The concept of the zero-divisor graph of a commutative ring has been studied by many authors, and the $k$-zero-divisor hypergraph of a commutative ring is a nice abstraction of this concept. Though some of the proofs in this paper are long and detailed, any reader familiar with zero-divisors will be able to read through the exposition and find many of the results quite interesting. Let $R$ be a commutative ring and $k$ an integer strictly larger than 2. A $k$-uniform hypergraph $H_k(R)$ with the vertex set $Z(R,k)$, the set of all $k$-zero-divisors in $R$, is associated to $R$, where each $k$-subset of $Z(R,k)$ that satisfies the $k$-zero-divisor condition is an edge in $H_k(R)$. It is shown that if $R$ has two prime ideals $P_1$ and $P_2$ with zero their only common point, then $H_k(R)$ is a bipartite (2-colorable) hypergraph with partition sets $P_1 - Z'$ and $P_2 - Z'$, where $Z'$ is the set of all zero divisors of $R$ which are not $k$-zero-divisors in $R$. If $R$ has a nonzero nilpotent element, then a lower bound for the clique number of $H_3(R)$ is found. Also, we have shown that $H_3(R)$ is connected with diameter at most 4 whenever $x^2 \neq 0$ for all 3-zero-divisors $x$ of $R$. Finally, it is shown that for any finite nonlocal ring $R$, the hypergraph $H_3(R)$ is complete if and only if $R$ is isomorphic to $Z_2 \times Z_2 \times Z_2$.

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1. Introduction

The notion of a zero-divisor graph $\Gamma(R)$ of a commutative ring $R$ was first introduced by Beck in [1] and was further investigated in [2], where the authors were interested in colorings of $\Gamma(R)$, though their vertex set included the zero element. In [3–9] the authors, using the set of nonzero zero divisors of $R$ as vertex set of $\Gamma(R)$, were interested in examining the interplay between the ring-theoretic properties of $R$ and the graph-theoretic properties...
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of $\Gamma(R)$. In this paper, we extend the concept of a zero-divisor of a commutative ring $R$ to that of a $k$-zero-divisor and investigate the interplay between the ring-theoretic properties of $R$ and the graph-theoretic properties of its associated $k$-uniform hypergraph $H_k(R)$. In this section, we define and study some examples of $k$-zero-divisors and recall some definitions from graph theory. In Section 2, we define and study some basic properties of the $k$-uniform hypergraph $H_k(R)$ and $k$-zero-divisors of a commutative ring $R$. Finally, in the last section, we merely concentrate on the properties of 3-zero-divisor hypergraphs.

Definition 1.1. Let $R$ be a commutative ring and $k \geq 2$ a fixed integer. A nonzero nonunit element $a_1$ in $R$ is said to be a $k$-zero-divisor in $R$ if there exist $k-1$ distinct nonunit elements $a_2, a_3, \ldots, a_k$ in $R$ different from $a_1$ such that $a_1 a_2 a_3 \cdots a_k = 0$ and the product of no elements of any proper subset of $A = \{a_1, a_2, \ldots, a_k\}$ is zero. Clearly, a 2-zero-divisor in $R$ is a zero divisor, but the converse is not true in general. For example, 2 is a zero divisor in $\mathbb{Z}_4$, but it is not a 2-zero-divisor.

Remark 1.2. In the literature, on zero-divisor graphs, the edges are defined to be between the distinct nonzero zero-divisors in order to construct a graph with no loops. Here, we assume distinctness of the elements in Definition 1.1 for $k$-zero-divisors in order to have a $k$-uniform hypergraph, for any fixed integer $k \geq 3$. Note that the graph constructed by 2-zero-divisors is exactly the same as the zero-divisor graph of a ring.

Example 1.3. The element 2 in $\mathbb{Z}_{30}$ is a 3-zero-divisor since $2 \cdot 3 \cdot 5 = 0$, and the product of no elements of any proper subset of $\{2, 3, 5\}$ is zero.

By $Z(R,k)$ we denote the set of all $k$-zero-divisors of $R$. It is not difficult to show that the statement “the product of no elements of any proper subset of $A$ is zero” or the statement “the product of no elements of any $(k-1)$-subset of $A$ is zero” can be used in Definition 1.1 equivalently. Clearly, from Definition 1.1, every element of the set $\{a_2, a_3, \ldots, a_k\}$ is a $k$-zero-divisor in $R$. It is clear that every $k$-zero-divisor in $R$ is also a zero divisor in $R$, but, the converse is not true in general. For example, the element 2 is a zero divisor, but not a 3-zero-divisor in $\mathbb{Z}_{10}$.

We review some basic graph-theoretic definitions, and for the necessary definitions and notations of hypergraphs, we refer the reader to standard texts of graph theory such as [10]. A hypergraph is a pair $(V, E)$ of disjoint sets, where the elements of $E$ are nonempty subsets (of any cardinality) of $V$. The elements of $V$ are the vertices, and the elements of $E$ are the edges of the hypergraph. The hypergraph $H = (V, E)$ is called $k$-uniform whenever every edge $e$ of $H$ is of size $k$. A $k$-uniform hypergraph $H$ is called complete if every $k$-subset of the vertices is an edge of $H$. The definition of a clique and the clique number of a $k$-uniform hypergraph are taken from [11, 12] as follows.

Let $H$ be a $k$-uniform hypergraph. A subset $A$ of $V(H)$ is called a clique of $H$ if every $k$-subset of $A$ is an edge of $H$. The clique number of $H$, denoted by $\omega(H)$, is defined to be

$$\omega(H) = \max \{ |A| \mid A \text{ is a clique} \} \frac{k-1}{k-1}. \quad (1.1)$$
An $r$-coloring of a hypergraph $H = (V, E)$ is a map $c : V \to \{1, 2, \ldots, r\}$ such that for every edge $e$ of $H$, there exist at least two vertices $x$ and $y$ in $e$ with $c(x) \neq c(y)$. The smallest integer $r$ such that $H$ has an $r$-coloring is called the chromatic number of $H$ and is denoted by $\chi(H)$. In [11], it is shown that for any $k$-uniform hypergraph $H$, $\chi(H) \geq \lceil \omega(H) \rceil$. A path in a hypergraph $H$ is an alternating sequence of distinct vertices and edges of the form $v_1, e_1, v_2, e_2, \ldots, v_k$ such that $v_i, v_{i+1}$ is in $e_i$ for all $1 \leq i \leq k - 1$. The number of edges of a path is its length. The distance between two vertices $x$ and $y$ of $H$, denoted by $d_H(x, y)$, is the length of the shortest path from $x$ to $y$. If no such path between $x$ and $y$ exists, we set $d_H(x, y) = \infty$. The greatest distance between any two vertices in $H$ is called the diameter of $H$ and is denoted by $\text{diam}(H)$. The hypergraph $H$ is said to be connected whenever $\text{diam}(H) < \infty$. A cycle in a hypergraph $H$ is an alternating sequence of distinct vertices and edges of the form $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1$ such that $v_i, v_{i+1}$ are in $e_i$ for all $1 \leq i \leq k - 1$ with $v_k, v_1 \in e_k$. The girth of a hypergraph $H$ containing a cycle, denoted by $\text{gr}(H)$, is the smallest size of the length of cycles of $H$.

