Research Article

On the Generalized Ulam-Gavruta-Rassias Stability of Mixed-Type Linear and Euler-Lagrange-Rassias Functional Equations

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In this paper, the mixed-type linear and Euler-Lagrange-Rassias functional equations introduced by J. M. Rassias is generalized to the following \( n \)-dimensional functional equation:

\[
\sum_{i=1}^{n} x_i + (n - 2) \sum_{i=1}^{n} f(x_i) = \sum_{1 \leq i < j \leq n} f(x_i - x_j)
\]

when \( n > 2 \). We prove the general solutions and investigate its generalized Ulam-Gavruta-Rassias stability.

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1. Introduction

In 1940, Ulam [1] proposed the famous Ulam stability problem of linear mappings. In 1941, Hyers [2] considered the case of approximately additive mappings \( f : E \to E' \), where \( E \) and \( E' \) are Banach spaces and \( f \) satisfies Hyers inequality

\[
\| f(x + y) - f(x) - f(y) \| \leq \varepsilon
\]

for all \( x, y \in E \). It was shown that the limit \( L(x) = \lim_{n \to \infty} 2^{-n} f(2^nx) \) exists for all \( x \in E \) and that \( L : E \to E' \) is the unique additive mapping satisfying \( \| f(x) - L(x) \| \leq \varepsilon \). In 1982–1998, Rassias [3–9] generalized the result to include the following theorem.

Theorem 1.1. Let \( X \) be a real-normed linear space and let \( Y \) be a real-complete-normed linear space. Assume in addition that \( f : X \to Y \) is an approximately additive mapping for which there exist constants \( \theta \geq 0 \) and \( p, q \in \mathbb{R} \) such that \( r = p + q \neq 1 \), and \( f \) satisfies the Cauchy-Gavruta-Rassias inequality

\[
\| f(x + y) - f(x) - f(y) \| \leq \theta \| x \|^p \| y \|^q
\]

for all \( x, y \in X \). Then, there exists a unique additive mapping \( L : X \to Y \) satisfying

\[
f(x) - L(x) \leq \frac{\theta}{|2^r - 2|} \| x \|^r \quad \forall \ x \in X.
\]
If in addition \( f : X \to Y \) is a mapping such that the transformation \( t \mapsto f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \), then \( L \) is \( \mathbb{R} \)-linear mapping.

In 2002, Rassias [10] established the Ulam stability of the following mixed-type functional equation:

\[
\sum_{i=1}^{3} f(x_i) + \sum_{i=1}^{3} f(x_i) = \sum_{1 \leq i < j \leq 3} f(x_i + x_j)
\]

on restricted domains. In this paper, we will generalize Rassias’ work to the following \( n \)-dimensional mixed-type functional equation:

\[
\sum_{i=1}^{n} f(x_i) + (n-2) \sum_{i=1}^{n} f(x_i) = \sum_{1 \leq i < j \leq n} f(x_i + x_j)
\]

when \( n > 2 \), and will investigate its generalized Ulam-Gavruta-Rassias stability.

2. The general solution

**Theorem 2.1.** Let \( n > 2 \) be a positive integer, and let \( X \) and \( Y \) be vector spaces.

A function \( f : X \to Y \) satisfies the functional equation

\[
f\left(\sum_{i=1}^{n} x_i\right) + (n-2) \sum_{i=1}^{n} f(x_i) = \sum_{1 \leq i < j \leq n} f(x_i + x_j)
\]

if and only if the even part of \( f \), defined by \( f_e(x) = (1/2)(f(x) + f(-x)) \) for all \( x \in X \), satisfies the classical quadratic functional equation, which is also a special Euler-Lagrange-Rassias equation [7, 9],

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y),
\]

and the odd part of \( f \), defined by \( f_o(x) = (1/2)(f(x) - f(-x)) \) for all \( x \in X \), satisfies the Cauchy functional equation

\[
f(x + y) = f(x) + f(y).
\]

**Proof.** For the if part of the proof, suppose that \( f : X \to Y \) satisfies (2.1), we can uniquely express \( f \) as \( f(x) = f_e(x) + f_o(x) \) for all \( x \in X \), where the even part, \( f_e \), and the odd part, \( f_o \), are defined as in the theorem. We will show that \( f_e \) satisfies (2.2) and \( f_o \) satisfies (2.3).

