Let $\mathcal{F}^*(p, \alpha)$ be the class of functions $f(z)$ which are analytic and $p$-valently starlike of order $\alpha$ in the open unit disk $E$. The object of the present paper is to derive an interesting condition for $f(z)$ to be in the class $\mathcal{F}^*(p, \alpha)$.

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1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots \})$$

which are analytic in the open unit disk $E = \{ z : z \in \mathbb{C}, |z| < 1 \}$. A function $f(z) \in \mathcal{A}(p)$ is said to be $p$-valently starlike of order $\alpha (0 \leq \alpha < p)$ in $E$ if and only if

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in E).$$

(1.2)

We denote by $\mathcal{F}^*(p, \alpha)$ the subclass of $\mathcal{A}(p)$ consisting of functions which are $p$-valently starlike of order $\alpha$ in $E$. We only call a function $f(z) \in \mathcal{F}^*(p, 0)$ to be $p$-valently starlike in $E$. Further, a function $f(z) \in \mathcal{A}(p)$ is said to be $p$-valently convex of order $\alpha (0 \leq \alpha < p)$ in $E$ if and only if

$$1 + \Re \left( \frac{zf'''(z)}{f''(z)} \right) > \alpha \quad (z \in E).$$

(1.3)
We denote by \( \mathcal{C}(p, \alpha) \) the subclass of \( \mathcal{A}(p) \) consisting of all \( p \)-valently convex functions of order \( \alpha \) in \( E \). We also call a function \( f(z) \in \mathcal{C}(p, 0) \) \( p \)-valently convex function. From the definition, it is trivial that if \( f(z) \) is a \( p \)-valently convex function, then \( zf'(z) \) is \( p \)-valently starlike in \( E \).

2. Preliminaries

In this paper, we need the following lemmas.

**Lemma 2.1.** If \( M(z) = z^p + \sum_{n=p+1}^\infty a_nz^n \) (1 \( \leq p \) and 1 \( \leq k \)) and \( N(z) = z^p + \sum_{n=p+1}^\infty a_nz^n \) are analytic in \( E \) and \( N(z) \) satisfies \( \text{Re}(N(z)/zN'(z)) > \delta \) (0 \( \leq \delta < 1/p \)), then

\[
\text{Re}\left(\frac{M'(z)}{N'(z)}\right) > \beta \quad \text{implies} \quad \text{Re}\left(\frac{M(z)}{N(z)}\right) > \frac{2\beta + k\delta}{2 + k\delta}.
\] (2.1)

**Remark 2.2.** This lemma holds to be true for \( N(z) \) which is multivalently starlike in \( E \).

We owe the above lemma to Ponnusamy and Karunakaran [1].

**Lemma 2.3.** Let \( f(z) \in \mathcal{F}^*(p, 0) \). Then

\[
\frac{F(z)}{p + 1} = \int_0^z f(t)dt \in S^*(p + 1, 0)
\] (2.2)

or

\[
\text{Re} \left( \frac{zF'(z)}{F(z)} \right) > 0 \quad (z \in E).
\] (2.3)

The proof of this result can be found in [2].

3. Main result

**Theorem 3.1.** If \( f(z) \in \mathcal{A}(p) \) satisfies the following condition:

\[
\text{Re} \left( \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right) > \alpha \quad (z \in E)
\] (3.1)

for \( \alpha(0 \leq \alpha < 1) \), then \( f(z) \in \mathcal{F}^*(p, \alpha + p - 1) \) or

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha + p - 1 \quad (z \in E).
\] (3.2)

**Proof.** From assumption (3.1), \( f^{(p-1)}(z) \) is univalently starlike of order \( \alpha \) and it is trivial that

\[
(zf^{(p-1)}(z) - f^{(p-2)}(z))' = zf^{(p)}(z).
\] (3.3)

Note that \( f^{(p-2)}(z) \) is starlike in \( E \) by Lemma 2.3. Therefore, applying Lemma 2.1 and (3.3), we have

\[
\text{Re} \left( \frac{zf^{(p-1)}(z) - f^{(p-2)}(z)}{f^{(p-2)}(z)} \right) > \alpha \quad (z \in E).
\] (3.4)
From Lemma 2.3, \( f^{(p-2)}(z) \) is 2-valently starlike in \( E \). Now then, it is trivial that
\[
(z f^{(p-2)}(z) - 2 f^{(p-3)}(z))' = z f^{(p-1)}(z) - f^{(p-2)}(z).
\] (3.5)

