is convergent for any $f \in E(I)$ and any $x \in J$. We suppose that $\psi_x^i \in E(I)$ for any $x \in I \cap J$ and any $i \in \{0, 1, \ldots, s+2\}$.

In what follows $s \in \mathbb{N}_0$, $s$ is even.

**Definition 2.1.** For $m \in \mathbb{N}$, define the operator $L_m : E(I) \to F(J)$ by

$$(L_m f)(x) = \sum_{k=0}^{\infty} \varphi_{m,k}(x)A_{m,k}(f)$$

for any $f \in E(I)$ and $x \in J$.

**Proposition 2.2.** The operators $(L_m)_{m \geq 1}$ are linear and positive on $E(I \cap J)$.

**Proof.** The proof follows immediately. \hfill \Box

**Definition 2.3.** For $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$, define $T_i$ by

$$(T_iL_m)(x) = m! (L_m \psi_x^i)(x) = m! \sum_{k=0}^{\infty} \varphi_{m,k}(x)A_{m,k}(\psi_x^i)$$

for any $x \in I \cap J$.

**Theorem 2.4.** If $f \in E(I)$ is an $s$-times differentiable function in $x \in I \cap J$, with $f^{(s)}$ continuous in $x$, and if there exist $\alpha_s, \alpha_{s+2} \in [0, \infty)$ and $m(s) \in \mathbb{N}$ such that

$$\alpha_{s+2} < \alpha_s + 2$$

and $(T_sL_m)(x)/m^{\alpha_s}$, $(T_{s+2}L_m)(x)/m^{\alpha_{s+2}}$ are bounded for any $m \in \mathbb{N}$, $m \geq (s)$, then

$$\lim_{m \to \infty} m^{s-\alpha} \left| (L_m f)(x) - \sum_{i=0}^{s} \frac{1}{i!m^i} (T_iL_m)(x) f^{(i)}(x) \right| = 0.$$  

Assume that $f$ is an $s$ times differentiable function on $I$ with $f^{(s)}$ continuous on $I$ and an interval $K \subset I \cap J$ exists such that there exist $m(s) \in \mathbb{N}$ and the constants $k_j(K) \in \mathbb{R}$ depending on $K$, so that for any $m \in \mathbb{N}$, $m \geq m(s)$ and $x \in K$, one has

$$\frac{(T_jL_m)(x)}{m^{\alpha_j}} \leq k_j(K),$$

where $j \in \{s, s+2\}$. Then, the convergence given in (2.4) is uniform on $K$ and

$$m^{s-\alpha} \left| (L_m f)(x) - \sum_{i=0}^{s} \frac{1}{i!m^i} (T_iL_m)(x) f^{(i)}(x) \right| \leq \frac{1}{s!} (k_s(K) + k_{s+2}(K)) \omega \left( f^{(s)}; \frac{1}{\sqrt{m^{\alpha_s-\alpha_{s+2}}}} \right)$$

for any $x \in K$ and $m \geq m(s)$. 

