An element \( a \) in a ring \( R \) is called left morphic if there exists \( b \in R \) such that \( 1_R(a) = Ra \) and \( 1_R(b) = Ra \). \( R \) is called left morphic if every element of \( R \) is left morphic. An element \( a \) in a ring \( R \) is called left \( \pi \)-morphic (resp., left \( G \)-morphic) if there exists a positive integer \( n \) such that \( a^n \) (resp., \( a^n \) with \( a^n \neq 0 \)) is left morphic. \( R \) is called left \( \pi \)-morphic (resp., left \( G \)-morphic) if every element of \( R \) is left \( \pi \)-morphic (resp., left \( G \)-morphic). In this paper, the \( G \)-morphic problem and \( \pi \)-morphic problem of group rings are studied.

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1. Introduction

An element \( a \) in a ring \( R \) is said to be left morphic if \( R/Ra \cong 1_R(a) \), which is equivalent to that there exists \( b \in R \) such that \( 1_R(a) = Ra \) and \( 1_R(b) = Ra \), where \( 1_R(a) \) denotes the left annihilator of \( a \) in \( R \). \( R \) is called left morphic if every element of \( R \) is left morphic. Right morphic elements and rings are defined analogously. Nicholson and Sánchez Campos introduced and investigated left morphic rings in [1] (see also [2–4] for more detailed discussion).

Left morphic rings are generalized to left \( \pi \)-morphic rings and left \( G \)-morphic rings by Huang and Chen [5]. An element \( a \in R \) is called left \( \pi \)-morphic (resp., left \( G \)-morphic) if there exists a positive integer \( n \) such that \( a^n \) (resp., \( a^n \) with \( a^n \neq 0 \)) is left morphic. \( R \) is called left \( \pi \)-morphic (resp., left \( G \)-morphic) if every element of \( R \) is left \( \pi \)-morphic (resp., left \( G \)-morphic). \( R \) is called \( \pi \)-morphic (resp., \( G \)-morphic) if it is left and right \( \pi \)-morphic (resp., left and right \( G \)-morphic). Moreover, they find examples which show that left \( \pi \)-morphic rings are proper generalizations of left morphic rings, and left \( G \)-morphic elements need not be left morphic.

Example 1.1 [5, Example 2.13]. Let \( R = F[x, \sigma]/(x^2) = \{a + xb \mid a, b \in F\} \), where \( F \) is a field with an isomorphism \( \sigma \) from \( F \) to a subfield \( \overline{F} \neq F \) and \( cx = x\sigma(c) \) for all \( c \in F \).
S = R ⊕ R, then λ = (1, xb) ∈ S (where b ∈ F, but b /∈ F) is left G-morphic, but not left morphic.

The question of when a group ring is morphic was studied by Chen et al. [6]. In this paper, we investigate when a group ring is π-morphic (resp., G-morphic). In Section 2, several general results about π-morphic and G-morphic group rings are obtained. In Section 3, necessary and sufficient conditions for RG to be left G-morphic are also given, where R = Z_n, G is a finite Abelian group. In particular, we prove that if G is a finite Abelian group or a finite p-group, r ≥ 1, then Z^r_p = G is π-morphic.

All rings in this paper are associative rings with identity. Let R be a ring and let G be a group. We denote by RG the group ring of G over R. The following concepts in group rings play very important roles in our discussion and will be used frequently later. For any element u = Σai gi ∈ RG, where ai ∈ R, gi ∈ G, the augmentation of u, denoted by ε(u), is defined by ε(u) = Σai. The augmentation ideal of RG, denoted by Δ(G), is defined by Δ(G) = {u ∈ RG | ε(u) = 0}. If G is a cyclic group generated by g, then Δ(G) = RG(1 − g).