2. $k$-zero-divisor hypergraphs

In this section, we define and study some properties of the $k$-uniform hypergraph $H_k(R)$, the $k$-zero-divisors of a commutative ring $R$, and provide some examples.

Definition 2.1. A ring $R$ is said to be a $k$-integral domain whenever $Z(R, k)$, the set of all $k$-zero-divisors of $R$, is the empty set.

Example 2.2. Let $(R, M)$ be a local ring with maximal ideal $M \neq 0$ such that $M^2 = 0$. Then $R$ is a $3$-integral domain which is not an integral domain.

Example 2.3. For any integer $k \geq 3$, we have the following results.

1. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime decomposition of $n$, where $p_i \neq p_j$ whenever $i \neq j$ and $1 \leq \alpha_i$ for all $i, j = 1, 2, \ldots, r$. Then $Z_n$ is a $k$-integral domain whenever $\sum_{i \leq r} \alpha_i \leq k - 1$.

2. Let $n_i = p_1^{\alpha_{i1}} p_2^{\alpha_{i2}} \cdots p_t^{\alpha_{it}}$ be the prime decomposition of $n_i$ for distinct primes $p_{ji}$'s and $1 \leq \alpha_{ji}$ for all $1 \leq i \leq t$ and $j = 1, 2, \ldots, r$. Then $Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_t}$ is a $k$-integral domain whenever

$$\sum_{j \leq r_1} \alpha_{j_1} + \sum_{j \leq r_2} \alpha_{j_2} + \cdots + \sum_{j \leq r_t} \alpha_{j_t} \leq k - 1. \quad (2.1)$$

3. Let $F$ be a field and let $f(x)$ be a polynomial in $F[x]$ such that $f(x) = P_1(x)^{\alpha_1} P_2(x)^{\alpha_2} \cdots P_r(x)^{\alpha_r}$, where $P_i(x) \in F[x]$ are distinct irreducible polynomials and $1 \leq \alpha_i$ for all $1 \leq i \leq r$. Then $F[x]/(f(x))$ is a $k$-integral domain whenever $\sum_{i \leq r} \alpha_i \leq k - 1$.

4. Let $R_i$ be an integral domain for each $i = 1, 2, \ldots, n$. Then $R = R_1 \times R_2 \times \cdots \times R_n$ is a $k$-integral domain whenever $n \leq k - 1$.

By [13], it is true that a nonintegral domain with a finite number of zero divisors is finite. Similarly, we pose the following question for the rings with a finite number of $k$-zero-divisors.
Question 1. Does the finiteness of $k$-zero-divisors in a non-$k$-integral domain $R$ imply the finiteness of zero-divisors or, equivalently, finiteness of $R$?

Definition 2.4. For any fixed integer $k \geq 3$, an ideal $P$ of a ring $R$ is said to be $k$-prime whenever for any set $A = \{a_1, a_2, \ldots, a_k\}$ of nonzero, distinct, and nonunit elements of $R$, $a_1 a_2 \cdots a_k \in P$ implies that the product of the elements of a proper subset of $A$ is in $P$.

Note that by this definition, every prime ideal of $R$ is a $k$-prime ideal of $R$.

Example 2.5. Let $(R_1, M_1)$ and $(R_2, M_2)$ be two local rings with nonzero maximal ideals $M_1$ and $M_2$, respectively. We show that $M_1 \times M_2$ is a $3$-prime ideal in $R = R_1 \times R_2$ which is not a prime ideal in $R$. Let $(a_1, b_1), (a_2, b_2),$ and $(a_3, b_3)$ be arbitrary elements in $R_1 \times R_2$, where for each $1 \leq i \leq 3$, $(a_i, b_i)$ is a nonzero nonunit in $R$. Clearly, $(a_1, b_1) \cdot (a_2, b_2) \cdot (a_3, b_3) = (a_1 a_2 a_3, b_1 b_2 b_3) \in M_1 \times M_2$ implies that at least one of the elements $a_i$’s ($b_j$’s) belongs to $M_1 (M_2)$ for some $i (j)$ in $\{1, 2, 3\}$. In this case, there always exists a proper subset of $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ such that the product of its elements belongs to $M_1 \times M_2$. But since $(1, 0) \cdot (0, 1) \in M_1 \times M_2$ and neither of the elements $(1, 0)$ and $(0, 1)$ is in $M_1 \times M_2$, then $M_1 \times M_2$ is a $3$-prime ideal in $R_1 \times R_2$ which is not a prime ideal in $R$.

The following theorem is similar to the well-known fact on the relationship between prime ideals and integral domains.

Theorem 2.6. Let $P$ be an ideal in the ring $R$. Then $R/P$ is a $k$-integral domain if $P$ is a $k$-prime ideal.

The proof follows directly from the definition, and we leave it to the reader.

The converse of the above theorem is not true in general. For example, the ideal $(8)$ generated by 8 in $Z_{48}$ is not a 3-prime ideal, but $Z_{48}/\langle 8 \rangle$ is a 3-integral domain.

Next, we extend the concept of zero-divisor graph of a commutative ring $R$ to that of a $k$-zero-divisor hypergraph.

Definition 2.7. Let $R$ be a commutative ring (with $1 \neq 0$) and let $Z(R, k)$ be the set of all $k$-zero-divisors in $R$. Associate a $k$-uniform hypergraph $H_k(R)$ to $R$ with vertex set $Z(R, k)$, and for distinct elements $x_1, x_2, \ldots, x_k$ in $Z(R, k)$, the set $\{x_1, x_2, \ldots, x_k\}$ is an edge of $H_k(R)$ if and only if $x_1 x_2 \cdots x_k = 0$ and the product of elements of no $(k - 1)$-subset of $\{x_1, x_2, \ldots, x_k\}$ is zero.

Clearly, from the above definition we can conclude that for any $k \geq 3$, $H_k(R)$ is the empty set if and only if $R$ is a $k$-integral domain.

Theorem 2.8. Let $R$ be a non-$k$-integral domain. If there exist prime ideals $P_1$ and $P_2$ in $R$ such that $P_1 \cap P_2 = \{0\}$, then $\chi(H_k(R)) = 2$.