Setting \((x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)\) in (2.1), we see that \( f(0) = 0 \). Setting \((x_1, x_2, \ldots, x_n) = (x, y, -y, 0, 0, \ldots, 0)\) in (2.1), we get

\[
f(x) + (n-2)(f(x) + f(y) + f(-y)) = f(x - y) + f(x + y)
\]

\[
+ (n-3)(f(x) + f(y) + f(-y)),
\]

which is simplified to

\[
2f(x) + f(y) + f(-y) = f(x + y) + f(x - y)
\]
for all $x, y \in X$. Replacing $x$ and $y$ with $-x$ and $-y$, respectively, then taking half the sum and half the difference with (2.5), we have
\[
2f_ε(x) + f_ε(y) + f_ε(-y) = f_ε(x + y) + f_ε(x - y),
\]
\[
2f_o(x) + f_o(y) + f_o(-y) = f_o(x + y) + f_o(x - y).
\]
(2.6)

By the evenness of $f_ε$, we immediately see that $f_ε$ satisfies the classical quadratic functional equation given by (2.2). By the oddness of $f_o$, we see that $2f_o(x) = f_o(x + y) + f_o(x - y)$ which is recognized as the Jensen functional equation. Since $f_o(0) = 0$, if we put $y = x$ in the above equation, then $f(2x) = 2f(x)$. By another substitution, $(x, y) = ((x + y)/2, (x - y)/2)$, we derive the Cauchy functional equation $f_o(x + y) = f_o(x) + f_o(y)$.

Now for the only if part of the proof, suppose that the even part and the odd part of $f : X \to Y$ satisfy (2.2) and (2.3), respectively, that is, $f_ε(x + y) + f_ε(x - y) = 2f_ε(x) + 2f_ε(y)$ and $f_o(x + y) = f_o(x) + f_o(y)$. We will show that $f$ satisfies (2.1). Noting that a linear combination of two solutions of (2.1) yields just another solution, we will in turn prove that each part of $f$ satisfies (2.1).

First, consider the odd part and make use of the linearity of the Cauchy functional equation. The left-hand side of (2.1) is
\[
f_o\left(\sum_{i=1}^{n} x_i\right) + (n - 2) \sum_{i=1}^{n} f_o(x_i) = \sum_{i=1}^{n} f_o(x_i) + (n - 2) \sum_{i=1}^{n} f_o(x_i) = (n - 1) \sum_{i=1}^{n} f_o(x_i),
\]
(2.7)
and the right-hand side of (2.1) is
\[
\sum_{1 \leq i < j \leq n} f_o(x_i + x_j) = \frac{2}{n} \left(\frac{n}{2}\right) \sum_{i=1}^{n} f_o(x_i) = (n - 1) \sum_{i=1}^{n} f_o(x_i).
\]
(2.8)
Thus, we have established (2.1) on the odd part of $f$.

For the even part, we will show by mathematical induction that (2.1) holds for every positive integer $n$. For $n = 1$, we take $\sum_{1 \leq i < j \leq 1} f_ε(x_i + x_j)$ as 0; then $f_ε(x_1) + (1 - 2)f_ε(x_1) = 0$, which is trivially true. For $n = 2$, we have $f_ε(x_1 + x_2) + 0 = f_ε(x_1 + x_2)$, which is again trivially true. For $n \geq 3$, we assume that (2.1) holds for every number of variables from 1 to $n - 1$, that is,
\[
f_ε\left(\sum_{i=1}^{k} x_i\right) + (k - 2) \sum_{i=1}^{k} f_ε(x_i) = \sum_{1 \leq i < j \leq k} f_ε(x_i + x_j)
\]
(2.9)
for $k = 1, 2, \ldots, n - 1$. For each $i, j = 1, 2, \ldots, n$ with $i \neq j$, we have
\[
f_ε(x_i - x_j) + f_ε(x_i + x_j) = 2(f_ε(x_i) + f_ε(x_j)).
\]
(2.10)
Then,
\[
\sum_{1 \leq i < j \leq n} (f_ε(x_i - x_j) + f_ε(x_i + x_j)) = 2 \sum_{1 \leq i < j \leq n} (f_ε(x_i) + f_ε(x_j)) = \frac{4}{n} \left(\frac{n}{2}\right) \sum_{i=1}^{n} f_ε(x_i).
\]
(2.11)
Thus,

$$\sum_{1 \leq i < j \leq n} (f_e(x_i - x_j) + f_e(x_i + x_j)) = 2(n - 1) \sum_{i=1}^{n} f_e(x_i). \quad (2.12)$$