For any finite subgroup H of G,  H is defined to be H = ∑_{h∈H} h. When H is a normal subgroup, H is a central element in RG. For any group element g ∈ G of finite order, define  by  where o(g) is the order of g. It is not hard to verify that if o(g) < ∞, then  and if |G| < ∞, then  So if G is a finite cyclic group, then  is always left morphic in RG. For more background knowledge about group rings, we refer readers to [7, 8].

2. General results

In this section, several general results about π-morphic and G-morphic group rings are given.

Theorem 2.1. Let R be a ring and let G be a locally finite group. If RG is left π-morphic (resp., left G-morphic), then R is left π-morphic (resp., left G-morphic).

Proof. For any a ∈ R, since a is left π-morphic (resp., left G-morphic) in RG, there exist a positive integer n (resp., a^n ≠ 0) and u ∈ RG such that I_{RG(a^n)} = RGu and I_{RG(u)} = RGa^n. Let u = Σ_a gi and H = ⟨g, g⟩. Since G is a locally finite group, H is a finite group. Since a^n u = ua^n = 0, we have a^n ε(u) = ε(a^n u) = 0 and ε(u)a^n = ε(ua^n) = 0, where ε(u) is the augmentation of u. Thus Rb ⊆ I_{R(a^n)} and R a^n ⊆ I_{R(b)}, where b = ε(u). Next we show that in fact, Rb = I_{R(a^n)} and R a^n = I_{R(b)}. So a is left π-morphic (resp., left G-morphic) in R, and thus R is left π-morphic (resp., left G-morphic).

Let x ∈ I_{R(a^n)}. Then x ∈ I_{RG(a^n)} = RGu, so x = vu, v ∈ RG. Taking the augmentation on both sides, we obtain x = ε(x) = ε(vu) = ε(v)ε(u) = ε(v)b ∈ Rb. Therefore, I_{R(a^n)} ⊆ Rb, and thus I_{R(a^n)} = Rb. Next, let y ∈ I_{R(b)}. Then yb = 0. Let H = ∑_{h∈H} h. Since u ∈ RH, we have Hu = ε(u)H = bH. Thus yHu = ybH = H, so yH ∈ I_{RG(u)} = RGa^n. Hence yH = Σ a g a^n. Comparing the coefficients of the identity on both sides, we obtain that y = a g a^n ∈ R a^n, and so I_{R(b)} ⊆ R a^n. This implies that I_{R(b)} = R a^n. Therefore, a is left π-morphic (resp., left G-morphic) and so is R.

Corollary 2.2. If G = H × K is a locally finite group and RG is left π-morphic (resp., left G-morphic), then RH and RK are both left π-morphic (resp., left G-morphic).
Proof. Note that $RG = R(H \times K) \cong (RH)K$. By Theorem 2.1, $RH$ is left $\pi$-morphic (resp., left $G$-morphic). Similarly $RK$ is left $\pi$-morphic (resp., left $G$-morphic).

Theorem 2.3. Let $G$ be a locally finite group. If $RH$ is left $\pi$-morphic (resp., left $G$-morphic) for every finite subgroup $H$ of $G$, then $RG$ is left $\pi$-morphic (resp., left $G$-morphic).

Proof. Let $u = \sum_{i=1}^{n} a_i g_i$. Now we show that $u$ is left $\pi$-morphic (resp., left $G$-morphic) in $RG$. Denote $H = \langle g_1, \ldots, g_n \rangle$. Since $G$ is locally finite, $H$ is a finite group. By the assumption, $RH$ is left $\pi$-morphic (resp., left $G$-morphic). Since $u \in RH$, there exist a positive integer $n$ (resp., $u^n \neq 0$) and $v \in RH$ such that $I_{RH}(u^n) = RHc$ and $I_{RH}(c) = RHu^n$. Since $u^n c = cu^n = 0$, we have $RGc \subseteq I_{RG}(u^n)$ and $RGu^n \subseteq I_{RG}(c)$. We next show that the other inclusions also hold.