Proof. According to Taylor’s Theorem, we have
\[
f(t) = \sum_{i=0}^s \frac{(t-x)^i}{i!} f^{(i)}(x) + (t-x)^s \mu(t-x),
\] (2.7)
where \( \mu \) is a bounded function and \( \lim_{t \to x} \mu(t-x) = 0. \)
Hence, from (2.7), we have
\[
A_{m,k}(f) = \sum_{i=0}^s \frac{f^{(i)}(x)}{i!} A_{m,k}(\psi_x^i) + A_{m,k}(\psi_x^s \mu_x),
\] (2.8)
where \( \mu_x : I \to \mathbb{R}, \mu_x(t) = \mu(t-x), \) for any \( t \in I \cap J. \)
Multiplying by \( \varphi_{m,k}(x) \) and summing over \( k \in \mathbb{N}_0, \) we obtain
\[
(L_m f)(x) = \sum_{i=0}^s \frac{f^{(i)}(x)}{i!}(L_m \psi_x^i)(x) + \sum_{k=0}^\infty \varphi_{m,k}(x) A_{m,k}(\psi_x^s \mu_x).
\] (2.9)
Thus,
\[
ms^{s-a_s} \left[ (L_m f)(x) - \sum_{i=0}^s \frac{f^{(i)}(x)}{i!m^i} (T_i L_m)(x) \right] = (R_m f)(x),
\] (2.10)
where
\[
(R_m f)(x) = ms^{s-a_s} \sum_{k=0}^\infty \varphi_{m,k}(x) A_{m,k}(\psi_x^s \mu_x).
\] (2.11)
Then,
\[
| (R_m f)(x) | \leq ms^{s-a_s} \sum_{k=0}^\infty \varphi_{m,k}(x) | A_{m,k}(\psi_x^s \mu_x) |
\] (2.12)
and taking Lemma 1.2 into account, we obtain
\[
| (R_m f)(x) | \leq ms^{s-a_s} \sum_{k=0}^\infty \varphi_{m,k}(x) | A_{m,k}(\psi_x^s \mu_x) |.
\] (2.13)
According to the relation (1.4), for any \( \delta > 0 \) and \( t \in I \cap J, \) we have
\[
| \mu_x(t) | = | \mu(t-x) | \leq \frac{1}{s!} \left[ 1 + \delta^{-2} \psi_x^s(t) \right] \omega(f^{(s)}; \delta),
\] (2.14)
and so
\[
(\psi_x^s | \mu_x |)(t) \leq \frac{1}{s!} \left[ \psi_x^s(t) + \delta^{-2} \psi_x^{s+2}(t) \right] \omega(f^{(s)}; \delta).
\] (2.15)
From (2.13) and (2.15), it results that
\[
\left| (R_m f)(x) \right| \leq \frac{1}{s!} m^{a_0-a_0} \left[ \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^s) + \delta^{-2} \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^{s+2}) \right] \omega(f^{(s)}; \delta).
\]
(2.16)

Thus,
\[
\left| (R_m f)(x) \right| \leq \frac{1}{s!} \left[ \left( \frac{T_s L_m}{m^{a_0}} \right)(x) + \delta^{-2} \frac{T_{s+2} L_m}{m^{a_{s+2}}}(x) \right] \omega(f^{(s)}; \delta).
\]
(2.17)

Considering \( \delta = 1/\sqrt{m^{2+a_0-a_{s+2}}} \), the inequality above becomes
\[
\left| (R_m f)(x) \right| \leq \frac{1}{s!} \left[ \left( \frac{T_s L_m}{m^{a_0}} \right)(x) + \frac{T_{s+2} L_m}{m^{a_{s+2}}}(x) \right] \omega(f^{(s)}; \frac{1}{\sqrt{m^{2+a_0-a_{s+2}}}}).
\]
(2.18)

Taking into account that \( (T_s L_m)/m^{a_0} \) and \( (T_{s+2} L_m)/m^{a_{s+2}} \) are bounded for any \( m \in \mathbb{N}, m \geq m(s) \), and considering the fact that
\[
\lim_{m \to \infty} \omega(f^{(s)}; \frac{1}{\sqrt{m^{2+a_0-a_{s+2}}}}) = \omega(f^{(s)};0) = 0,
\]
we have that
\[
\lim_{m \to \infty} (R_m f)(x) = 0.
\]
(2.20)

From (2.10) and (2.20), (2.4) follows.

If in addition (2.5) takes place then, (2.18) becomes
\[
\left| (R_m f)(x) \right| \leq \frac{1}{s!} \left( k_s(K) + k_{s+2}(K) \right) \omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+a_0-a_{s+2}}}}\right),
\]
(2.21)

for \( m \geq m(s) \) and \( x \in K \). Therefore, the convergence from (2.4) is uniform on \( K \). Now, (2.10) and (2.21) yield (2.6).