For each $j, k = 1, 2, \ldots, n$ with $j \neq k$, we have

$$f_e\left(\sum_{i=1}^{n} x_i - 2x_j\right) + f_e\left(\sum_{i=1}^{n} x_i - 2x_k\right) = 2f_e\left(\sum_{i=1}^{n} x_i - x_j - x_k\right) + 2f_e(x_j - x_k). \quad (2.13)$$

Write down the above equation for every possible pair $(j, k)$ and note that there are \(\binom{n}{2}\) such pairs; so each $f_e(\sum_{i=1}^{n} x_i - 2x_j)$ appears $n - 1$ times in all \(\binom{n}{2}\) equations. Adding up the equations, we get

$$(n - 1) \sum_{j=1}^{n} f_e\left(\sum_{i=1}^{n} x_i - 2x_j\right) = 2 \sum_{1 \leq j < k \leq n} f_e\left(\sum_{i=1}^{n} x_i - x_j - x_k\right) + 2 \sum_{1 \leq j < k \leq n} f_e(x_j - x_k). \quad (2.14)$$

For each $j = 1, 2, \ldots, n$, we have

$$f_e\left(\sum_{i=1}^{n} x_i\right) + f\left(\sum_{i=1}^{n} x_i - 2x_j\right) = 2f_e\left(\sum_{i=1}^{n} x_i - x_j\right) + 2f_e(x_j). \quad (2.15)$$

Sum the above equation for all $j$’s and substitute the result from (2.12) and (2.14), then rearrange the resulting equation

$$nf_e\left(\sum_{i=1}^{n} x_i\right) + \frac{2}{n - 1} \sum_{1 \leq j < k \leq n} f_e\left(\sum_{i=1}^{n} x_i - x_j - x_k\right) \quad (2.16)$$

$$= 2 \sum_{j=1}^{n} f_e\left(\sum_{i=1}^{n} x_i - x_j\right) + \frac{2}{n - 1} \sum_{1 \leq i < j \leq n} f_e(x_i + x_j) - 2 \sum_{i=1}^{n} f_e(x_i).$$

Note that $\sum_{j=1}^{n} f_e(\sum_{i=1}^{n} x_i - x_j)$ is the sum of $f$ of $x_j$’s taken $n - 1$ variables at a time, and $\sum_{1 \leq j < k \leq n} f_e(\sum_{i=1}^{n} x_i - x_j - x_k)$ is the sum of $f$ of $x_j$’s taken $n - 2$ variables at a time. From the induction assumption, (2.1) holds for $n - 1$ and $n - 2$ variables, that is,

$$\sum_{j=1}^{n} f_e\left(\sum_{i=1}^{n} x_i - x_j\right) + (n - 1)(n - 3) \sum_{i=1}^{n} f_e(x_i) = (n - 2) \sum_{1 \leq i < j \leq n} f_e(x_i + x_j), \quad (2.17)$$

$$\sum_{1 \leq j < k \leq n} f_e\left(\sum_{i=1}^{n} x_i - x_j - x_k\right) + \frac{(n - 1)(n - 2)(n - 4)}{2} \sum_{i=1}^{n} f_e(x_i)$$

$$= \frac{(n - 2)(n - 3)}{2} \sum_{1 \leq i < j \leq n} f_e(x_i + x_j).$$
Substitute (2.17) into (2.16) and simplify, we will finally establish (2.1) on the even part of \( f \). Thus, \( f \) satisfies (2.1) and the proof is complete.

3. The Ulam-Gavruta-Rassias stability

Rassias [10] established the Ulam stability of (2.1) in the special case when \( n = 3 \) on restricted domains. The following theorem provides a general condition for which a true general solution discussed in Theorem 2.1 exists near an approximate solution. For convenience, we define

\[
Df(x_1,x_2,\ldots,x_n) = f \left( \sum_{i=1}^{n} x_i \right) + (n-2) \sum_{i=1}^{n} f(x_i) - \sum_{1 \leq i < j \leq n} f(x_i + x_j). \tag{3.1}
\]

From now on, we will refer to the even part and the odd part of a function by subscripts \( e \) and \( o \), respectively.