Proof. Since $P_1 \cap P_2 = \{0\}$, then $P_1 \cup P_2$ is equal to the set of all zero divisors of $R$. On the other hand, since each $k$-zero-divisor is also a zero divisor, each $k$-zero-divisor must belong to the prime ideals $P_1$ or $P_2$. Consider the function $c : V(H_k(R)) \to \{1, 2\}$ given by

$$c(x) = \begin{cases} 1, & x \in P_1, \\ 2, & x \in P_2. \end{cases}$$

(2.2)
In order to prove that \( c \) is a 2-coloring of \( H_k(R) \), we need to show that there is no edge \( e \) in \( H_k(R) \) such that every vertex of \( e \) obtains the same color. Without loss of generality, let \( e = \{x_1, x_2, \ldots, x_k\} \) be an edge of \( H_k(R) \) such that \( c(x_1) = c(x_2) = \cdots = c(x_k) = 1 \). Since \( x_1 x_2 \cdots x_k = 0 \in P_2 \) and \( P_2 \) is a prime ideal of \( R \), then \( x_i \in P_2 \) for at least one \( 1 \leq i \leq k \), which is a contradiction. Therefore, \( \chi(H_k(R)) \leq 2 \). On the other hand, since \( R \) is not a \( k \)-integral domain, then \( H_k(R) \) has at least one edge, which implies that \( \chi(H_k(R)) \geq 2 \), and the proof is complete. \( \square \)

**Remark 2.9.** From the above theorem, it is clear that \( H_k(R) \) is a bipartite hypergraph with partition sets \( V(H_k(R)) \cap P_1 \) and \( V(H_k(R)) \cap P_2 \). Note that in [4], it is shown that for any reduced ring \( R \), the zero-divisor graph \( \Gamma(R) \) is bipartite if and only if there exist two distinct prime ideals \( P_1 \) and \( P_2 \) of \( R \) such that \( P_1 \cap P_2 = \{0\} \). In addition, if \( \Gamma(R) \) is bipartite, then it is a complete bipartite graph.

**Remark 2.10.** By considering the ring \( R = Z_2 \times Z_2 \times Z_2 \), we see that \( \chi(H_3(R)) = 2 \). But there are no prime ideals \( P_1 \) and \( P_2 \) in \( R \) satisfying the condition of Theorem 2.8. Therefore, the converse of Theorem 2.8 is not true in general.

**Theorem 2.11.** Let \( R = R_1 \times R_2 \times \cdots \times R_n \), where \( R_i \) is an integral domain for each \( i = 1, 2, \ldots, n \).

1. If \( n = k \), then \( \chi(H_k(R)) = 2 \).
2. If \( n = k + t \), then \( \chi(H_k(R)) \leq 2 + t \) for all \( t \geq 0 \).

**Proof.** Let \( k = n \). We claim that

\[
Z(R, k) = \{(a_1, a_2, \ldots, a_k) \mid \text{exactly one of the } a_i \text{'s is zero for } 1 \leq i \leq k \}. \tag{2.3}
\]

It is obvious that any \( k \)-zero-divisor must have at least one zero component. Let \( x_1 = (a_{11}, a_{12}, \ldots, a_{1k}) \) be a \( k \)-zero-divisor with at least two zero components. Without loss of generality, assume that \( a_{11} = a_{12} = 0 \). Consequently, there exist \( x_2, x_3, \ldots, x_k \in V(H_k(R)) \) such that \( \{x_1, x_2, x_3, \ldots, x_k\} \in E(H_k(R)) \), where \( x_i = (a_{i1}, a_{i2}, \ldots, a_{ik}) \) for all \( 1 \leq i \leq k \). Thus, \( \prod_{i \geq 1} a_{ij} = 0 \) for each \( j \geq 3 \). Now since \( R_j \) is an integral domain, then for each fixed \( j \geq 3 \), there exists at least one \( i \) with \( 1 \leq i \leq k \) such that \( a_{ij} = 0 \). Let \( I \) be the set of all \( i \)’s such that \( a_{ij} = 0 \) for the smallest \( i \) in the set \( \{1, 2, \ldots, k\} \). Thus, we have \( x_1 \prod_{i \in I} x_i = 0 \) and since \( |I| \leq k - 2 \), we have a contradiction. Now let \( x_1 = (a_1, a_2, \ldots, a_k) \in R \) such that exactly one and only one of the components is zero. Without loss of generality, assume that \( a_1 = 0 \). Let \( x_i = (1, 1, \ldots, 1, 0, 1, 1, \ldots, 1) \), where the \( i \)th component is the only zero component of \( x_i \) for \( 2 \leq i \leq k \). It is obvious that \( \{x_1, x_2, \ldots, x_k\} \in E(H_k(R)) \) and the claim is true. Consider the function \( c : V(H_k(R)) \to \{1, 2\} \) given by

\[
c(x) = \begin{cases} 
1 & \text{the first component of } x \text{ is zero,} \\
2 & \text{otherwise.} 
\end{cases} \tag{2.4}
\]

It is easy to see that \( c \) is a 2-coloring of \( H_k(R) \), and since \( H_k(R) \) has at least one edge, \( \chi(H_k(R)) = 2 \).

For the proof of part 2, assume \( n = k + t \) with \( t \geq 0 \) a fixed integer. The proof is by induction on \( t \). From part 1, the first step of induction for \( t = 0 \) is true. Now, assume that
In this section, we only focus on some graph-theoretic properties of
the following conditions:

\[ i \]  

Consequently, for the rest of the proof, we can always assume that

\[ i = 2 \]  

or

\[ i = 3 \]

From this, it is not difficult to show that \( \alpha' \) is a \((t + 3)\)-coloring of \( H_k(R_1 \times R_2 \times \cdots \times R_{k+1}) \), and the proof is complete.

As a very special case of the above theorem, it is easy to show that the chromatic number of \( H_3(Z_2^3) \) and \( H_3(Z_3^3) \) is 3. Note that the chromatic number of \( H_3(Z_2^3) \) is strictly less than \( 2 + (5 - 3) \), and the chromatic number of \( H_3(Z_3^3) \) equal to 3 shows that the bound is sharp.

### 3. 3-zero-divisor hypergraphs

In this section, we only focus on some graph-theoretic properties of \( H_3(R) \). We show that

\[ H_3(R) \]

is connected with diameter at most 4 provided that \( x^2 \neq 0 \) for all 3-zero-divisors \( x \) in \( R \). We find a necessary and sufficient condition for its completeness, and we also find a lower bound for its clique number.