Let $v \in I_{RG}(u^n)$ and let $\{1, g_1', g_2', \ldots\}$ be a left coset representative of $H$ in $G$. That is, $G = H \cup g_1'H \cup g_2'H \cup \cdots$. Now $v$ can be written as $v = \sum g_i b_i$, where $b_i \in RH$. Since $0 = v u^n = \sum g_i (b_i u^n)$ and $b_i u^n \in RH$, we obtain that $b_i u^n = 0$ for all $i$. So $b_i \in I_{RH}(u^n) = RHc$, and thus $b_i = c_i c$ for some $c_i \in RH$. It follows that $v = \sum g_i b_i = \sum (g_i c_i) c \in RGc$, so $I_{RG}(u^n) \subseteq RGc$, and thus $I_{RG}(c) = RGc$. Similarly, we can prove that $I_{RG}(c) = RGc$. This shows that $u$ is left $\pi$-morphic (resp., left $G$-morphic) in $RG$, and therefore $RG$ is left $\pi$-morphic (resp., left $G$-morphic).

Recall that a group $G$ is called a semidirect product of $H$ by $K$, denoted by $G = H \rtimes K$, if $H, K$ are subgroups of $G$ such that (1) $H \leq G$; (2) $HK = G$; (3) $H \cap K = 1$.

Theorem 2.4. Let $G = H \rtimes K$, $|H| < \infty$. If $RG$ is left $\pi$-morphic (resp., left $G$-morphic), then $RK$ is also left $\pi$-morphic (resp., left $G$-morphic).

Proof. We show that for any $a \in RK$, $a$ is left $\pi$-morphic (resp., left $G$-morphic) in $RK$. Since $a$ is left $\pi$-morphic (resp., left $G$-morphic) in $RG$, there exist a positive integer $n$ (resp., $a^n \neq 0$) and $u \in RG$ such that $I_{RG}(a^n) = RGu$ and $I_{RG}(u) = RGa^n$. Let $u = \sum u_i k_i$, where $u_i \in RH$, $k_i \in K$ (since $G = H \rtimes K$, the expression of $u$ is unique) and $a^n = \sum a_j k_j$ where $a_j \in R$. Denote $b = \sum \epsilon(u_i) k_i$, so $b \in RK$. We will show that $I_{RK}(a^n) = RKb$ and $I_{RK}(b) = RKa^n$. So $a$ is left $\pi$-morphic (resp., left $G$-morphic) in $RK$, and thus $RK$ is left $\pi$-morphic (resp., left $G$-morphic).

Let $\omega : G \to G/H$ be the natural group homomorphism. We extend $\omega$ to a ring homomorphism (still denote it by $\omega$). That is, $\omega : RG \to R(G/H)$ defined by $\omega(\sum a_i g_i) = \sum a_i \omega(g_i)$. Clearly, $\ker(\omega) \cap RK = \{0\}$ and $\omega(v) = \epsilon(v)$ for all $v \in RH$. Since $0 = a^n u$, we have $0 = \omega(a^n) \omega(u) = \omega(a^n) \omega(\sum u_i k_i) = \omega(a^n) \sum \epsilon(u_i) \omega(k_i) = \omega(a^n) \sum \epsilon(u_i) \omega(k_i) = \omega(a^n) b$. Since $a^n b \in RK$, we conclude that $a^n b = 0$. Similarly, $b a^n = 0$. This shows that $RGb \subseteq I_{RK}(a^n)$ and $R Ga^n \subseteq I_{RK}(b)$. We next show that the other inclusions also hold.

Let $x \in I_{RK}(a^n)$. Then $x \in RG(a^n) = RGu$. So $x = vu$. Let $v = \sum v_j k_j$ and $c = \sum \epsilon(v_j) k_j$, where $v_j \in RH$, $k_j \in K$. Then $\omega(x) = \omega(v) \omega(u) = \sum \epsilon(v_j) \omega(k_j) \sum \epsilon(u_i) \omega(k_i) = \omega(c b)$. Thus $x - cb \in \ker(\omega) \cap RK = \{0\}$. Therefore $x = cb \in RKb$. This shows that $I_{RK}(a^n) \subseteq RKb$, and thus $I_{RK}(a^n) = RKb$.