**Theorem 3.1.** Let \( n > 2 \) be a positive integer, let \( X \) be a real vector space, let \( Y \) be a Banach space, let \( \phi : X^n \to [0, \infty) \) be an even function. Define \( \varphi(x) = \phi(x,x,-x,0,\ldots,0) \) for all \( x \in X \). If

\[
\sum_{i=0}^{\infty} 2^{-i}\phi(2^i x) \text{ converges, } \lim_{m \to \infty} 2^{-m}\phi(2^m x_1,\ldots,2^m x_n) = 0 \tag{3.2}
\]

or

\[
\sum_{i=1}^{\infty} 4^i\phi(2^{-i} x) \text{ converges, } \lim_{m \to \infty} 4^m\phi(2^{-m} x_1,\ldots,2^{-m} x_n) = 0 \tag{3.3}
\]

for all \( x_1,x_2,\ldots,x_n \in X \), and a function \( f : X \to Y \) satisfies \( f(0) = 0 \) and

\[
\|Df(x_1,x_2,\ldots,x_n)\| \leq \phi(x_1,x_2,\ldots,x_n) \tag{3.4}
\]

for all \( x_1,x_2,\ldots,x_n \in X \), then there exists a unique function \( T : X \to Y \) that satisfies functional equation (2.1) and, if condition (3.2) holds,

\[
\|f_e(x) - T_e(x)\| \leq \frac{1}{4} \sum_{i=0}^{\infty} 4^{-i}\phi(2^i x), \quad \|f_o(x) - T_o(x)\| \leq \frac{1}{2} \sum_{i=0}^{\infty} 2^{-i}\phi(2^i x) \tag{3.5}
\]

or, if condition (3.3) holds,

\[
\|f_e(x) - T_e(x)\| \leq \frac{1}{4} \sum_{i=1}^{\infty} 4^i\phi(2^{-i} x), \quad \|f_e(x) - T_e(x)\| \leq \frac{1}{2} \sum_{i=1}^{\infty} 2^i\phi(2^{-i} x). \tag{3.6}
\]
The function $T$ is given by

$$T(x) = \begin{cases} \lim_{m \to \infty} 4^{-m} f_e(2^m x) + 2^{-m} f_o(2^m x) & \text{if condition (3.2) holds,} \\ \lim_{m \to \infty} 4^m f_e(2^{-m} x) + 2^m f_o(2^{-m} x) & \text{if condition (3.3) holds} \end{cases} \tag{3.7}$$

for all $x \in X$.

Proof. We will prove the theorem for a function $\phi$ satisfying condition (3.2) and accordingly inequality (3.5). A proof for conditions (3.3) and (3.6) can be reproduced in a similar manner. Setting $(x_1, x_2, \ldots, x_n) = (x, x, -x, 0, 0, \ldots, 0)$ in (3.4) and simplifying, we have $\|3 f(x) + f(-x) - f(2x)\| \leq \phi(x)$. Replacing $x$ by $-x$, we have $\|3 f(-x) + f(x) - f(-2x)\| \leq \phi(-x) = \phi(x)$. Then,

$$\begin{align*}
\|4 f_e(x) - f_e(2x)\| &= \frac{1}{2} \| (3 f(x) + f(-x) - f(2x)) + (3 f(-x) + f(x) - f(-2x)) \| \\
&\leq \frac{1}{2} \| 3 f(x) + f(-x) - f(2x)\| + \frac{1}{2} \| 3 f(-x) + f(x) - f(-2x)\| \\
&\leq \frac{1}{2} \phi(x) + \frac{1}{2} \phi(x) = \phi(x),
\end{align*} \tag{3.8}$$

$$\begin{align*}
\|2 f_o(x) - f_o(2x)\| &= \frac{1}{2} \| (3 f(x) + f(-x) - f(2x)) - (3 f(-x) + f(x) - f(-2x)) \| \\
&\leq \frac{1}{2} \| 3 f(x) + f(-x) - f(2x)\| + \frac{1}{2} \| 3 f(-x) + f(x) - f(-2x)\| \\
&\leq \frac{1}{2} \phi(x) + \frac{1}{2} \phi(x) = \phi(x).
\end{align*}$$