In the following, we suppose that for any \( k \in \mathbb{N}_0 \) and \( m \in \mathbb{N} \), we have
\[
A_{m,k}(e_0) = 1,
\]
(2.22)

and for any \( x \in I \cap J \) and \( m \in \mathbb{N} \)
\[
\sum_{k=0}^{\infty} \varphi_{m,k}(x) = 1.
\]
(2.23)

**Remark 2.5.** Taking (2.22) and (2.23) into account, it results that
\[
(T_0 L_m)(x) = 1
\]
(2.24)

for any \( x \in I \cap J \) and \( m \in \mathbb{N} \).
Remark 2.6. In Theorem 2.4, we choose the smallest $\alpha_s$ and $\alpha_{s+2}$, if they exist.

Remark 2.7. Taking (2.24) into account, we choose $\alpha_0 = 0$.

Remark 2.8. For $s = 0, s = 2$, respectively, we state two corollaries which we will use in the section Main results.

Corollary 2.9. If $f \in E(I)$ is a continuous function in $x \in I \cap J$, and if there exist $\alpha_2$ and $m(0) \in \mathbb{N}$ such that

\[ 0 \leq \alpha_2 < 2 \]  

and $(T_2 L_m)(x)/m^{\alpha_2}$ is bounded for any $m \in \mathbb{N}, m \geq m(0)$, then

\[ \lim_{m \to \infty} (L_m f)(x) = f(x). \]  

Assume that $f$ is continuous on $I$ and an interval $K \subset I \cap J$ exists, such that there exist $m(0) \in \mathbb{N}$ and $k_2(K)$ so that for any $m \in \mathbb{N}, m \geq m(0)$, and $x \in K$, one has

\[ \frac{(T_2 L_m)(x)}{m^{\alpha_2}} \leq k_2(K). \]  

Then, the convergence given in (2.26) is uniform on $K$ and

\[ |(L_m f)(x) - f(x)| \leq (1 + k_2(K))\omega \left( f; \frac{1}{\sqrt{m^{2-\alpha_2}}} \right) \]  

for any $x \in K$ and $m \geq m(0)$.

Corollary 2.10. If $f \in E(I)$ is a two-times differentiable function in $x \in I \cap J$, with $f^{(2)}$ continuous in $x$, and if there exist $\alpha_2, \alpha_4$ and $m(2) \in \mathbb{N}$ such that

\[ 0 \leq \alpha_2 < 2, \]

\[ 0 \leq \alpha_4 < \alpha_2 + 2, \]  

$(T_2 L_m)(x)/m^{\alpha_2}$ and $((T_4 L_m)(x))/m^{\alpha_4}$ are bounded for any $m \in \mathbb{N}, m \geq m(2)$, then

\[ \lim_{m \to \infty} \frac{m^{2-\alpha_2}}{m^{2-\alpha_4}} \left[ (L_m f)(x) - f(x) - \frac{1}{m} (T_1 L_m)(x) f^{(1)}(x) - \frac{1}{2m^2} (T_2 L_m)(x) f^{(2)}(x) \right] = 0. \]  

Assume that $f$ is a two-times differentiable function on $I$ with $f^{(2)}$ continuous on $I$ and an interval $K \subset I \cap J$ exists, such that there exist $m(2) \in \mathbb{N}$ and $k_j(K)$, so that for any $m \geq m(2)$ and $x \in K$, one has

\[ \frac{(T_j L_m)(x)}{m^{\alpha_j}} \leq k_j(K), \]  

where $j \in \{2, 4\}$. Then, the convergence given in (2.30) is uniform on $K$.

Remark 2.11. Theorem 2.4, Corollary 2.9, and 2.10 are Voronovskaja-type theorems.
3. Main results

In this section, we construct a general class of linear positive operators. Let \( \mathbb{R}_0 = [0, \infty) \) and \( J \) be an interval with \( J \subset \mathbb{R}_0 \). Let the sequence \( (a_m)_{m \geq 1} \) so that \( a_m x \in J \) for any \( m \in \mathbb{N} \) and \( x \in J \). The indefinitely differentiable functions \( a, b : J \to \mathbb{R} \) have the property:

\[
b(x) > 0 \quad (3.1)
\]

for any \( x \in \mathbb{R}_0 \),

\[
a(1) \neq 0 \quad (3.2)
\]

and for any compact \( K \subset J \) the constants \( M_1(K), M_2(K) \) depending on \( K \) exist, such that

\[
|a^{(k)}(x)| \leq M_1(K),
\]

\[
|b^{(k)}(x)| \leq M_2(K)
\]

for any \( x \in K \) and \( k \in \mathbb{N}_0 \).

