Let $X$ be a Banach space and let $L^\Phi(I,X)$ denote the space of Orlicz $X$-valued integrable functions on the unit interval $I$ equipped with the Luxemburg norm. In this paper, we present a distance formula $\text{dist}_\Phi(f_1, f_2, L^\Phi(I,G))$, where $G$ is a closed subspace of $X$, and $f_1, f_2 \in L^\Phi(I,X)$. Moreover, some related results concerning best simultaneous approximation in $L^\Phi(I,X)$ are presented.

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1. Introduction

A function $\Phi : (-\infty, \infty) \to [0, \infty]$ is called an Orlicz function if it satisfies the following conditions:

1. $\Phi$ is even, continuous, convex, and $\Phi(0) = 0$;
2. $\Phi(x) > 0$ for all $x \neq 0$;
3. $\lim_{x \to 0} \Phi(x)/x = 0$ and $\lim_{x \to \infty} \Phi(x)/x = \infty$.

We say that a function $\Phi$ satisfies the $\Delta_2$ condition if there are constants $k > 1$ and $x_0 > 0$ such that $\Phi(2x) \leq k\Phi(x)$ for $x > x_0$. Examples of Orlicz functions that satisfy the $\Delta_2$ conditions are widely available such as $\Phi(x) = |x|^p$, $1 \leq p < \infty$, and $\Phi(x) = (1 + |x|) \log(1 + |x|) - |x|$. In fact, Orlicz functions are considered to be a subclass of Young functions defined in [1].

Let $X$ be a Banach space and let $(I, \mu)$ be a measure space. For an Orlicz function $\Phi$, let $L^\Phi(I,X)$ be the Orlicz-Bochner function space that consists of strongly measurable functions $f : I \to X$ with $\int_I \Phi(\alpha \|f\|) d\mu(t) < \infty$ for some $\alpha > 0$. It is known that $L^\Phi(I,X)$ is a Banach space under the Luxemburg norm.
is shown that if \( \Phi(x) = |x|^p \), \( 1 \leq p < \infty \), the space \( L^p(I,X) \) is simply the \( p \)-Lebesgue Bochner function space with the Luxemburg norm.

\[
\|f\|_\Phi = \inf \left\{ k > 0, \int_I \Phi \left( \frac{1}{k} \|f\| \right) \, d\mu(t) \leq 1 \right\}. \tag{1.1}
\]

It should be remarked that if \( \Phi(x) = (1 + |x|) \log(1 + |x|) - |x| \), then the space \( L^p(I,X) \) is the well-known Zygmund space, \( L \log L^+ \). For excellent monographs on \( L^p(I,X) \), we refer the readers to [1–3].

For a function \( F = (f_1, f_2) \in (L^p(I,X))^2 \), we define \( \|F\| \) by

\[
\|F\| = \|f_1(\cdot)\| + \|f_2(\cdot)\|_\Phi. \tag{1.3}
\]

In this paper, for a given closed subspace \( G \) of \( X \) and \( F = (f_1, f_2) \in (L^p(I,X))^2 \), we show the existence of a pair \( G_0 = (g_0, g_0) \in (L^p(I,G))^2 \) such that

\[
\|F - G_0\| = \inf_{g \in G} \|F - (g, g)\|. \tag{1.4}
\]

If such a function \( g \) exists, it is called a best simultaneous approximation of \( F = (f_1, f_2) \). The problem of best simultaneous approximation can be viewed as a special case of vector-valued approximation. Recent results in this area are due to Pinkus [4], where he considered the problem when a finite-dimensional subspace is a unicity space. Characterization results for linear problems were given in [5] based on the derivation of an expression for the directional derivative, and these results generalize the earlier results presented in [6]. Results on best simultaneous approximation in general Banach spaces may be found in [7, 8]. Related results on \( L^p(I,X) \), \( 1 \leq p < \infty \), are given in [9]. In [9], it is shown that if \( G \) is a reflexive subspace of a Banach space \( X \), then \( L^p(I,G) \) is simultaneously proximinal in \( L^p(I,X) \). If \( L^p(I,X) = L^1(I,X) \), Abu-Sarhan and Khalil [10] proved that if \( G \) is a reflexive subspace of the Banach space \( X \) or \( G \) is a 1-summand subspace of \( X \), then \( L^1(I,G) \) is simultaneously proximinal in \( L^1(I,X) \).

It is the aim of this work to prove a distance formula \( \text{dist}_\Phi(f_1, f_2, L^p(I,G)) \), where \( f_1, f_2 \in L^p(I,X) \), similar to that of best approximation. This will allow us to generalize some recent results on \( L^1(I,X) \) to \( L^p(I,X) \).

Throughout this paper, \( X \) is a Banach space, \( \Phi \) is an Orlicz function, and \( L^p(I,X) \) is the Orlicz-Bochner function space equipped with the Luxemburg norm.