Let $y \in I_{RK}(b)$. Then $yb = 0$. Since $H \subseteq G$, $\hat{H} = \sum_{h \in H} h$ is central in $RG$. Now we have $y \hat{H} u = y \hat{H} \sum u_i k_i = y \hat{H} \epsilon(u_i) k_i = yb \hat{H} = yb \hat{H} = 0$. So $y \hat{H} \in I_{RG}(u) = RGa^n$. Thus $\hat{H} y = y \hat{H} = wa^n$, where $w = \sum h_j u_j$, $h_j \in H$, $u_j \in RK$. Hence
Since $H \cap K = \{1\}$, the expression of $wa^n$ is unique. Comparing the coefficients of the identity $h_0 = e$ in (2.1), we obtain $y = u_0a^n \in R\!K\!a^n$. Thus $I_{R\!K}(b) \subseteq R\!K\!a^n$, and therefore $I_{R\!K}(b) = R\!K\!a^n$. \hfill $\square$

From now on, we always assume that $G$ is a finite group.

**Proposition 2.5.** Assume that $p$ is a prime number and $r > 1$. If $\mathbb{Z}_{p^r} G$ is left $G$-morphic, then $p$ does not divide $|G|$.

**Proof.** Assume that $p \mid |G|$. Then there exists $g \in G$ such that $o(g) = p$. Let $u = p^{r-1} \hat{G}$, where $\hat{G} = \sum_{g \in G} g$. Since $u$ is left $G$-morphic in $\mathbb{Z}_{p^r} G$, there exists a positive integer $n$ such that $u^n$ is left morphic in $\mathbb{Z}_{p^r} G$. Since $u^2 = 0$, $u$ is left morphic in $\mathbb{Z}_{p^r} G$. By Chen et al. [6, Theorem 2.7], this is impossible. So $p \nmid |G|$. \hfill $\square$

**Theorem 2.6.** Assume that $p$ is a prime number and $G$ is a finite $p$-group. $\mathbb{Z}_{p^r} G$ is left $G$-morphic if and only if $G$ is a cyclic group and $r = 1$.

**Proof.** “⇒” It follows from Proposition 2.5 that $r = 1$. Since $R = \mathbb{Z}_p$ is a field and $G$ is a finite $p$-group, $RG$ is a local ring by Nicholson theorem [9]. Because $RG$ is left Artinian, the Jacobson radical $J(RG)$ is nilpotent. Since $RG$ is left $G$-morphic, $RG$ is left special by Huang and Chen [5, Theorem 2.8]. So it is left morphic. According to Chen et al. [6, Theorem 2.9], $G$ is a cyclic group.

“⇐” If $G = \langle g \rangle$, clearly $\mathbb{Z}_{p^r} G$ is a special ring. Therefore it is left $G$-morphic. \hfill $\square$

**Theorem 2.7.** Assume that $p$ is a prime number and $G$ is a finite $p$-group, $r \geq 1$, then $\mathbb{Z}_{p^r} G$ is $\pi$-morphic.

**Proof.** Since $R = \mathbb{Z}_{p^r}$ is local and $G$ is a finite $p$-group, $RG$ is a local ring by Nicholson’s theorem [9]. Because $R$ is Artinian and $G$ is a finite group, $RG$ is Artinian by Connell [10, Theorem 1], and so the Jacobson radical $J(RG)$ is nilpotent. According to Huang and Chen [5, Lemma 2.10], every element of $RG$ is either nilpotent or invertible. So $RG$ is $\pi$-morphic. \hfill $\square$

**Remark 2.8.** By Theorem 2.6, when $r > 1$ and $G$ is a finite $p$-group, $\mathbb{Z}_{p^r} G$ is not left $G$-morphic, but by the above theorem, it is $\pi$-morphic.