Then, it is known that

\[
a(x) = \sum_{n=0}^{\infty} \frac{1}{n!} a^{(n)}(0) x^n,
\]

\[
b(x) = \sum_{p=0}^{\infty} \frac{1}{p!} b^{(p)}(0) x^p
\]

for any \( x \in J \).

If \( u, x, ux \in J \), we calculate

\[
a(u)b(ux) = \left( \sum_{n=0}^{\infty} \frac{1}{n!} a^{(n)}(0) u^n \right) \left( \sum_{p=0}^{\infty} \frac{1}{p!} b^{(p)}(0)(ux)^p \right)
\]

and we take it to the form

\[
a(u)b(ux) = \sum_{k=0}^{\infty} p_k(x) u^k, \quad (3.6)
\]

where

\[
p_k(x) = \sum_{i=0}^{k} \frac{1}{i!(k-i)!} a^{(i)}(0) b^{(k-i)}(0) x^{k-i}. \quad (3.7)
\]
Remark 3.1. If $u = 1$, then from (3.6), we obtain

$$a(1)b(a_mx) = \sum_{k=0}^{\infty} p_k(a_mx)$$

(3.8)

for any $m \in \mathbb{N}$ and $x \in J$.

Remark 3.2. We consider that the conditions $a^{(i)}(0)b^{(k-i)}(0)/a(1) \geq 0$, $i \in \{0, 1, \ldots, k\}$ and $k \in \mathbb{N}_0$, hold and then it results that $a(1)p_k(x) \geq 0$ for any $x \in J$ and any $k \in \mathbb{N}_0$.

In the following, let a fixed function $w : \mathbb{R}_0 \to (0, \infty)$, called the weight function, and the set functions

$$E(w) = \{ f \mid f : \mathbb{R}_0 \to \mathbb{R} \text{ such that } wf \text{ is bounded on } [0, \infty) \}. \quad (3.9)$$

For $f \in E(w)$, there exists a positive constant $M$ such that $w(x)|f(x)| \leq M$ for any $x \in \mathbb{R}_0$. For $m \in \mathbb{N}$ and $x \in J$, and taking in the end (3.8) into account, we have

$$\left| \frac{1}{a(1)b(a_mx)} \sum_{k=0}^{\infty} p_k(a_mx)f\left( \frac{k}{m} \right) \right| \leq \frac{1}{a(1)b(a_mx)} \sum_{k=0}^{\infty} p_k(a_mx) \left| f\left( \frac{k}{m} \right) \right|$$

$$\leq \frac{M}{w(x)} \frac{1}{a(1)b(a_mx)} \sum_{k=0}^{\infty} p_k(a_mx) = \frac{M}{w(x)},$$

(3.10)

from where it results that the series $(1/a(1)b(a_mx))\sum_{k=0}^{\infty} p_k(a_mx)f(k/m)$ is convergent.

Definition 3.3. For $m \in \mathbb{N}$, define the operator $L_m : E(w) \to F(J)$ by

$$(L_m f)(x) = \frac{1}{a(1)b(a_mx)} \sum_{k=0}^{\infty} p_k(a_mx)f\left( \frac{k}{m} \right)$$

(3.11)

for any $f \in E(w)$ and $x \in J$, where $F(J)$ is a subset of the set of real functions defined on $J$.

Remark 3.4. The operators $(L_m)_{m \geq 1}$ are linear and positive on $E(w)$. 
In the following, we consider that for any \( x \in J \), we have \( \psi_x^i \in E(w) \), \( i \in \{1,2,3,4\} \).