Rewrite the inequality on $f_e$ as $\| f_e(x) - 4^{-1} f_e(2x)\| \leq 4^{-1} \phi(x)$ for all $x \in X$. Suppose that $\| f_e(x) - 4^{-m} f_e(2^m x)\| \leq (1/4) \sum_{i=0}^{m-1} 4^{-i} \phi(2^i x)$ for a positive integer $m$. Then,

$$\begin{align*}
\| f_e(x) - 4^{-(m+1)} f_e(2^{m+1} x)\| &\leq \| f_e(x) - 4^{-m} f_e(2^m x)\| + \| 4^{-m} f_e(2^m x) - 4^{-(m+1)} f_e(2^{m+1} x)\| \\
&\leq \| f_e(x) - 4^{-m} f_e(2^m x)\| + 4^{-m} \| f_e(2^m x) - 4^{-1} f_e(2 \cdot 2^m x)\| \\
&\leq \frac{1}{4} \sum_{i=0}^{m-1} 4^{-i} \phi(2^i x) + 4^{-m} \phi(2^m x) = \frac{1}{4} \sum_{i=0}^{m} 4^{-i} \phi(2^i x). \tag{3.9}
\end{align*}$$

Hence, $\| f_e(x) - 4^{-m} f_e(2^m x)\| \leq (1/4) \sum_{i=0}^{m-1} 4^{-i} \phi(2^i x)$ for every positive integer $m$.

If we rewrite the inequality for $f_o$ as $\| f_o(x) - 2^{-1} f_o(2x)\| \leq 2^{-1} \phi(x)$ and repeat the same steps as in the case of $f_e$, we will have $\| f_o(x) - 2^{-m} f_o(2^m x)\| \leq (1/2) \sum_{i=0}^{m-1} 2^{-i} \phi(2^i x)$ for every positive integer $m$. 


The convergence of the sequence \( \{4^{-m} f_e(2^m x)\} \) can be settled as follows. For every positive integer \( p \),

\[
\|4^{-m} \cdot p \| f_e(2^m x) - 4^{-m} f_e(2^m x)\| = 4^{-m} \|4^{-p} f_e(2^p \cdot 2^m x) - f_e(2^m x)\| \\
\leq 4^{-m} \cdot \frac{1}{4} \sum_{i=0}^{p-1} 4^{-i} \phi(2^i \cdot 2^m x) \\
\leq \frac{1}{4} \sum_{i=0}^{\infty} 4^{-(i+m)} \phi(2^{i+m} x).
\]

(3.10)

By the definition of \( \phi \) and condition (3.2), the right-hand side approaches 0 as \( m \) goes to infinity, hence, we have a Cauchy sequence in a Banach space. Let \( T_e(x) = \lim_{m \to \infty} 4^{-m} f_e (2^m x) \) for all \( x \in X \), and thus \( \| f_e(x) - T_e(x) \| \leq (1/4) \sum_{i=0}^{\infty} 4^{-i} \phi(2^i x) \). We can similarly show that \( \{2^{-m} f_o(2^m x)\} \) converges; so let \( T_o(x) = \lim_{m \to \infty} 2^{-m} f_o (2^m x) \) for all \( x \in X \), and thus \( \| f_o(x) - T_o(x) \| \leq (1/2) \sum_{i=0}^{\infty} 2^{-i} \phi(2^i x) \). Define \( T(x) = T_e(x) + T_o(x) \) for all \( x \in X \).

In order to show that \( T \) satisfies (2.1), we will in turn show that \( T_e \) and \( T_o \) satisfy (2.1). For convenience, define \( Df_e \) and \( Df_o \) as the even part and the odd part of \( Df \) in (3.1), respectively. For \( T_e \),

\[
4^{-m} \|Df_e(2^m x_1 , \ldots , 2^m x_n)\| \\
= 4^{-m} \cdot \frac{1}{2} \|Df(2^m x_1 , \ldots , 2^m x_n) + Df(-2^m x_1 , \ldots , -2^m x_n)\| \\
\leq 4^{-m} \phi(2^m x_1 , \ldots , 2^m x_n).
\]

(3.11)

As \( m \) tend to infinity, the left-hand side approaches \( \|DT_e(x_1 , \ldots , x_n)\| \) and, by condition (3.2), the right-hand side approaches 0. Thus,

\[
DT_e(x_1 , x_2 , \ldots , x_n) = T_e \left( \sum_{i=1}^{n} x_i \right) + (n-2) \sum_{i=1}^{n} T_e(x_i) - \sum_{1 \leq i < j \leq n} T_e(x_i + x_j) = 0,
\]

(3.12)

which shows that \( T_e \) satisfies (2.1).