2. Distance formula

Let \( G \) be a closed subspace of \( X \). For \( x, y \in X \), define

\[
\text{dist}(x, y, G) = \inf_{z \in G} \|x - z\| + \|y - z\|. \tag{2.1}
\]
For \( f_1, f_2 \in L^\Phi(I,X) \), we define \( \text{dist}_\Phi(f_1, f_2, L^\Phi(I,G)) \) by
\[
\text{dist}_\Phi(f_1, f_2, L^\Phi(I,G)) = \inf_{g \in L^\Phi(I,G)} ||(f_1, f_2) - (g, g)|| \\
= \inf_{g \in L^\Phi(I,G)} |||f_1(\cdot) - g(\cdot)|| + ||f_2(\cdot) - g(\cdot)|||_\Phi. 
\] (2.2)

Our main result is the following.

**Theorem 2.1.** Let \( G \) be a subspace of the Banach space \( X \) and let \( \Phi \) be an Orlicz function that satisfies the \( \Delta_2 \) condition. If \( f_1, f_2 \in L^\Phi(I,X) \), then the function \( \text{dist}(f_1(\cdot), f_2(\cdot), G) \) belongs to \( L^\Phi(I) \) and
\[
||\text{dist}(f_1(\cdot), f_2(\cdot), G)||_\Phi = \text{dist}_\Phi(f_1, f_2, L^\Phi(I,G)). 
\] (2.3)

**Proof.** Let \( f_1, f_2 \in L^\Phi(I,X) \). Then there exist two sequences \( (f_{n,1}), (f_{n,2}) \) of simple functions in \( L^\Phi(I,X) \) such that
\[
||f_{n,1}(t) - f_1(t)|| \to 0, \quad ||f_{n,2}(t) - f_2(t)|| \to 0, \quad \text{as } n \to \infty 
\] (2.4) for almost all \( t \) in \( I \). The continuity of \( \text{dist}(x, y, G) \) implies that
\[
|\text{dist}(f_{n,1}(t), f_{n,2}(t), G) - \text{dist}(f_1(t), f_2(t), G)| \to 0, \quad \text{as } n \to \infty. 
\] (2.5)

Set \( H_n(t) = \text{dist}(f_{n,1}(t), f_{n,2}(t), G) \). Then each \( H_n \) is a measurable function. Thus \( \text{dist}(f_1(\cdot), f_2(\cdot), G) \) is measurable and
\[
\text{dist}(f_1(t), f_2(t), G) \leq ||f_1(t) - z|| + ||f_2(t) - z|| 
\] (2.6) for all \( z \) in \( G \). Therefore,
\[
\text{dist}(f_1(t), f_2(t), G) \leq ||f_1(t) - g(t)|| + ||f_2(t) - g(t)|| 
\] (2.7) for all \( g \in L^\Phi(I,G) \). Thus
\[
||\text{dist}(f_1(\cdot), f_2(\cdot), G)||_\Phi \leq |||f_1(\cdot) - g(\cdot)|| + ||f_2(\cdot) - g(\cdot)|||_\Phi 
\] (2.8) for all \( g \in L^\Phi(I,G) \). Hence \( \text{dist}(f_1(\cdot), f_2(\cdot), G) \in L^\Phi(I) \) and
\[
||\text{dist}(f_1(\cdot), f_2(\cdot), G)||_\Phi \leq \text{dist}_\Phi(f_1, f_2, L^\Phi(I,G)). 
\] (2.9)

Fix \( \epsilon > 0 \). Since the set of simple functions are dense in \( L^\Phi(I,X) \), there exist simple functions \( f_i^* \) in \( L^\Phi(I,X) \) such that \( ||f_i - f_i^*||_\Phi \leq \epsilon/6 \) for \( i = 1, 2 \). Assume that \( f_i^*(t) = \sum_{k=1}^n x_k^i \chi_{A_k}(t) \) with \( A_k \)'s are measurable sets, \( x_k^i \in X, k = 1, 2, \ldots, n, i = 1, 2, A_k \cap A_j = \phi, k \neq j, \) and \( \bigcup_{k=1}^n A_k = I \). We can assume that \( \mu(A_k) > 0 \) and \( \Phi(1) \leq 1 \). For each \( k = 1, 2, \ldots, n \), let \( y_k \in G \) be such that
\[
||x_k^1 - y_k|| + ||x_k^2 - y_k|| \leq \text{dist}(x_k^1, x_k^2, G) + \frac{\epsilon}{3}. 
\] (2.10)
Then
\[\int_I \Phi \left( \frac{\|f_1^*(t) - g(t)\| + \|f_2^*(t) - g(t)\|}{\|F\|_\Phi} \right) d\mu(t)\]
\[= \sum_{k=1}^{n} \int_{A_k} \Phi \left( \frac{\|x_k^1 - y_k\| + \|x_k^2 - y_k\|}{\|F\|_\Phi} \right) d\mu(t)\]
\[< \sum_{k=1}^{n} \int_{A_k} \Phi \left( \frac{\text{dist}(x_k^1, x_k^2, G) + \epsilon/3}{\|F\|_\Phi} \right) d\mu(t)\]
\[= \int_I \Phi \left( \frac{\|f_1(t) - f_1^*(t)\| + \|f_2(t) - f_2^*(t)\| + \text{dist}(f_1(t), f_2(t), G) + \epsilon/3}{\|F\|_\Phi} \right) d\mu(t)\]
\[= \int_I \Phi \left( \frac{F(t)}{\|F\|_\Phi} \right) d\mu(t) \leq 1.\] (2.12)