### 3. Abelian group rings

In this section, we discuss when an Abelian group ring $RG$ is left $\pi$-morphic (resp., left $G$-morphic).

**Lemma 3.1** [6, Lemma 3.1]. $(R_1 \oplus R_2 \oplus \cdots \oplus R_s)G \cong \oplus_{i=1}^s R_i G$.

**Lemma 3.2.** If $R = R_1 \oplus R_2 \oplus \cdots \oplus R_s$ is left $\pi$-morphic (resp., left $G$-morphic), then each $R_i$ is left $\pi$-morphic (resp., left $G$-morphic).

**Proof.** For any $r_i \in R_i$, $r = (0, \ldots, 0, r_i, 0, \ldots, 0) \in R$. Since $R$ is left $\pi$-morphic (resp., left $G$-morphic), there exist $u = (u_1, \ldots, u_{i-1}, u_i, \ldots, u_s) \in R$, where $u_k \in R_k$, $k = 1, \ldots, s$, and
a positive integer \( n \) (resp., \( r^n \neq 0 \)) such that \( I_R(u) = Rr^n \) and \( I_R(r^n) = Ru \), so we have \( I_R(u) = Rr^n \) and \( I_R(r^n) = Ru \). Then \( r_i \) is left \( \pi \)-morphic (resp., left \( G \)-morphic) in \( R_i \), and thus \( R_i \) is left \( \pi \)-morphic (resp., left \( G \)-morphic).

\[ \square \]
\[ Z_{p_i}G \oplus Z_{n_i}G. \] By Lemma 3.2, \( Z_{p_i}G \) is \( G \)-morphic. Since \( Z_{p_i}G \cong Z_{p_i}(C_{q_i^p} \times C_{q_i^m}) \), we conclude that \( Z_{p_i}(C_{q_i^p} \times C_{q_i^m}) = Z_{p_i}(C_{p_i^2} \times C_{p_i^2}) \) is \( G \)-morphic. This contradicts the result of Theorem 2.6. Therefore, \( p^2 \nmid \alpha \), and thus \( G_{p_i} \) is cyclic.

**Remark 3.6.** According to Proposition 3.4 and Theorem 3.5, the following group rings are not \( G \)-morphic:

\[
Z_4C_2, \quad Z_4C_4, \quad Z_4(C_2 \times C_2), \quad Z_2(C_2 \times C_2), \quad Z_2(C_2 \times C_4). \tag{3.1}
\]

But by Theorem 2.7, the above group rings are all \( \pi \)-morphic.

**Lemma 3.7.** Let \( R \) be a ring and let \( G \) be a group. If \( a \in R \) is left morphic in \( R \), then \( a \) is left morphic in \( RG \).

**Proof.** If \( a \in R \) is left morphic, there exists \( b \in R \) such that \( l_R(a) = Rb \) and \( l_R(b) = Ra \)

Since \( ba = ab = 0 \), we have \( Rb \subseteq l_{RG}(a) \) and \( Ra \subseteq l_{RG}(b) \). We next show that the other inclusions also hold.

Let \( x \in l_{RG}(a) \), \( x = \sum r_i g_j \), where \( r_i \in R \), \( g_j \in G \). Then \( \sum r_i g_j a = 0 \) or \( \sum (r_i a) g_j = 0 \), so all \( r_i a = 0 \). Thus \( r_j \in Rb \) and \( r_j = r'_j b, r'_j \in R \). Therefore, \( x = \sum (r'_j b) g_j = \sum r'_j g_j b \in R G b \). This shows that \( l_{RG}(a) \subseteq R G b \), and thus \( l_{RG}(a) = R G b \).

Using a similar proof, we can show that \( l_{RG}(b) \subseteq R G a \), and thus \( l_{RG}(b) = R G a \). So \( a \) is left morphic in \( RG \).