We can similarly show that \( T_o \) satisfies (2.1) by considering

\[
2^{-m} \|Df_o(2^m x_1 , \ldots , 2^m x_n)\| \\
= 2^{-m} \cdot \frac{1}{2} \|Df(2^m x_1 , \ldots , 2^m x_n) - Df(-2^m x_1 , \ldots , -2^m x_n)\| \\
\leq 2^{-m} \phi(2^m x_1 , \ldots , 2^m x_n),
\]

(3.13)

and take the limit as \( m \to \infty \). Hence, \( T = T_e + T_o \) satisfies (2.1) as desired.
To prove the uniqueness of $T$, suppose that there exists another function $S : X \to Y$ such that $S$ satisfies (2.1) and satisfies the inequality (3.5) with $T$ replaced by $S$. Then,

$$\|S(x) - T(x)\| \leq \|S(x) - f(x)\| + \|T(x) - f(x)\|$$

and

$$\leq \|S_e(x) - f_e(x)\| + \|S_o(x) - f_o(x)\|$$

$$+ \|T_e(x) - f_e(x)\| + \|T_o(x) - f_o(x)\|.$$ (3.14)

It is straightforward to show that every solution of the quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ has the quadratic property $f(nx) = n^2 f(x)$ and every solution of the linear functional equation $f(x + y) = f(x) + f(y)$ has the linear property $f(nx) = nf(x)$ for every positive integer $n$ and for every $x$ in the domain. We thus obtain

$$\|S(x) - T(x)\| \leq 4^{-m}\|S_e(2^m x) - f_e(2^m x)\| + 2^{-m}\|S_o(2^m x) - f_o(2^m x)\|$$

$$+ 4^{-m}\|T_e(2^m x) - f_e(2^m x)\| + 2^{-m}\|T_o(2^m x) - f_o(2^m x)\|$$

$$\leq 2\left(4^{-m} \cdot \frac{1}{4} \sum_{i=0}^{\infty} 4^{-i} \varphi(2^i \cdot 2^m x) + \frac{1}{2^m} \cdot \frac{1}{2} \sum_{i=0}^{\infty} 2^{-i} \varphi(2^i \cdot 2^m x)\right)$$

$$= \frac{1}{2} \sum_{i=0}^{\infty} 4^{-(i+m)} \varphi(2^{i+m} x) + \sum_{i=0}^{\infty} 2^{-(i+m)} \varphi(2^{i+m} x)$$

for all $x \in X$. As $m$ goes to infinity, the right-hand side approaches 0, and $S(x) = T(x)$ for all $x \in X$. This completes the proof. □

The following corollary proves the Hyers-Ulam stability of (2.1).

**Corollary 3.2.** If a function $f : X \to Y$ satisfies $f(0) = 0$ and the functional equation

$$\|Df(x_1, x_2, \ldots, x_n)\| \leq \varepsilon$$

(3.16)

for some $\varepsilon > 0$ and for all $x_1, x_2, \ldots, x_n \in X$, then there exists a unique function $T : X \to Y$ that satisfies functional equation (2.1) and, for all $x \in X$,

$$\|f_e(x) - T_e(x)\| \leq \varepsilon / 3, \quad \|f_o(x) - T_o(x)\| \leq \varepsilon.$$ (3.17)

**Proof.** Let $\varphi(x_1, x_2, \ldots, x_n) = \varepsilon$, then condition (3.2) in Theorem 3.1 holds. Hence, it follows from the theorem that there exists a unique function $T : X \to Y$ such that

$$\|f_e(x) - T_e(x)\| \leq \frac{1}{4} \sum_{i=0}^{\infty} 4^{-i} \cdot \varepsilon = \frac{\varepsilon}{3}, \quad \|f_o(x) - T_o(x)\| \leq \frac{1}{2} \sum_{i=0}^{\infty} 2^{-i} \varepsilon = \varepsilon.$$ (3.18)

The following corollary proves the Hyers-Ulam-Rassias stability of (2.1).
Corollary 3.3. Let \( p \) be a positive real number with \( 0 < p < 1 \) or \( p > 2 \). If a function \( f : X \to Y \) satisfies the inequality

\[
\|Df(x_1, x_2, \ldots, x_n)\| \leq \varepsilon \sum_{i=1}^{n} \|x_i\|^p
\]  