Then, from Lemma 2.1, 2-valently starlikeness of \( f^{(p-2)}(z) \), and (3.5), we have
\[
\text{Re} \frac{z f^{(p-2)}(z) - 2 f^{(p-3)}(z)}{f^{(p-3)}(z)} > \alpha \quad (z \in E).
\] (3.6)

Further, it is trivial that
\[
(z f^{(p-3)}(z) - 3 f^{(p-4)}(z))' = z f^{(p-2)}(z) - 2 f^{(p-3)}(z),
\] (3.7)
and applying the same method and reason as above, we have
\[
\text{Re} \frac{z f^{(p-3)}(z) - 3 f^{(p-4)}(z)}{f^{(p-4)}(z)} > \alpha \quad (z \in E),
\] (3.8)
where \( f^{(p-3)}(z) \) is 3-valently starlike in \( E \). Applying the mathematical induction, we have
\[
\text{Re} \frac{zf'(z) - (p-1)f(z)}{f(z)} > \alpha \quad (z \in E),
\] (3.9)
where \( f(z) \) is \( p \)-valently starlike in \( E \). This shows that
\[
\text{Re} \frac{zf'(z)}{f(z)} > \alpha + p - 1 \quad (z \in E),
\] (3.10)
or \( f(z) \in S^*(p, \alpha + p - 1) \).

Our main result shows the following.

**Corollary 3.2.** Let \( f(z) \in A(p), 2 \leq p, 0 \leq \alpha < 1, \)
\[
\text{Re} \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > \alpha \quad (z \in E),
\] (3.11)
and put \( f(z) = z^{p-1} f_1(z) \) where
\[
f_1(z) = z + \sum_{n=p+1}^{\infty} a_n z^{n-p+1}.
\] (3.12)

Then \( f_1(z) \) is univalently starlike of order \( \alpha \) in \( E \).

**Proof.** From the definition of \( f_1(z) \) and Theorem 3.1, it follows that
\[
\text{Re} \frac{zf'(z)}{f(z)} = p - 1 + \text{Re} \frac{zf_1'(z)}{f_1(z)} > \alpha + p - 1.
\] (3.13)
This completes the proof. □
Corollary 3.3. Let \( f(z) \in \mathcal{A}(p), \ 2 \leq p, \ 0 \leq \alpha < 1, \ \text{and} \)

\[
1 + \text{Re} \left( \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \right) > \alpha \quad (z \in \mathbb{E}). \tag{3.14}
\]

Then one has

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \beta(\alpha) + p - 1 \quad (z \in \mathbb{E}), \tag{3.15}
\]

where

\[
\beta(\alpha) = \begin{cases} 
\frac{1 - 2\alpha}{2^{1 - 2\alpha}[1 - 2^{2\alpha}]} & \text{if } \alpha \neq \frac{1}{2}, \\
\frac{1}{2 \log 2} & \text{if } \alpha = \frac{1}{2}.
\end{cases} \quad (3.16)
\]

Proof. Putting

\[
g(z) = \frac{f^{(p-1)}(z)}{p!} = z + \sum_{n=2}^{\infty} b_n z^n, \quad (3.17)
\]

then from assumption (3.14), \( g(z) \) is univalently convex of order \( \alpha \), and therefore from Wilken-Feng result [3] and Theorem 3.1, we have

\[
\text{Re} \left( \frac{zg'(z)}{g(z)} \right) = \text{Re} \left( \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right) > \beta(\alpha) \quad (z \in \mathbb{E}), \tag{3.18}
\]

and therefore it follows that

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \beta(\alpha) + p - 1 \quad (z \in \mathbb{E}). \tag{3.19}
\]

We will give here an open problem.

Problem 3.4. Let \( f(z) \in \mathcal{F}^*(p, \alpha) \) and \( 0 \leq \alpha < p \).

Then

\[
\frac{F(z)}{p+1} = \int_0^z f(t) dt \in S^*(p+1, \beta(p, \alpha)). \tag{3.20}
\]

What is the best \( \beta(p, \alpha) \)?
References


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