Recall that if \( n = p^u m_1 \), \((n_1, p) = 1 \), we denote that \( p^u \mid n \).

**Lemma 3.8.** Let \( p \) be a prime number, \( r \geq 1 \), \( p^r \mid m \), and \( 1 \leq n \leq m \).

1. If \( (p, n) = 1 \), then \( p^r \mid C_m^n \).
2. If \( p^r \mid n \), \( r \geq t \), then \( p^{r-t} \mid C_m^n \).

**Proof.** Let \( m = m_1 p^r, (m_1, p) = 1 \). Then

\[
C_m^n = \frac{m(m-1) \cdots (m-(n-1))}{1 \cdots (n-1)n} = \frac{m}{n} C_{m-1}^{n-1} = \frac{m_1 p^r}{n} C_{m-1}^{n-1}. \tag{3.2}
\]

1. If \( (p, n) = 1 \), then \( (p^r, n) = 1 \), so \( p^r \mid C_m^n \).
2. If \( p^r \mid n \), \( t \leq r \), then \( n = n_1 p^t \), where \( (p, n_1) = 1 \), so

\[
C_m^n = \frac{m_1 p^r}{n} C_{m-1}^{n-1} = \frac{m_1 p^r}{n_1 p^t} C_{m-1}^{n-1} = \frac{m_1 p^{r-t}}{n_1} C_{m-1}^{n-1}. \tag{3.3}
\]

We have \( p^{r-t} \mid C_m^n n_1 \). Since \( (p, n_1) = 1 \), \((p^{r-t}, n_1) = 1 \), so \( p^{r-t} \mid C_m^n \).

**Proposition 3.9.** Let \( p \) be a prime number and let \( G \) be a finite Abelian group. If for some \( r, t \geq 1 \), \( x \in Z_{p^r}(C_{p^r} \times G) = Z_{p^r}((g) \times G) \), then \( x p^t \in Z_{p^r}(C_{p^r-t} \times G) = Z_{p^r}((g^t) \times G) \).
Proof. For $x \in \mathbb{Z}_{p'}(C_{p'} \times G) = (\mathbb{Z}_{p'} G)C_{p'} = (\mathbb{Z}_{p'} G)(g)$, $x = r_0 + r_1 g + \cdots + r_{p'-1} g^{p'-1}$, where $r_i \in \mathbb{Z}_{p'} G$. Since

$$(x_1 + x_2 + \cdots + x_s)^k = \sum_{k_1=0}^{k} \sum_{k_2=0}^{k} \cdots \sum_{k_{s-1}=0}^{k} C_{k_1}^{k_1} C_{k_2}^{k_2} \cdots C_{k_{s-1}}^{k_{s-1}} x_1^{k_1} x_2^{k_2} \cdots x_{s-1}^{k_{s-1}} x_s^{k_{s-1}},$$

$x^{p'} = (r_0 + r_1 g + \cdots + r_{p'-1} g^{p'-1})^{p'}$

$$= \sum_{n=0}^{p'-1} \sum_{n'_{i=1}}^{n_{p'-1}} C_{n_1}^{n_1} C_{n_2}^{n_2} \cdots C_{n_{p'-1}}^{n_{p'-1}} (r_{n_1} g)^{n_1-n_2} \cdots (r_{p'-1} g^{p'-1})^{n_{p'-1}}.$$ (3.4)

Claim 3.10. Let $n_i$ be the first number in $n_1, \ldots, n_{p'-1}$ such that $n_i$ is not divisible by $p$. Then $p' \mid C_{n_1}^{n_1} \cdots C_{n_{i-1}}^{n_{i-1}} C_{n_i}^{n_i}$.

Proof. If $i = 1$, then $(n_1, p) = 1$, and by Lemma 3.8, $p' \mid C_{n_1}^{n_1}$.