(3.19)

for some \( \varepsilon > 0 \) and for all \( x_1, x_2, \ldots, x_n \in X \), then there exists a unique function \( T : X \to Y \) that satisfies functional equation (2.1) and, for all \( x \in X \),

\[
\|f_c(x) - T_c(x)\| \leq \frac{3\varepsilon}{4|1 - 2^{p-2}|} \|x\|^p, \quad \|f_0(x) - T_0(x)\| \leq \frac{3\varepsilon}{2|1 - 2^{p-1}|} \|x\|^p.
\]  

(3.20)

Proof. Substituting \( x_1 = x_2 = \cdots = x_n = 0 \) into (3.19), we get

\[
f(0) + (n - 2) \cdot nf(0) = \binom{n}{2} f(0).
\]  

(3.21)

Since \( n > 2 \), it follows that \( 1 + n(n - 2) > \binom{n}{2} \), hence, \( f(0) = 0 \).

Let \( \phi(x_1, x_2, \ldots, x_n) = \varepsilon \sum_{i=1}^{n} \|x_i\|^p \). If \( 0 < p < 1 \), then condition (3.2) in Theorem 3.1 holds and it follows that

\[
\|f_c(x) - T_c(x)\| \leq \frac{1}{4} \sum_{i=0}^{\infty} 4^{-i} (3\varepsilon \cdot 2^{ip}) \|x\|^p = \frac{3\varepsilon}{4(1 - 2^{p-2})} \|x\|^p,
\]  

(3.22)

\[
\|f_0(x) - T_0(x)\| \leq \frac{1}{2} \sum_{i=0}^{\infty} 2^{-i} (3\varepsilon \cdot 2^{ip}) \|x\|^p = \frac{3\varepsilon}{2(1 - 2^{p-1})} \|x\|^p.
\]  

If \( p > 1 \), we apply Theorem 3.1 with condition (3.3) to get a similar result. \( \square \)

The following corollary proves the Ulam-Gavruta-Rassias stability of (2.1).

Corollary 3.4. Let \( p_1, p_2, \ldots, p_n \) be nonnegative real numbers and \( r = \sum_{i=1}^{n} p_i \) with \( 0 < r < 1 \) or \( r > 2 \). If a function \( f : X \to Y \) satisfies the inequality

\[
\|Df(x_1, x_2, \ldots, x_n)\| \leq \varepsilon \prod_{i=1}^{n} \|x_i\|^{p_i}
\]  

(3.23)

for some \( \varepsilon > 0 \) and for all \( x_1, x_2, \ldots, x_n \in X \), then there exists a unique function \( T : X \to Y \) that satisfies functional equation (2.1) and, for \( n = 3 \),

\[
\|f_c(x) - T_c(x)\| \leq \frac{\varepsilon}{4|1 - 2^{r-2}|} \|x\|^r, \quad \|f_0(x) - T_0(x)\| \leq \frac{\varepsilon}{2|1 - 2^{r-1}|} \|x\|^r
\]  

(3.24)

for all \( x \in X \).

Proof. We can show that \( f(0) = 0 \) by the same substitution used in the proof of Corollary 3.3. Let \( \phi(x_1, x_2, \ldots, x_n) = \varepsilon \prod_{i=1}^{n} \|x_i\|^{p_i} \). According to Theorem 3.1, if \( 0 < r < 1 \), then condition (3.2) holds, and if \( r > 2 \), then condition (3.3) holds. If \( n > 3 \), then the desired result
immediately follows. However, for \( n = 3 \), we have
\[
\| f_e(x) - T_e(x) \| \leq \frac{1}{4} \sum_{i=0}^{\infty} 4^{-i} (\epsilon \cdot 2^i \|x\| r) = \frac{\epsilon}{4(1 - 2r^{-2})} \|x\| r,
\]
and
\[
\| f_0(x) - T_0(x) \| \leq \frac{1}{2} \sum_{i=0}^{\infty} 2^{-i} (\epsilon \cdot 2^i \|x\| r) = \frac{\epsilon}{2(1 - 2^{-1})} \|x\| r,
\]
when \( 0 < r < 1 \), and a similar result when \( r > 1 \).

\[
(3.25)
\]

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