Consequently,
\[
\left\|\|f_1^*(\cdot) - g(\cdot)\| + \|f_2^*(\cdot) - g(\cdot)\|\right\|_\Phi \leq \left\|\|f_1(\cdot) - f_1^*(\cdot)\| + \|f_2(\cdot) - f_2^*(\cdot)\|\right\|_\Phi + \text{dist}(f_1(\cdot), f_2(\cdot), G) + \frac{\epsilon}{3}.\] (2.13)

Notice that
\[
\text{dist}_\Phi(f_1, f_2, L^\Phi(I, G)) \leq \text{dist}_\Phi(f_1^*, f_2^*, L^\Phi(I, G)) + \|f_1 - f_1^*\|_\Phi + \|f_2 - f_2^*\|_\Phi \leq \epsilon + \|\|f_1^*(\cdot) - g(\cdot)\| + \|f_2^*(\cdot) - g(\cdot)\|\|_\Phi
\leq \frac{\epsilon}{3} + \|\text{dist}(f_1(\cdot), f_2(\cdot), G) + |f_1(\cdot) - f_1^*(\cdot)|\|_\Phi
\leq \frac{\epsilon}{3} + \|\text{dist}(f_1(\cdot), f_2(\cdot), G) + |f_1(\cdot) - f_1^*(\cdot)|\|_\Phi
\leq \frac{2\epsilon}{3} + \|\text{dist}(f_1(\cdot), f_2(\cdot), G)\|_\Phi
\leq \epsilon + \|\text{dist}(f_1(\cdot), f_2(\cdot), G)\|_\Phi,
\] (2.14)
which (since $\epsilon$ is arbitrary) implies that

$$\text{dist}_{\Phi} (f_1, f_2, L^\Phi (I, G)) \leq \| \text{dist} (f_1 (\cdot), f_2 (\cdot), G) \|_{\Phi}. \tag{2.15}$$

Hence by (2.9) and (2.15) the proof is complete. \hfill \Box

A direct consequence of Theorem 2.1 is the following result.

**Theorem 2.2.** Let $G$ be a closed subspace of the Banach space $X$ and let $\Phi$ be an Orlicz function that satisfies the $\Delta_2$ condition. For $g \in L^\Phi (I, G)$ to be a best simultaneous approximation of a pair of elements $(f_1, f_2)$ in $L^\Phi (I, G)$, it is necessary and sufficient that $g(t)$ is a best simultaneous approximation of $(f_1(t), f_2(t))$ in $G$ for almost all $t \in I$.

### 3. Proximinality of $L^\Phi (I, G)$ in $L^\Phi (I, X)$

A closed subspace $G$ of $X$ is called 1-summand in $X$ if there exists a closed subspace $Y$ such that $X = G \oplus Y$, that is, any element $x \in X$ can be written as $x = g + y$, $g \in G$, $y \in Y$, and $\| x \| = \| g \| + \| y \|$. It is known that a 1-summand subspace $G$ of $X$ is proximinal in $X$, and $L^1 (I, G)$ is proximinal in $L^1 (I, X)$, [11].

Our first result in this section is the following.

**Theorem 3.1.** If $G$ is simultaneously proximinal in $X$, then every pair of simple functions admits a best simultaneous approximation in $L^\Phi (I, G)$.