**Theorem 3.1.** Let \( H_3(R) \) be the 3-zero-divisor hypergraph of a ring \( R \) such that \( x^2 \neq 0 \) for every 3-zero-divisor \( x \in R \). Then \( H_3(R) \) is connected and

\[ \text{diam} \ (H_3(R)) \leq 4. \]  

**Proof.** For the proof of the theorem, it is enough to show that for each two edges \( e_1 = \{a_1, a_2, a_3\} \) and \( e_2 = \{b_1, b_2, b_3\} \) of \( H_3(R) \), there exist edges \( e_3 \) and \( e_4 \) which satisfy one of the following conditions:

\[ e_3 \cap e_1 \neq \emptyset, \quad e_3 \cap e_2 \neq \emptyset, \]  

\[ \text{or} \]

\[ e_3 \cap e_1 \neq \emptyset, \quad e_4 \cap e_2 \neq \emptyset, \quad e_4 \cap e_3 \neq \emptyset. \]  

Consequently, for the rest of the proof, we can always assume that \( a_i \neq b_j \) and \( a_i \neq -b_j \) for all \( i, j \in \{1, 2, 3\} \). Let \( G \) be the bipartite graph constructed as follows: \( V(G) = e_1 \cup e_2 \) and \( a_i b_j \in E(G) \) if and only if \( a_i b_j = 0 \) in the ring \( R \).

Suppose \( G \) has two isolated vertices, one in \( e_1 \) and the other in \( e_2 \). For example, \( \text{deg}_G(a_1) = \text{deg}_G(b_3) = 0 \). If there exists an element \( c \in \{a_1, a_2, b_1, b_2\} \) such that \( a_3 b_3 c = 0 \), then \( e_3 = \{a_3, b_3, c\} \) satisfies \((\ast_1)\). Suppose that this is not the case. If \( a_3 b_3 \notin \{a_1, a_2, b_1, b_2\} \), then \( e_3 = \{a_1, a_2, a_3, b_3\} \) and \( e_4 = \{b_1, b_2, a_3 b_3\} \) satisfy \((\ast_2)\). Otherwise without loss of generality, assume that \( a_3 b_3 = a_1 \). Then \( e_3 = \{a_1, b_1, b_2\} \) satisfies \((\ast_1)\). The rest of our proof depends on the number of edges of \( G \).

**Case 1.** Suppose \( |E(G)| \leq 2 \). Then \( G \) has two isolated vertices, one in \( e_1 \) and the other in \( e_2 \).
**Case 2.** Suppose \(|E(G)| = 3\). We study this case for four different subcases as follows.

**Case 2.1.** Assume the degree of each vertex of \(G\) is one and

\[
E(G) = \{a_1b_1,a_2b_2,a_3b_3\}.
\]  

(3.2)

Consider the set \(\{a_1,a_2b_3,b_1 + b_2\}\). If \(a_1 = a_2b_3\), then \(a_1b_2 = 0\) is a contradiction. If \(a_1 = b_1 + b_2\), then \(b_1a_2a_3 = 0\), and \(e_3 = \{b_1,a_2,a_3\}\) satisfies (\(\ast_1\)). If \(b_1 + b_2 = a_2b_3\), then \(a_1b_2a_3 = 0\), and \(e_3 = \{a_1,b_2,a_3\}\) satisfies (\(\ast_1\)). Otherwise, \(e_3 = \{a_1,a_2b_3,b_1 + b_2\}\) is an edge. Similarly if we consider the set \(\{b_1,a_2b_3,a_1 + a_3\}\), then we find an edge \(e_3\) which satisfies (\(\ast_1\)) or \(e_4 = \{b_1,a_2b_3,a_1 + a_3\}\) is an edge with \(e_3\) and \(e_4\) satisfying (\(\ast_2\)).

**Case 2.2.** Assume that the degree of exactly one of the vertices of \(G\) is one. Without loss of generality, suppose that

\[
E(G) = \{a_1b_1,a_2b_2,a_3b_3\}.
\]  

(3.3)

Consider the set \(\{a_2,a_3b_1,a_1 + b_3\}\). If \(a_2 = a_3b_1\), then \(a_1a_2 = 0\) implies a contradiction. If \(a_2 = a_1 + b_3\), then \(a_2b_2b_1 = 0\), and \(e_3 = \{a_2, b_2, b_1\}\) satisfies (\(\ast_1\)). If \(a_1 + b_3 = a_3b_1\), then \(a_2b_1b_2b_1 = 0\). In this case if \(a_3 = b_1b_2\), then \(a_1a_3 = 0\), also, \(b_1 = b_1b_2\) implies that \(b_1b_2 = 0\), which in both cases we have a contradiction. Therefore, \(e_3 = \{a_3, b_1b_2, b_1\}\) is an edge which satisfies (\(\ast_1\)). If none of the above conditions holds, then the set \(e_3 = \{a_2, a_3b_1, a_1 + b_3\}\) is an edge. Now consider the set \(\{b_2,a_3b_1,b_3\}\). Similarly, we find an edge \(e_3\) which satisfies (\(\ast_1\)), or \(e_4 = \{a_2,a_3b_1,a_1 + b_3\}\) is an edge where \(e_3\) and \(e_4\) satisfy (\(\ast_2\)).

**Case 2.3.** Let the degree of two vertices of \(G\) be two. Without loss of generality, suppose that

\[
E(G) = \{a_1b_1,a_1b_2,a_2b_2\}.
\]  

(3.4)

In this case, \(\deg_G(a_3) = \deg_G(b_3) = 0\), and the proof is complete.

**Case 2.4.** Assume that the degree of one vertex of \(G\) is three. Without loss of generality, suppose

\[
E(G) = \{a_1b_1,a_1b_2,a_1b_3\}.
\]  

(3.5)

Suppose that \(a_1^2a_2 \neq 0\). Consider the set \(\{a_1a_2 - b_1,a_1,a_3\}\). If \(a_1a_2 - b_1 = a_1\), then \(b_2b_1 = 0\) is a contradiction. If \(a_1a_2 - b_1 = a_3\), then \(a_3b_2b_2 = 0\), and therefore \(e_3 = \{a_3, b_2, b_3\}\) is an edge satisfying (\(\ast_1\)). In the other case, \(e_3 = \{a_1a_2 - b_1,a_1,a_3\}\) is an edge. Similarly, if we consider the set \(\{a_1a_2 - b_1,b_2,b_3\}\), we will find an edge \(e_3\) that satisfies (\(\ast_1\)), or \(e_4 = \{a_1a_2 - b_1,b_2,b_3\}\) is an edge with \(e_3\) and \(e_4\) that satisfy (\(\ast_2\)). Now let \(a_1^2a_2 = 0\). Consider the set \(\{a_1 - b_1,a_1,a_2\}\). If \(a_1 - b_1 = a_2\), then \(a_2b_3b_2 = 0\), and therefore \(e_3 = \{a_2, b_2, b_3\}\) is an edge satisfying (\(\ast_1\)). In the other case, \(e_3 = \{a_1 - b_1,a_1,a_2\}\) is an edge. Similarly, if we consider the set \(\{a_1 - b_1,b_2,b_3\}\), we will find a contradiction, or \(e_4 = \{a_1a_2 - b_1,b_2,b_3\}\) is an edge with \(e_3\) and \(e_4\) that satisfy (\(\ast_2\)).

