Spectral theory from the second-order $q$-difference operator $\Delta_q$ is developed. We give an integral representation of its inverse, and the resolvent operator is obtained. As application, we give an analogue of the Poincare inequality. We introduce the Zeta function for the operator $\Delta_q$ and we formulate some of its properties. In the end, we obtain the spectral measure.

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1. Basic definitions

Consider $0 < q < 1$. In what follows, the standard conventional notations from [1] will be used

$$
\mathbb{R}_q = \{ \mp q^n, n \in \mathbb{Z}\}, \quad \mathbb{R}_q^+ = \{ q^n, n \in \mathbb{Z}\},
$$

$$
(a,q)_0 = 1, \quad (a,q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \quad [n]_q = \frac{1 - q^n}{1 - q}.
$$

The $q$-shift operator is

$$
\Lambda_q f(x) = f(qx).
$$
Next, we introduce two concepts of $q$-analysis: the $q$-derivative and the $q$-integral. The $q$-derivative (see [2]) of a function $f$ is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x},$$

and the second-order $q$-difference operator is

$$\Delta_q f(x) = \left[ 1 - \frac{q}{q} \right] \Lambda_q^{-1} D_q^2 f(x) = \frac{1}{x^2} \left[ f(q^{-1}x) - \frac{1 + q}{q} f(x) + \frac{1}{q} f(qx) \right].$$

The product rule for the $q$-derivative is

$$D_q (fg)(x) = D_q f(x)g(x) + \Lambda_q f(x)D_q g(x).$$

Jackson’s $q$-integral (see [3]) in the interval $[a, b]$ is defined by

$$\int_a^b f(x) dqx = (1 - q) \sum_{n=0}^\infty q^n [ b f(bq^n) - a f(aq^n) ].$$

Also the rule of $q$-integration by parts is given by

$$\int_a^b D_q f(x)g(x) dqx = \left[ f(b)g(b) - f(a)g(a) \right] - \int_a^b \Lambda_q f(x)D_q g(x) dqx.$$

The Hahn-Exton $q$-Bessel function of order $\alpha > -1$ (see [4–6]) is defined by

$$J_{\alpha}^{(3)}(x, q) = \frac{(q^{\alpha+1}, q)_{\infty}}{(q, q)_{\infty}} x^{\alpha} \phi_1(0, q^{\alpha+1}, q; qx^2).$$

The $q$-trigonometric functions (see [7]) are defined on $\mathbb{C}$ by

$$\cos(x, q^2) = 1 \phi_1(0, q^2; q^2 x^2)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q, q)_{2n}} x^{2n} = \frac{(q^2, q^2)_{\infty}}{(q, q^2)_{\infty}} \sqrt{x} J^{(3)}_{-1/2}(x, q^2),$$

$$\sin(x, q^2) = (1 - q)^{-1} x \phi_1(0, q^3, q^2; q^2 x^2)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q, q)_{2n+1}} x^{2n+1} = \frac{(q^2, q^2)_{\infty}}{(q, q^2)_{\infty}} \sqrt{x} J^{(3)}_{1/2}(x, q^2).$$

Both functions $\cos(x, q^2)$ and $\sin(x, q^2)$ are analytic. In [7], it is proved that

$$\Delta_q f(x) = \begin{cases} -\lambda^2 f(x), & \text{if } f(x) = \cos(\lambda x, q^2) \\ -\frac{1}{q} \lambda^2 f(x), & \text{if } f(x) = \sin(\lambda x, q^2). \end{cases}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q, q)_{2n}} x^{2n+1} = \frac{(q^2, q^2)_{\infty}}{(q, q^2)_{\infty}} \sqrt{x} J^{(3)}_{1/2}(x, q^2).$$

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Let $\mathcal{L}_{q,2}$ be the space of all real-valued functions defined on

$$[0,1]_q = \{q^n, \ n = 0, 1, \ldots\},$$

such that

$$\|f\|_{\mathcal{L}_{q,2}} = \left(\int_0^1 |f(x)|^2 d_q x\right)^{1/2} < \infty.$$  \hfill (1.12)

Then, $\mathcal{L}_{q,2}$ is a separable Hilbert space with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)d_q x.$$  \hfill (1.13)