Now we set $i > 1$. Let $n_k = n_k' p^{u_k}$, $1 \leq k \leq i - 1$, where $(n_k', p) = 1$. Since $C_{n_{k-1}}^{n_{k-1}} = C_{n_{k-1}-n_k}^{n_{k-1}-n_k}$, we can assume that $u_k \leq u_{k-1}$. By Lemma 3.8, we have $p^{u_i-1-u_k} \mid C_{n_k}^{n_k}, 1 \leq k \leq i - 1$, and $p^{u_i-1} \mid C_{n_i}^{n_i}$ because $(p, n_i) = 1$. So

$$p^{(r_{u_1-1}+u_{1-2}+\cdots+(u_{i-2}-u_{i-1})+u_{i-1})} \mid C_{n_1}^{n_1} C_{n_2}^{n_2} \cdots C_{n_{i-1}}^{n_{i-1}}.$$ (3.5)

Hence, $p' \mid C_{n_1}^{n_1} C_{n_2}^{n_2} \cdots C_{n_{i-1}}^{n_{i-1}}$.

By the above claim, if there exists $n_i$ such that $p \mid n_i$, then $C_{n_1}^{n_1} \cdots C_{n_{i-1}}^{n_{i-1}} = 0$ in $\mathbb{Z}_{p'}$. So assume that $p \mid n_j, j = 1, \ldots, p' - 1$, and then we have

$$x^{p'} = \sum_{p \mid n_1, 0 \leq n_1 \leq p'} \sum_{p \mid n_2, 0 \leq n_2 \leq n_1} \cdots \sum_{p \mid n_{p'-1}, 0 \leq n_{p'-1} \leq n_{p'-2}} C_{n_1}^{n_1} C_{n_2}^{n_2} \cdots C_{n_{p'-1}}^{n_{p'-1}} (r_{n_1} g)^{n_1-n_2} \cdots (r_{p'-1} g^{p'-1})^{n_{p'-1}}.$$ (3.6)

$$= \sum c_i (g^p)^i \in (\mathbb{Z}_{p'} G)(g^p) = (\mathbb{Z}_{p'} G)C_{p'}.$$ □

Theorem 3.11. If $p$ is a prime number, $r \geq 1$, and $G$ is a finite Abelian group, then $\mathbb{Z}_{p'} G$ is $\pi$-morphic.

Proof

Case 1. If $(p, |G|) = 1$, then $(p', |G|) = 1$. By Chen et al. [6, Corollary 3.13], $\mathbb{Z}_{p'} G$ is morphic, so $\mathbb{Z}_{p'} G$ is $\pi$-morphic.

Case 2. If $p \mid |G|$, then $G = C_{p^1} \times \cdots \times C_{p^i} \times H$, where $(p, |H|) = 1$. Now if $x \in \mathbb{Z}_p G = \mathbb{Z}_p (C_{p^1} \times \cdots \times C_{p^i} \times H)C_{p^1}$, then $x^{p'} \in \mathbb{Z}_{p'} (C_{p^1} \times \cdots \times C_{p^i} \times H)C_{p^1}$ by Proposition 3.9. So we have $x^{k_1} \in \mathbb{Z}_{p'} (C_{p^2} \times \cdots \times C_{p^i} \times H)$ for some $k_1$. Continuing the process, we get $x^n \in \mathbb{Z}_{p'} H$ for some $n$. By Chen et al. [6, Corollary 3.13], $\mathbb{Z}_p H$ is morphic. So $x^n$ is
Thus $x^n$ is morphic in $\mathbb{Z}_p H$ by Lemma 3.7. Hence $x$ is $\pi$-morphic in $\mathbb{Z}_p G$.

Acknowledgments

The authors would like to thank the referee for careful reading of the paper and for pointing out some typos and minor errors. The research was supported by the National Natural Science Foundation of China (no. 10571026), the Natural Science Foundation of Jiangsu Province (no. BK2005207), and the Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutes of MOE, China. The authors would like to thank Professor Yuanlin Li for reading of the paper and making some valuable comments and suggestions.

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