**Case 3.** Suppose \(|E(G)| = 4\). We study this case using four different subcases as follows.
Case 3.1. Assume the degree of one vertex of $G$ is three. Without loss of generality, suppose that

$$E(G) = \{a_1b_1,a_1b_2,a_1b_3,a_2b_3\}. \quad (3.6)$$

Consider the set $\{a_3b_1,a_2,a_1 + b_3\}$. If $a_3b_1 = a_2$, then $a_2b_3b_1 = 0$, and therefore $e_3 = \{a_3,b_1,b_3\}$ is an edge satisfying $(*)_1$. If $a_3b_1 = a_1 + b_3$, then $a_1^2 = 0$ is a contradiction. If $a_2 = a_1 + b_3$, then $b_2^2 = 0$ is a contradiction. In the other case, $e_3 = \{a_3b_1,a_2,a_1 + b_3\}$ is an edge. Similarly, if we consider the set $\{a_3b_1,b_2,b_3\}$, we will find an edge $e_3$ that satisfies $(*)_1$, or $e_4 = \{a_3b_1,b_2,b_3\}$ is an edge with $e_3$ and $e_4$ that satisfy $(*)_2$.

Case 3.2. Assume that the degree of four vertices of $G$ is two. Without loss of generality, suppose that

$$E(G) = \{a_1b_1,a_1b_2,a_2b_1,a_2b_2\}. \quad (3.7)$$

In this case, $\deg_{G}(a_3) = \deg_{G}(b_3) = 0$, and the proof is complete.

Case 3.3. Let the degree of three vertices of $G$ be two. Suppose without loss of generality that

$$E(G) = \{a_1b_1,a_1b_2,a_2b_2,a_2b_3\}. \quad (3.8)$$

Consider the set $\{a_3b_3,a_1,a_2\}$. If $a_3b_3 = a_1$ or $a_2$, then $a_3b_3b_2 = 0$, and therefore $e_3 = \{a_3,b_2,b_3\}$ is an edge that satisfies $(*)_1$. In the other case, $e_3 = \{a_3,b_3,a_1,a_2\}$ is an edge. Similarly, if we consider the set $\{a_3b_3,b_1,b_2\}$, we will find an edge $e_3$ that satisfies $(*)_1$, or $e_4 = \{a_3b_3,b_1,b_2\}$ is an edge with $e_3$ and $e_4$ that satisfy $(*)_2$.

Case 3.4. Assume that the degree of two vertices of $G$ is two. In this case, there might be two different nonisomorphic cases. Without loss of generality, for one case we can assume that

$$E(G) = \{a_1b_1,a_1b_2,a_2b_2,a_3b_3\}, \quad (3.9)$$

and in the other case

$$E(G) = \{a_1b_1,a_1b_2,a_2b_3,a_3b_3\}. \quad (3.10)$$

In the first case, consider the set $\{a_3b_1,a_2,a_1 + b_2\}$. If $a_3b_1 = a_2$, then $a_3b_1b_2 = 0$, and therefore $e_3 = \{a_3,b_1,b_2\}$ is an edge that satisfies $(*)_1$. If $a_3b_1 = a_1 + b_2$, then $a_1^2 = 0$ is a contradiction. Also, $a_2 = a_1 + b_2$ implies that $b_2^2 = 0$, which is a contradiction. In the other case, $e_3 = \{a_1 + b_2,a_2,b_1a_3\}$ is an edge. Similarly, if we consider the set $\{a_3b_3,b_1,a_1 + b_2\}$, we will find an edge $e_3$ that satisfies $(*)_1$, or $e_4 = \{a_3b_3,b_1,a_1 + b_2\}$ is an edge with $e_3$ and $e_4$ that satisfy $(*)_2$.

Similarly, for the second case, by considering the sets $\{a_1 + b_1,a_2,a_3\}$ and $\{a_1 + b_1,a_2,a_3\}$, we find an edge $e_3$ that satisfies $(*)_1$, or two edges $e_3$ and $e_4$ that satisfy $(*)_2$.
Case 4. Suppose $|E(G)| = 5$. We continue our investigation for five different nonisomorphic subcases as follows.

Case 4.1. Without loss of generality, we can assume that

$$E(G) = \{a_1 b_1, a_1 b_2, a_1 b_3, a_2 b_1, a_2 b_2\}. \quad (3.11)$$

Consider the set $\{a_3 b_3, a_2, a_1 + b_2\}$. If $a_3 b_3 = a_2$, then $a_3 b_3 b_2 = 0$, and therefore $e_3 = \{a_3, b_2, b_3\}$ is an edge that satisfies ($*_1$). If $a_3 b_3 = a_1 + b_2$, then $a_3^2 = 0$, which is a contradiction. If $a_1 + b_2 = a_2$, then $b_1 b_2 = 0$ is a contradiction. In the other case, $e_3 = \{a_1 + b_2, a_2, a_3 b_3\}$ is an edge. Similarly, if we consider the set $\{a_3 b_3, b_1, b_2\}$, we will find an edge $e_3$ which satisfies ($*_1$), or $e_4 = \{a_3 b_3, b_1, b_2\}$ is an edge with $e_3$ and $e_4$ that satisfy ($*_2$).

Case 4.2. Assume without loss of generality that

$$E(G) = \{a_1 b_1, a_1 b_2, a_1 b_3, a_2 b_1, a_3 b_2\}. \quad (3.12)$$

Consider the set $\{a_1 + b_1, a_2, b_2\}$. If $a_1 + b_1 = a_2$, then $b_1^2 = 0$ is a contradiction. If $a_1 + b_1 = b_2$, then $a_1^2 = 0$ implies a contradiction. In the other case, $e_3 = \{a_1 + b_2, a_2, a_3 b_3\}$ is an edge that satisfies ($*_1$).

Case 4.3. Assume without loss of generality that

$$E(G) = \{a_1 b_1, a_1 b_2, a_1 b_3, a_2 b_1, a_3 b_1\}. \quad (3.13)$$

Consider the set $\{a_1 + b_1, a_2, b_2\}$. If $a_1 + b_1 = a_2$, then $b_1^2 = 0$ is a contradiction. If $a_1 + b_1 = b_2$, then $a_2 a_3 b_2 = 0$, and $e_3 = \{a_2, b_2, a_3\}$ is an edge which satisfies ($*_1$). In the other case, $e_3 = \{a_1 + b_1, a_2, b_2\}$ is an edge that satisfies ($*_1$).