In the following, we denote by $\mathcal{D}$ the subspace of $\mathcal{L}_{q,2}$ defined by

$$\mathcal{D} = \{ f \in \mathcal{L}_{q,2}, \Delta_q f \in \mathcal{L}_{q,2}, \lim_{n \to \infty} f(q^n) = f(1) = 0 \}.$$  \hfill (1.14)

2. Eigenfunctions of $\Delta_q$ in $\mathcal{D}$

**Theorem 2.1.** $(\Delta_q, \mathcal{D})$ has an infinite sequence of nonzero real eigenvalues

$$\{ \eta_n = -\lambda_n^2 \}_{n \in \mathbb{N}^*},$$

where $0 < \lambda_1 < \lambda_2 < \cdots$ are the positive zeros of the following function:

$$x \mapsto \sin(\sqrt{q} x, q^2).$$

The corresponding set of eigenfunctions is

$$\{ \sin(\sqrt{q} \lambda_n x, q^2) \}_{n \in \mathbb{N}^*}.$$  \hfill (2.3)

**Proof.** For $f, g \in \mathcal{D}$, using the $q$-integration by parts we write

$$\int_0^1 \Delta_q^{-1}D_q^2 f(x)g(x)d_q x = q^2 \int_0^1 D_q^2 \Delta_q^{-1} f(x)g(x)d_q x
$$

$$= -q^2 \int_0^1 \Delta_q D_q ^{-1} f(x)D_q g(x)d_q x
$$

$$= -q \int_0^1 D_q f(x)D_q g(x)d_q x,$$

hence

$$\int_0^1 \Delta_q f(x)g(x)d_q x = \int_0^1 f(x)\Delta_q g(x)d_q x.$$  \hfill (2.5)

In particular, we can write

$$\int_0^1 \Delta_q f(x)f(x)d_q x = -\frac{(1-q)^2}{q} \int_0^1 [D_q f(x)]^2 d_q x.$$  \hfill (2.6)
Let $\sigma_p$ be the sequence of eigenvalues of $(\Delta_q, \mathcal{D})$. Then

$$\sigma_p \subset \mathbb{R}.$$  \hspace{1cm} (2.7)

Let $f \in \mathcal{D}$ be a function satisfying the $q$-differential equation

$$\Delta_q f(x) = -\lambda^2 f(x), \quad \forall x \in \mathbb{R}_q^+.$$  \hspace{1cm} (2.8)

Using (1.4), we write

$$q^{-2n}\left[ f(q^{n-1}) - \frac{(1+q)}{q} f(q^n) + f(q^{n+1}) \right] = -\lambda^2 f(q^n), \quad \forall n \in \mathbb{Z}.$$  \hspace{1cm} (2.9)

Therefore the set of solution of (2.8) is a vector space over $\mathbb{R}$ of dimension 2. From (1.10), it follows that $f$ can be written in the form

$$f(x) = a \sin(\sqrt{q}\lambda x, q^2) + b \cos(\lambda x, q^2), \quad a, b \in \mathbb{R}.$$  \hspace{1cm} (2.10)

If $f$ satisfies

$$\lim_{n \to \infty} f(q^n) = f(1) = 0,$$  \hspace{1cm} (2.11)

then $f(x) = a \sin(\sqrt{q}\lambda x, q^2)$ and $\sin(\sqrt{q}\lambda, q^2) = 0$. In [5], it is proved that The Hahn-Exton $q$-Bessel function of order $\alpha > -1$ has a countably infinite number of positive simple zeros. This finishes the proof. \hspace{1cm} $\square$

3. The inverse of $\Delta_q$ in the space $\mathcal{D}$

Given $x = q^s \in \mathbb{R}_q^+$, we define

$$[x,1]_q = \{q^n, \, n = 0 \cdots s\}, \quad [0,x]_q = \{q^n, \, n = s \cdots \}.$$  \hspace{1cm} (3.1)

We introduce the operator

$$u_k : \mathcal{L}_{q,2} \longrightarrow u_k(\mathcal{L}_{q,2})$$  \hspace{1cm} (3.2)

defined by

$$u_k(f)(x) = \frac{q}{(1-q)^2} \int_0^1 k(x,y) f(y) dq_y,$$  \hspace{1cm} (3.3)

where

$$k(x,y) = \begin{cases} 
  x(y-1) & \text{if } y \in [x,1]_q \\
  y(x-1) & \text{if } y \in [0,x]_q 
\end{cases}.$$  \hspace{1cm} (3.4)