**Definition 3.5.** For \( m \in \mathbb{N} \) and \( i \in \{1,2,3,4\} \), define \( T_i \) by

\[
(T_i L_m)(x) = m^i (L_m \psi_x^i)(x) = m^i \frac{1}{a(1) b(a_m x)} \sum_{k=0}^{\infty} p_k(a_m x) \left( \frac{k}{m} - x \right)^i
\]  

(3.12)

for any \( x \in J \).

**Lemma 3.6.** One has

\[
(L_m e_0)(x) = 1,
\]

(3.13)

\[
(L_m e_1)(x) = \frac{a_m b^{(1)}(a_m x)}{b(a_m x)} x + \frac{1}{m} \frac{a^{(1)}(1)}{a(1)},
\]

\[
(L_m e_2)(x) = \left( \frac{a_m}{m} \right)^2 \frac{b^{(2)}(a_m x)}{b(a_m x)} x^2 + \frac{1}{m} \frac{a_m a(1) + 2a^{(1)}(1) b^{(1)}(a_m x)}{a(1) b(a_m x)} x + \frac{1}{m^2} \frac{a^{(1)}(1) + a^{(2)}(1)}{a(1)},
\]

(3.14)

for any \( x \in J \) and \( m \in \mathbb{N} \).

**Proof.** The relation (3.13) results from (3.8). The proof of relations (3.14) follows immediately by differentiating (3.6) with respect to \( u \), and after that take 1 for \( u \) and \( a_m x \) for \( x \). \( \square \)
Lemma 3.7. For \( x \in J \) and \( m \in \mathbb{N} \), the following hold

\[
(T_0 L_m)(x) = 1,
\]

\[
(T_1 L_m)(x) = -m \left( 1 - \frac{a_m}{m} \frac{b^{(1)}(a_m x)}{b(a_m x)} \right) x + \frac{a^{(1)}(1)}{a(1)},
\]

\[
(T_2 L_m)(x) = -m^2 \left[ 1 - \left( \frac{a_m}{m} \right)^2 \frac{b^{(2)}(a_m x)}{b(a_m x)} \right] x^2
\]

\[+ m^2 \left( 1 - \frac{a_m}{m} \frac{b^{(1)}(a_m x)}{b(a_m x)} \right) \left( 2x^2 - \frac{1}{m} \frac{a(1) + 2a^{(2)}(1)}{a(1)} x \right)
\]

\[+ mx + \frac{a^{(1)}(1) + a^{(2)}(1)}{a(1)},
\]

\[
(T_4 L_m)(x) = -m^4 \left[ 1 - \left( \frac{a_m}{m} \right)^4 \frac{b^{(4)}(a_m x)}{b(a_m x)} \right] x^4
\]

\[+ m^4 \left[ 1 - \left( \frac{a_m}{m} \right)^2 \frac{b^{(2)}(a_m x)}{b(a_m x)} \right] \left( 4x^4 - \frac{1}{m} \frac{6a(1) + 4a^{(1)}(1)}{a(1)} x^3 \right)
\]

\[+ m^4 \left[ 1 - \left( \frac{a_m}{m} \right)^2 \frac{b^{(2)}(a_m x)}{b(a_m x)} \right] \left( -6x^4 + 4 \frac{1}{m} \frac{3a(1) + 3a^{(1)}(1)}{a(1)} x^3 \right.
\]

\[\left. - \frac{1}{m^2} \frac{7a(1) + 18a^{(1)}(1) + 6a^{(2)}(1)}{a(1)} x^2 \right]
\]

\[+ m^4 \left( 1 - \frac{a_m}{m} \frac{b^{(1)}(a_m x)}{b(a_m x)} \right) \left( 4x^4 - \frac{6}{m} \frac{a(1) + 2a^{(1)}(1)}{a(1)} x^3 \right.
\]

\[+ 4 \frac{1}{m^2} \frac{a(1) + 6a^{(1)}(1) + 3a^{(2)}(1)}{a(1)} x^2
\]

\[\left. - \frac{1}{m^3} \frac{a(1) + 14a^{(1)}(1) + 18a^{(2)}(1) + 4a^{(3)}(1)}{a(1)} x \right]
\]

\[+ 3m^2 x^2 + \frac{a(1) + 10a^{(1)}(1) + 6a^{(2)}(1)}{a(1)} mx
\]

\[+ \frac{a^{(1)}(1) + 7a^{(2)}(1) + 6a^{(3)}(1) + a^{(4)}(1)}{a(1)}.
\]

(3.15)

(3.16)

(3.17)

Proof. The proof follows immediately from (3.12) and Lemma 3.6.