Case 4.4. Without loss of generality, we can assume that

$$E(G) = \{a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2, a_3 b_3\}. \quad (3.14)$$

Consider the set $\{a_3 + b_1, a_1, b_3\}$. If $a_3 + b_1 = a_1$ or $a_3 + b_1 = b_3$, then $a_1 a_2 b_3 = 0$, and $e_3 = \{a_1, a_2, b_3\}$ is an edge that satisfies ($*_1$). In the other case, $e_3 = \{a_3 + b_1, a_1, b_3\}$ is an edge which satisfies ($*_1$).

Case 4.5. Assume without loss of generality that

$$E(G) = \{a_1 b_1, a_1 b_2, a_2 b_2, a_3 b_3\}. \quad (3.15)$$

Consider the set $\{a_1 + b_2, a_2, b_1\}$. If $a_1 + b_2 = a_2$, then $b_1^2 = 0$. If $a_1 + b_2 = b_1$, then $a_2^2 = 0$, which is a contradiction. Therefore $e_3 = \{a_1 + b_2, a_2, b_1\}$ is an edge that satisfies ($*_1$).

Case 5. Suppose $|E(G)| = 6$. We study three different nonisomorphic subcases as follows.

Case 5.1. Without loss of generality, we can assume that

$$E(G) = \{a_1 b_1, a_1 b_2, a_1 b_3, a_2 b_1, a_2 b_2, a_3 b_1\}. \quad (3.16)$$
Consider the sets \( \{a_1 + b_1, a_2, a_3\} \) and \( \{a_1 + b_1, b_2, b_3\} \). If \( a_1 + b_1 = a_2 \), then \( b_1 b_2 = 0 \). If \( a_1 + b_1 = a_3 \), then \( b_1^2 = 0 \). Also, \( a_1 + b_1 = b_2 \) or \( a_1 + b_1 = b_3 \) implies that \( a_1^2 = 0 \), and in either case, we have a contradiction. Therefore, \( e_3 = \{a_1 + b_1, a_2, a_3\} \) and \( e_4 = \{a_1 + b_1, b_2, b_3\} \) are two edges that satisfy (*)₂.

**Case 5.2.** Without loss of generality, we can assume that

\[
E(G) = \{a_1 b_1, a_1 b_2, a_1 b_3, a_2 b_1, a_2 b_2, a_2 b_3, a_3 b_1, a_3 b_2, a_3 b_3\}. \tag{3.17}
\]

Consider the set \( \{a_1 + b_3, a_3, b_1\} \). If \( a_1 + b_3 = a_3 \), then \( b_3^2 = 0 \). Also \( a_1^2 = 0 \) whenever \( a_1 + b_3 = b_1 \), which is a contradiction. Therefore, \( e_3 = \{a_1 + b_3, a_3, b_1\} \) is an edge that satisfies (*₁).

**Case 5.3.** Assume without loss of generality that

\[
E(G) = \{a_1 b_1, a_1 b_3, a_2 b_1, a_2 b_2, a_3 b_2, a_3 b_3\}. \tag{3.18}
\]

In this case, similar to the above subcase, \( e_3 = \{a_1 + b_3, a_3, b_1\} \) is an edge which satisfies (*₁).

**Case 6.** Suppose that \( 7 \leq |E(G)| \leq 9 \). In this case, there always exist two vertices with degree three, one from \( e_1 \) and the other from \( e_2 \). Let \( d_G(a_1) = d_G(b_1) = 3 \). Consider the sets \( \{a_1 + b_1, a_2, a_3\} \) and \( \{a_1 + b_1, b_2, b_3\} \). If \( a_1 + b_1 = a_2 \) or \( a_3 \), then \( b_1^2 = 0 \); and if \( a_1 + b_1 = b_2 \) or \( b_3 \), then \( a_1^2 = 0 \), which is a contradiction in all cases. Therefore, \( e_3 = \{a_1 + b_1, a_2, a_3\} \) and \( e_4 = \{a_1 + b_1, b_2, b_3\} \) are two edges that satisfy (*)₂.

**Remark 3.2.** From the above theorem and the fact that

\[
\text{gr}(H_3(R)) \leq 2 \text{diam}(H_3(R)) + 1, \tag{3.19}
\]

we can conclude that the diameter and girth of any hypergraph \( H_3(R) \) containing a cycle and satisfying the conditions in the above theorem are bounded by 4 and 9, respectively. Note that a similar result for a zero-divisor graph \( \Gamma(R) \) is studied in [5, 8, 9, 14] as follows.

1. \( \Gamma(R) \) is connected and \( \text{diam}(\Gamma(R)) \leq 3 \).
2. If \( \Gamma(R) \) contains a cycle, then \( \text{gr}(\Gamma(R)) \leq 4 \).

**Lemma 3.3.** Let \( R \) be a finite ring with \( |R| \geq 4 \). Then \( R \cong Z_2 \times Z_2 \), or there exist two distinct elements \( x \) and \( y \) in \( R - \{0, 1\} \) such that \( xy \neq 0 \).

**Proof.** For the case \( |R| = 4 \), it is clear that \( R \) is isomorphic to either \( Z_2 \times Z_2 \), \( Z_2[x]/(x^2) \) or \( Z_4 \), which implies the desired result. Next, we study the case for \( |R| \geq 5 \) by a contrary method. Suppose \( R - \{0, 1\} = \{a_1, a_2, \ldots, a_m\} \), \( m \geq 3 \), and \( a_i a_j = 0 \) for all \( 1 \leq i \neq j \leq m \). It is clear that \( a_2 + 1 \) is different from 0 and 1. Otherwise, \( a_1 = 0 \) or \( a_2 = 0 \), which is a contradiction to the choice of \( a_1 \) and \( a_2 \). If \( a_2 + 1 \neq a_1 \), then \( a_1(a_2 + 1) = 0 \), and we have \( a_1 = 0 \), which is a contradiction. Thus, \( a_2 + 1 = a_1 \). Similarly, \( a_1 a_3 = 0 \), and \( a_3 + 1 = a_1 \) implies that \( a_3 = a_2 \), which is a contradiction. □

In the next theorem, we give a necessary and sufficient condition for a hypergraph \( H_3(R) \) to be complete. In the process of the following proof, we consider the obvious fact
that $H_3(Z_2 \times Z_2 \times Z_2)$ has only one edge, and necessarily it is a complete hypergraph. Note that for a detailed study of the completeness of a zero-divisor graph $\Gamma(R)$, the reader is referred to [5].

**Theorem 3.4.** Let $R$ be a finite nonlocal ring. Then $H_3(R)$ is complete if and only if $R = Z_2 \times Z_2 \times Z_2$.

**Proof.** The sufficient part of the theorem is trivial, because $H_3(R)$ has only one edge, and therefore is complete whenever $R = Z_2 \times Z_2 \times Z_2$. Suppose that $H_3(R)$ is complete. It is a well-known fact that any finite ring $R$ is isomorphic to the product of local rings. Thus, assume that $R = R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ is a local ring for all $i = 1, 2, \ldots, n$. Now, we study the following cases for different values of $n$.