**Theorem 3.1.** The operator $u_k$ is the inverse of $\Delta_q$ in the space $\mathcal{D}$.  \hspace{1cm} \square
Proof. For \( f \in \mathcal{S}_{q,2} \), we write

\[
uk(f)(x) = \frac{q}{(1-q)^2} \left[ (x-1) \int_0^x yf(y)dy + x \int_x^1 (y-1)f(y)dy \right].
\]  
(3.5)

Then

\[
uk(f) \in \mathcal{S}_{q,2}, \quad \lim_{n \to \infty} uk(f)(q^n) = uk(f)(1) = 0.
\]  
(3.6)

On the other hand, using (1.5) we write

\[
D_quk(f)(x) = \frac{q}{(1-q)^2} \left[ (x-1)xf(x) + \int_0^{qx} yf(y)dy - x(x-1) f(x) + \int_{qx}^1 (y-1)f(y)dy \right]
\]

\[
= \frac{q}{(1-q)^2} \left[ \int_0^{qx} yf(y)dy + \int_{qx}^1 (y-1)f(y)dy \right],
\]

(3.7)

which shows that

\[
\Delta_q \circ uk(f)(x) = f(x).
\]  
(3.8)

We conclude that

\[
uk(f) \in \mathcal{D}, \quad \Delta_q \circ uk = \text{id}_{\mathcal{S}_{q,2}}.
\]  
(3.9)

Similarly, we can prove that

\[
uk \circ \Delta_q f(x) = f(x), \quad \forall f \in \mathcal{D}.
\]  
(3.10)

Indeed

\[
uk \circ \Delta_q(f)(x) = \frac{q}{(1-q)^2} \int_0^1 k(x,y)\Delta_q f(y)dy,
\]  
(3.11)

with

\[
k(x,0) = k(x,1) = 0,
\]  
(3.12)

and we obtain

\[
\frac{q}{(1-q)^2} \int_0^1 k(x,y)\Delta_q f(y)dy
\]

\[
= \frac{q}{(1-q)^2} \int_0^1 \Delta_q k(x,y) f(y)dy = \frac{q}{1-q} \sum_{n=0}^{\infty} q^n \Delta_q k(x,q^n) f(q^n).
\]  
(3.13)
Next, we have
\[ \Delta qk(x,y) = \frac{1}{y^2} \left[ k(x,q^{-1}y) - \frac{1+q}{q} k(x,y) + \frac{1}{q} k(x,xy) \right], \quad (3.14) \]
which implies
\[ \Delta qk(x,x) = \frac{1}{x^2} \left[ k(x,q^{-1}x) - \frac{1+q}{q} k(x,x) + \frac{1}{q} k(x,qx) \right] 
= \frac{1}{x^2} \left[ x(q^{-1}x - 1) - \frac{1+q}{q} x(x - 1) + \frac{1}{q} qx(x - 1) \right] = \frac{1-q}{x}. \quad (3.15) \]
Now we will prove that
\[ \Delta qk(x,y) = 0 \quad \text{if } x \neq y. \quad (3.16) \]
For \( y \in [0,x]_q \), we have
\[ \Delta qk(x,y) = \frac{1}{y^2} \left[ (x-1)q^{-1}y - \frac{1+q}{q} (x-1)y + \frac{1}{q} (x-1)xy \right] = 0, \quad (3.17) \]
and if \( y \in [x,1]_q \),
\[ \Delta qk(x,y) = \frac{1}{y^2} \left[ (y-1)q^{-1}x - \frac{1+q}{q} (y-1)x + \frac{1}{q} (y-1)qx \right] = 0. \quad (3.18) \]
Therefore
\[ \frac{q}{(1-q)^2} \int_0^1 k(x,y) \Delta q f(y) d_q y = f(x). \quad (3.19) \]
This finishes the proof. \( \square \)

**Corollary 3.2.** The sequence
\[ \{ \sin (\sqrt{q} \lambda_n x, q^2) \}_{n \in \mathbb{N}}, \quad (3.20) \]
is an orthogonal basis of \( \mathcal{L}_{q,2} \).