□

Theorem 3.8. Let \( f : \mathbb{R}_0 \to \mathbb{R} \) be a function, \( f \in E(w) \). If \( x \in \mathbb{R}_0 \), \( f \) is continuous in \( x \), \( \alpha_2 \) and \( m(0) \in \mathbb{N} \) exist such that

\[ 1 \leq \alpha_2 < 2 \]

(3.18)

and \( m^{2-\alpha_2} |1 - (a_m/m)^i (b^{(i)}(a_m x)/b(a_m x))| \) is bounded for any \( m \in \mathbb{N} \), \( m \geq m(0) \), where \( i \in \{1, 2\} \), then

\[ \lim_{m \to \infty} (I_m f)(x) = f(x). \]

(3.19)
Assume that $f$ is continuous on $\mathbb{R}_0$ and a compact interval $K \subset \mathbb{R}_0$ exists, such that there exist $m(0) \in \mathbb{N}$ and $l_i(K)$ so that for any $m \in \mathbb{N}$, $m \geq m(0)$, and $x \in K$, one has

$$m^{2-a_2} \left| 1 - \left( \frac{a_m}{m} \right)^i \frac{b^{(i)}(a_m x)}{b(a_m x)} \right| \leq l_i(K),$$

(3.20)

where $i \in \{1, 2\}$.

Then, the convergence given in (3.19) is uniform in $K$ and

$$\left| (L_m f)(x) - f(x) \right| \leq M(K)\omega \left( f; \frac{1}{\sqrt{m^{2-a_2}}} \right)$$

(3.21)

for any $x \in K$ and any $m \geq m(0)$, where $M(K)$ is a constant depending on $K$.

Proof. Because $m^{2-a_2} \left| 1 - (a_m/m)^i (b^{(i)}(a_m x)/b(a_m x)) \right|$ is bounded for any $m \in \mathbb{N}$, $m \geq m(0)$, it results that $(T_2 L_m)(x)/m^{a_2}$ is bounded for any $m \in \mathbb{N}$, $m \geq m(0)$. Taking relation (3.16) into account, we apply now the Corollary 2.9. The proof is similar on a compact interval $K$. □

**Theorem 3.9.** Let $f : \mathbb{R}_0 \to \mathbb{R}$ be a function, $f \in E(w)$. If $x \in \mathbb{R}_0$, $f$ is a two times differentiable function in $x$ with $f^{(2)}$ continuous in $x$, $\alpha_2, \alpha_4$ and $m(2) \in \mathbb{N}$ exist such that

$$1 \leq \alpha_2 < 2,$$

$$2 \leq \alpha_4 < \alpha_2 + 2,$$

(3.22)

(3.23)

$m^{2-a_4} \left| 1 - (a_m/m)^i (b^{(i)}(a_m x)/b(a_m x)) \right|$ is bounded for any $m \in \mathbb{N}$, $m \geq m(2)$, where $i \in \{1, 2, 3, 4\}$, then

$$\lim_{m \to \infty} m^{2-a_4} \left[ (L_m f)(x) - f(x) - \frac{1}{m} (T_1 L_m)(x) f^{(1)}(x) - \frac{1}{2m^2} (T_2 L_m)(x) f^{(2)}(x) \right] = 0.$$  

(3.24)

In addition, if the limit $\lim_{m \to \infty} ((T_2 L_m)(x)/m^{a_2})$ exists and

$$\lim_{m \to \infty} \frac{(T_2 L_m)(x)}{m^{a_2}} = B_2(x) \in \mathbb{R},$$

(3.25)

then

$$\lim_{m \to \infty} m^{2-a_2} \left[ (L_m f)(x) - f(x) - \frac{1}{m} (T_1 L_m)(x) f^{(1)}(x) \right] = \frac{1}{2} B_2(x) f^{(2)}(x).$$