**Case 1.** Suppose $n \geq 4$. It is clear that $e_1 = \{x_1, x_2, x_3\}$ and $e_2 = \{y_1, y_2, y_3\}$ with

$$
\begin{align*}
x_1 &= (1, 1, 0, 0, \ldots, 0), & x_2 &= (1, 0, 1, 0, \ldots, 0), & x_3 &= (0, 1, 1, 0, \ldots, 0), \\
y_1 &= (1, 0, 0, 1, \ldots, 0), & y_2 &= (1, 1, 0, 0, \ldots, 0), & y_3 &= (0, 1, 0, 1, \ldots, 0)
\end{align*}
$$

(3.20)

are two edges of $H_3(R)$. Clearly, $H_3(R)$ is not complete since $\{x_1, x_2, y_1\}$ is not an edge of $H_3(R)$.

**Case 2.** Let $R = R_1 \times R_2 \times R_3$. Without loss of generality, suppose that $|R_1| \geq 3$. Let $x \in R_1 \setminus \{0, 1\}$. Obviously, $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\} \in E(H_3(R))$ and $\{(x, 1, 0), (1, 0, 1), (0, 1, 1)\} \notin E(H_3(R))$. But $\{(x, 1, 0), (1, 0, 1), (1, 1, 0)\} \notin E(H_3(R))$, which implies that $H_3(R)$ is not complete. Hence, we can conclude that $|R_1| \leq 2$ and $R = Z_2 \times Z_2 \times Z_2$.

**Case 3.** Let $R = R_1 \times R_2$. If $H_3(R)$ does not have any vertices, we do not have anything to prove. Therefore, first we assume that $|R_i| \geq 4$ for each $1 \leq i \leq 2$ and investigate the following subcases.

**Case 3.1.** The square of one of the components of some 3-zero-divisor of $R$ is zero. Let $(a, b)$ be a 3-zero-divisor in $R$ with $a^2 = 0$ and let $e = \{(a, b), (c, d), (f, g)\}$ be an edge of $H_3(R)$. Since $Z_2 \times Z_2$ is not a local ring, by Lemma 3.3 there exist distinct elements $x$ and $y$ in $R_2 \setminus \{0, 1\}$ such that $xy \neq 0$. Now, from the fact that $\{(a, 1), (a, x), (1, 0)\}$ and $\{(a, 1), (a, y), (1, 0)\}$ are in $E(H_3(R))$ and $\{(a, x), (a, y), (a, 1)\} \notin E(H_3(R))$, we can conclude that $H_3(R)$ is not complete.

**Case 3.2.** The square of none of the components of any 3-zero-divisor of $R$ is zero. Suppose that $e = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ is an edge of $H_3(R)$. In this case, there always exists $i \in \{1, 2, 3\}$, say $i = 1$, such that $a_1a_2 \neq 0$ and $a_1a_3 \neq 0$, or similarly, $b_1b_2 \neq 0$ and $b_1b_3 \neq 0$. Otherwise, the product of two elements of $e$ will be zero, which contradicts the definition for $e$ to be an edge in $H_3(R)$. Without loss of generality, we assume that $a_1a_2 \neq 0$ and $a_1a_3 \neq 0$. By using Lemma 3.3, similar to Case 3.1, there exist distinct elements $x$ and $y$ in $R_2 \setminus \{0, 1\}$ such that $xy \neq 0$. Since $\{(a_1, 0), (a_2, x), (a_3, 1)\}$ and $\{(a_1, 0), (a_2, y), (a_3, 1)\}$ are the edges of $H_3(R)$, and $\{(a_2, x), (a_2, y), (a_3, 1)\}$ is not an edge of $H_3(R)$, then $H_3(R)$ is not complete.

Next, we assume that the size of one of the rings $R_i$’s is 2, where $i = 1, 2$. Without loss of generality, assume that $R_2 = Z_2$. It is clear that $R$ does not have any 3-zero-divisors
whenever $R_1$ is an integral domain. Thus, $R_1$ has at least four elements. Obviously, the edges of $H_3(R)$ cannot be different from the following forms:

$$
\{(a,0),(b,0),(c,0)\}, \quad \{(a,1),(b,0),(c,0)\}, \quad \{(a,1),(b,1),(c,0)\}. \quad (3.21)
$$

**Case 3.3.** Let $H_3(R)$ have an edge of the form $\{(a,0),(b,0),(c,0)\}$.

Then $\{(a,1),(b,0),(c,0)\} \in E(H_3(R))$, $\{(a,0),(b,1),(c,0)\} \in E(H_3(R))$, and $\{(a,0),(b,0),(c,1)\} \in E(H_3(R))$. In this case, the completeness of $H_3(R)$ implies that $\{(a,1),(b,1),(c,1)\} \in E(H_3(R))$, which is a contradiction.

**Case 3.4.** Suppose $\{(a,1),(b,0),(c,0)\}$ is an edge of $H_3(R)$. Therefore, $b \neq c$, $ab \neq 0$, $ac \neq 0$, and $bc \neq 0$. In this subcase, we study two different cases:

(a) The first components of two elements of $\{(a,1),(b,0),(c,0)\}$ are equal. For example, assume $a = b$. Thus, $\{(a^2,1),(1,0),(c,1)\} \in E(H_3(R))$ whenever $a^2 \neq c$. In this case, $c \neq 1$, and the completeness of $H_3(R)$ implies that $\{(a,1),(1,0),(c,0)\} \in E(H_3(R))$, which contradicts $ac \neq 0$.

On the other hand if $a^2 = c$, we have $a^4 = 0$, which implies $a^3 \neq a$. Therefore, $\{(a,1), (a^3,1),(1,0)\}$ is an edge of $H_3(R)$, which contradicts $ac \neq 0$.

(b) Let $a \neq b$ and $a \neq c$. In this case, $\{(a,1),(b,1),(c,0)\}$ and $\{(a,1),(b,0),(c,1)\}$ are in $E(H_3(R))$. Consequently, the completeness of $H_3(R)$ implies that $\{(a,1),(b,1),(c,1)\} \in E(H_3(R))$, which is a contradiction.