**Proof.** Let $f_1, f_2$ be two simple functions in $L^\Phi (I, X)$. Then $f_1, f_2$ can be written as $f_1 (s) = \sum_{k=1}^n u_k^1 \chi_{I_k} (s)$, $f_2 (s) = \sum_{k=1}^n u_k^2 \chi_{I_k} (s)$, where $I_k$’s are disjoint measurable subsets of $I$ satisfying $\bigcup_{k=1}^n I_k = I$, and $\chi_{I_k}$ is the characteristic function of $I_k$. Since $f_1$ and $f_2$ represent classes of functions, we may assume that $\mu (I_k) > 0$ for each $1 \leq k \leq n$. By assumption, we know that for each $1 \leq k \leq n$ there exists a best simultaneous approximation $w_k$ in $G$ of the pair of elements $(u_k^1, u_k^2) \in X^2$ such that

$$\text{dist} (u_k^1, w_k, G) = \| u_k^1 - w_k \| + \| u_k^2 - w_k \|. \tag{3.1}$$

Set $g = \sum_{k=1}^n w_k \chi_{I_k} (s)$. Then, for any $\alpha > 0$ and $h \in L^\Phi (I, G)$, we obtain that

$$\int_I \Phi \left( \frac{\| f_1 (t) - h(t) \| + \| f_2 (t) - h(t) \|}{\alpha} \right) d\mu (t) = \sum_{k=1}^n \int_{I_k} \Phi \left( \frac{\| u_k^1 - h(t) \| + \| u_k^2 - h(t) \|}{\alpha} \right) d\mu (t) \leq \sum_{k=1}^n \int_{I_k} \Phi \left( \frac{\| u_k^1 - w_k \| + \| u_k^2 - w_k \|}{\alpha} \right) d\mu (t) \leq \int_I \Phi \left( \frac{\| f_1 (t) - g(t) \| + \| f_2 (t) - g(t) \|}{\alpha} \right) d\mu (t). \tag{3.2}$$

Taking the infimum over all such $\alpha$’s, we have that

$$\| \| f_1 (\cdot) - h(\cdot) \| + \| f_2 (t) - h(t) \| \|_{\Phi} \geq \| \| f_1 (\cdot) - g(\cdot) \| + \| f_2 (t) - g(\cdot) \| \|_{\Phi}. \tag{3.3}$$
for all \( h \in L^\Phi(I,G) \). Hence
\[
\text{dist}_\Phi(f_1,f_2,L^\Phi(I,G)) = \|\|f_1(\cdot) - g(\cdot)\| + \|f_2(\cdot) - g(\cdot)\|\|_\Phi \\
\geq \|\|f_1(\cdot) - h(\cdot)\| + \|f_2(\cdot) - h(\cdot)\|\|_\Phi.
\] (3.4)

Now we prove the following 2-dimensional analogous of [12, Theorem 4].

**Theorem 3.2.** Let \( G \) be a closed subspace of the Banach space \( X \) and let \( \Phi \) be an Orlicz function that satisfies the \( \Delta_2 \) condition. If \( L^1(I,G) \) is simultaneously proximinal in \( L^1(I,X) \), then \( L^\Phi(I,G) \) is simultaneously proximinal in \( L^\Phi(I,X) \).

**Proof.** Let \( f_1,f_2 \in L^\Phi(I,X) \). Then \( f_1,f_2 \in L^1(I,X) \); see [13]. By assumption, there exists \( g \in L^1(I,G) \) such that
\[
\|\|f_1(\cdot) - g(\cdot)\| + \|f_2(\cdot) - g(\cdot)\|\|_1 \leq \|\|f_1(\cdot) - h(\cdot)\| + \|f_2(\cdot) - h(\cdot)\|\|_1
\] (3.5)
for every \( h \in L^1(I,G) \). By Theorem 2.2 [10],
\[
\|f_1(t) - g(t)\| + \|f_2(t) - g(t)\| \leq \|f_1(t) - h(t)\| + \|f_2(t) - h(t)\|
\] (3.6)
for almost all \( t \) in \( I \). But \( 0 \in G \). Thus
\[
\|f_1(t) - g(t)\| + \|f_2(t) - g(t)\| \leq \|f_1(t)\| + \|f_2(t)\|
\] (3.7)
for almost all \( t \) in \( I \). But 0 is a limit point of \( G \). Thus
\[
\|g(t)\| \leq \|f_1(t)\| + \|f_2(t)\|
\] (3.8)
Hence \( g \in L^\Phi(I,G) \) and
\[
\|\|f_1(\cdot) - g(\cdot)\| + \|f_2(\cdot) - g(\cdot)\|\|_\Phi \leq \|\|f_1(\cdot) - h(\cdot)\| + \|f_2(\cdot) - h(\cdot)\|\|_\Phi
\] (3.9)
for all \( h \in L^1(I,G) \). □

**Theorem 3.3.** Let \( G \) be a 1-summand subspace of the Banach space \( X \). Then \( L^\Phi(I,G) \) is simultaneously proximinal in \( L^\Phi(I,X) \).

The proof follows from Theorem 3.2 and [10, Theorem 2.4].

**Theorem 3.4.** Let \( G \) be a reflexive subspace of the Banach space \( X \). Then \( L^\Phi(I,G) \) is simultaneously proximinal in \( L^\Phi(I,X) \).

The proof follows from Theorem 3.2 and [10, Theorem 3.2].

**References**


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