(3.26)

Assume that $f$ is a two times differentiable function on $\mathbb{R}_0$ with $f^{(2)}$ continuous on $\mathbb{R}_0$ and a compact interval $K \subset \mathbb{R}_0$ exists, such that there exist $m(2) \in \mathbb{N}$ and $l_i(K)$ so that for any $m \geq m(2)$ and $x \in K$, one has

$$m^{4-a_4} \left| 1 - \left( \frac{a_m}{m} \right)^i \frac{b^{(i)}(a_m x)}{b(a_m x)} \right| \leq l_i(K),$$

(3.27)

where $i \in \{1, 2, 3, 4\}$. Then, the convergence given in (3.24) is uniform on $K$. 


Proof. From (3.23), it results that $4 - \alpha_4 > 2 - \alpha_2$, and then we have that $m^{2-\alpha_4}\left(1 - \frac{(a_m/m)^i}{(b_0^i(a_{0m})/b(a_{0m}))}\right)$ is bounded for any $m \geq m(2)$. So $(T_{2L_m}(x)/m^{\alpha_2})$ is bounded for any $m \geq m(2)$. Using the same idea from the proof of Theorem 3.8, we have that $(T_{2L_m}(x)/m^{\alpha_2})$ and $(T_{4L_m}(x)/m^{\alpha_4})$ are bounded for any $m \in \mathbb{N}, m \geq m(2)$, and then we apply Corollary 2.10.

Now, we give some applications where $a_m = m$ for any $m \in \mathbb{N}$. In the following, by particularization and applying Theorems 3.8 and 3.9, we can obtain approximation theorems and Voronovskaja-type theorems for some known operators. Because every application is a simple substitute in the theorems of this section, we will not replace anything.

Application 3.10. If $a(x) = 1$ and $b(x) = e^x$, $x \in \mathbb{R}_0$, we obtain the Mirakjan-Favard-Szász operators (see [3–5]).

Application 3.11. If $a(x) = g(x) = \sum_{n=0}^{\infty} a_n x^n$ and $b(x) = e^x$, $x \in \mathbb{R}_0$, we obtain the operators considered by Jakimovski and Leviatan in the paper [1].

Application 3.12. If $a(x) = g(x) = 1$ and $b(x) = \cosh x = \sum_{k=0}^{\infty} (1/(2k)!)(x)^{2k}$, $x \in \mathbb{R}_0$, then we get the operators considered by Leśniewicz and Rempulska in the paper [6].

Application 3.13. If $a(x) = g(x) = 1$ and $b(x) = \sinh x = \sum_{k=0}^{\infty} (1/(2k+1)!(x)^{2k+1}$, $x \in \mathbb{R}_0$, we get the operators

$$
(A_m f)(x) = \begin{cases} 
1/(\sinh mx) \sum_{k=0}^{\infty} (mx)^{2k+1}/(2k+1)! f(2k+1)/m & \text{if } x > 0, \\
(0) & \text{if } x = 0,
\end{cases}
$$

where $m \in \mathbb{N}$ and $x \in \mathbb{R}_0$. The operators of this type are introduced and studied by Rempulska and Skorupka in the paper [7].

Application 3.14. If $a(x) = b(x) = g(x) = \cosh x$, $x \in \mathbb{R}_0$, we obtain the operators studied by Ciupa in [8].

Application 3.15. If $a(x) = g(x) = \sum_{n=0}^{\infty} a_n x^n$ and $b(x) = \cosh x$, $x \in \mathbb{R}_0$, we get the operators constructed by Ciupa in the paper [9], and studied in [9, 10].

Application 3.16. If $a(x) = 1$ and $b(x) = b_m((1/m)x)$, $x \in \mathbb{R}_0$ and $m \in \mathbb{N}$, we obtain the operators studied in the paper [11].

References


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