**Case 3.5.** Let all the edges of $H_3(R)$ be of the form $\{(a,1),(b,1),(c,0)\}$. Assume that $\{(a,1),(b,1),(c,0)\}$ and $\{(a',1),(b',1),(c',0)\}$ are two edges of $H_3(R)$. Therefore, by the completeness of $H_3(R)$, one of the sets

$$
\{(a,1),(b,1),(a',1)\}, \quad \{(a,1),(b,1),(b',1)\}, \quad \{(a,1),(c,0),(c',0)\} \quad (3.22)
$$

should be an edge of $H_3(R)$. This is a contradiction to the definition of an edge or to Case 3.4. Now, we can conclude that $H_3(R)$ has only one edge of the form $\{(a,1),(b,1),(c,0)\}$, where $ac \neq 0$ and $bc \neq 0$. Furthermore, if $ab \neq 0$, then $\{(a,1),(b,0),(c,0)\}$ is an edge of $H_3(R)$, which is a contradiction. Thus, $ab = 0$. Consequently, $c \neq 1$ implies that $\{(a,1),(b,1),(c,0)\}$, $\{(a,1),(b,1),(1,0)\}$ and $\{(a,1),(b,1),(-1,0)\}$ are edges in $H_3(R)$, which is a contradiction. Hence, we can conclude that $\{(a,1),(b,1),(1,0)\}$ is the only edge of $H_3(R)$ and $1 = -1$ in $R_1$. Next, we show that $a^2 = a$ and $b^2 = b$. Since $\{(a,1),(b,1),(a+1,0)\}$ is not an edge in $H_3(R)$, $ba = 0$, and $b \neq 0$, then $b(a+1) \neq 0$, and we must have $a(a+1) = 0$, which implies that $a^2 = a$. By a similar argument, we can conclude that $b^2 = b$. Suppose $x \in R_1 - \{0,1,a,b\}$. Since $\{(a,1),(b,1),(x,0)\}$ is not an edge of $H_3(R)$, then $ax = 0$ or $bx = 0$. Without loss of generality, suppose that $ax = 0$. Now, since $b+x \neq b$, $\{(a,1),(b+x,1),(1,0)\}$ is not an edge of $H_3(R)$. Therefore, $b+x = 0$ or $b+x = a$. If $b+x = 0$, we have $b = x$, which is a contradiction. Let $b+x = a$. Then $x = b+a$, and therefore $a(b+a) = 0$, which implies that $a = a^2 = 0$, a contradiction. Thus, $\{0,1,a,b\}$ are the only elements of $R_1$. Since $R_1$ is a local ring with 4 elements, then $R_1 = Z_4$ or $R_1 = Z_2[x]/(x^2)$. In either case, $R = R_1 \times Z_2$ does not have any edges, and $H_3(R)$ is not complete.
Finally, since the proof of the case $R_2 = Z_3$ is similar to the above argument, we leave the rest of the proof to the reader. □

**Remark 3.5.** Bounds for $\omega(\Gamma(R))$ are given by using nilpotent elements of $R$ as studied in [6] as follows. Let $R$ be a commutative ring and $0 \neq x \in \text{nil}(R)$, and let $n$ be the least positive integer such that $x^n = 0$.

1. If $n = 2t$, then $\omega(\Gamma(R)) \geq 2^t - 1$.
2. If $n = 2t + 1$, then $\omega(\Gamma(R)) \geq 2^t$.

Similarly, in the next theorem, we give a lower bound for the clique number of $H_3(R)$ using the index of nilpotence as studied in [6] for a zero-divisor graph $\Gamma(R)$.

**Theorem 3.6.** Let $x$ be an element of a commutative ring $R$ such that $x^n = 0$ and $x^{n-1} \neq 0$. Then

$$\omega(H_3(R)) \geq \begin{cases} 
2^{2t-2} & \text{if } n = 3t, \\
\frac{2^{2t-1} + 1}{2} & \text{otherwise.}
\end{cases} \tag{3.23}$$

**Proof.** For $n = 3t$, the set

$$A = \{x^i(1 + a_1x + a_2x^2 + \cdots + a_{2t-1}x^{2t-1}) \mid a_i \in \{0, 1\}, \ 1 \leq i \leq 2t - 1\} \tag{3.24}$$

is a clique of size $2^{2t-1}$.

Similarly, for $n = 3t + 1$ and $n = 3t + 2$, the set

$$A = \{x^{i+1}(1 + a_1x + a_2x^2 + \cdots + a_{2t-1}x^{2t-1}) \mid a_i \in \{0, 1\}, \ 1 \leq i \leq 2t - 1\} \cup \{x^i\} \tag{3.25}$$

is a clique of size $2^{2t-1} + 1$. □

**Theorem 3.7.** For any integer $m \geq 3$, there exists an integer $n$ such that

$$\omega(H_3(Z^n_2)) \geq \frac{m}{2}, \tag{3.26}$$

where $Z^n_2 = Z_2 \times Z_2 \times \cdots \times Z_2$ ($n$ times).

**Proof.** For $m = 3$, it is clear that the set $\{(1,1,0),(1,0,1),(0,1,1)\}$ is a clique of size 3 in $H_3(Z^n_2)$. Suppose that $\{a_1,a_2,\ldots,a_m\}$ is a clique of size $m$ in $H_3(Z^n_2)$. Let $n = n' + m$. We define $b_i$ in $H_3(Z^n_2)$ to be the $n$-tuple whose first $n'$ components are exactly $a_i$ and all the other components are 0, except the $(n' + i)$th component, which is 1 for all $1 \leq i \leq m$. Let $b_{m+1}$ be the $n$-tuple whose first $n'$ components are 0 and all the other $m$ components are 1. Now, it is easy to see that $\{b_1,b_2,\ldots,b_{m+1}\}$ is a clique of size $m + 1$ in $H_3(Z^n_2)$. Note that $n$ satisfies the recursion relation $x_m = x_{m-1} + m - 1$, where $m \geq 4$ and $x_3 = 3$. □

The following corollary is an immediate consequence of the above theorem.
Corollary 3.8. The chromatic number of $H_3(Z_n^2)$ goes to infinity as $n$ approaches infinity. That is,

$$\lim_{n \to \infty} \chi(H_3(Z_n^2)) = \infty. \quad (3.27)$$

We conclude this section by posing a question on the isomorphism of the rings of 3-zero-divisor hypergraphs. In [6], it is shown that for any finite reduced commutative rings $A$ and $B$ which are not fields, then $\Gamma(A) \cong \Gamma(B)$ as graphs if and only if $A \cong B$ as rings. Furthermore, in [3], this result is generalized to the case that if $A$ is a finite reduced ring which is not isomorphic to $Z_2 \times Z_2$ or $Z_6$ with $B$ a ring such that $\Gamma(A) \cong \Gamma(B)$, then $A \cong B$. Also, in [7], it is shown that $A$ and its total quotient ring $T(A)$ have isomorphic zero-divisor graphs.

**Question 2.** Let $A$ and $B$ be two commutative rings. Under what condition(s) does the isomorphism of $H_3(A)$ and $H_3(B)$ imply the isomorphism of $A$ and $B$?

**Acknowledgments**

The authors would like to thank the referees for interest in the subject and making useful suggestions and comments which led to improvement and simplification of the first draft. This research was in part supported by Grant no. 84050015 from Institute for Studies in Theoretical Physics and Mathematics (IPM).

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