**Proof.** Since \( u_k \) is a Hilbert-Schmidt operator, then \( u_k \) is normal and compact because \( \mathcal{L}_{q,2} \) is separable. The eigenfunctions of \( u_k \) are the elements of the sequence
\[ \{ \sin (\sqrt{q} \lambda_n x, q^2) \}_{n \in \mathbb{N}}, \quad (3.21) \]
associated with corresponding eigenvalues
\[ \eta_n = -\frac{1}{\lambda_n^2}, \quad (3.22) \]
and they form an orthogonal basis of \( \mathcal{L}_{q,2} \). \( \square \)
4. Resolvent operator and Green kernel

We introduce the $q$-hyperbolic sine and the $q$-hyperbolic cosine function as follows:

$$\sinh(x, q^2) = -i \sin(ix, q^2), \quad \cosh(x, q^2) = \cos(ix, q^2).$$  \hspace{1cm} (4.1)

For $z \in \mathbb{C}/\{\sigma_p\}$, we have the following result.

Theorem 4.1. The $q$-Sturm-Liouville problem

$$\Delta_q U(x) = zU(x) - f(x), \quad U \in \mathbb{D}, \hspace{1cm} (4.2)$$

has a unique solution in the form

$$U(x) = (z - \Delta_q)^{-1}f(x) = \int_0^1 G_{q,z}(x, y)\Lambda_q f(y) dy, \hspace{1cm} (4.3)$$

where $G_{q,z}$ is the Green kernel defined by

$$G_{q,z}(x, y) = -\frac{q^2}{(1-q)^2} \frac{1}{\sqrt{qz}\sinh(\sqrt{qz}, q^2)} \begin{cases}
U_1(x)U_2(qy), & y \in [x,1]_q \\
U_1(qy)U_2(x), & y \in [0,x]_q
\end{cases} \hspace{1cm} (4.4)$$

and $U_1$ and $U_2$ are defined by

$$U_1(x) = \sinh(\sqrt{qz}x, q^2), \hspace{1cm} (4.5)$$

$$U_2(x) = \cosh(\sqrt{qz}x, q^2) \sinh(\sqrt{qz}, q^2) - \sinh(\sqrt{qz}x, q^2) \cosh(\sqrt{qz}, q^2).$$

Proof. We will solve this $q$-problem using the $q$-analogue of the method of variation of constants. We write $U$ in the following form:

$$U(x) = U_1(x)V_1(x) + U_2(x)V_2(x). \hspace{1cm} (4.6)$$

Note that $U_1$ and $U_2$ form a fundamental solution set of the $q$-difference equation

$$\Delta_q U(x) = zU(x). \hspace{1cm} (4.7)$$

Using (1.5), we write

$$D_q U(x) = D_q U_1(x)V_1(x) + \Lambda_q U_1(x)D_q V_1(x) + D_q U_2(x)V_2(x) + \Lambda_q U_2(x)D_q V_2(x). \hspace{1cm} (4.8)$$

From the first condition

$$\Lambda_q U_1(x)D_q V_1(x) + \Lambda_q U_2(x)D_q V_2(x) = 0, \hspace{1cm} (4.9)$$

we get

$$D_q U(x) = D_q U_1(x)V_1(x) + D_q U_2(x)V_2(x). \hspace{1cm} (4.10)$$
Therefore
\[ D_q^2 U(x) = D_q V_1(x) D_q U_1(x) + \Lambda_q V_1(x) D_q^2 U_1(x) + D_q V_2(x) D_q U_2(x) + \Lambda_q V_2(x) D_q^2 U_2(x). \] (4.11)

From the second condition
\[ D_q V_1(x) D_q U_1(x) + D_q V_2(x) D_q U_2(x) = - \frac{q^2}{(1-q)^2} \Lambda_q f(x), \] (4.12)
we obtain
\[ D_q^2 U(x) = \Lambda_q V_1(x) D_q^2 U_1(x) + \Lambda_q V_2(x) D_q^2 U_2(x) - \frac{q^2}{(1-q)^2} \Lambda_q f(x). \] (4.13)

Conditions (4.9) and (4.12) form a linear system
\[
\begin{pmatrix}
\Lambda_q U_1(x) & \Lambda_q U_2(x) \\
D_q U_1(x) & D_q U_2(x)
\end{pmatrix}
\begin{pmatrix}
D_q V_1(x) \\
D_q V_2(x)
\end{pmatrix}
= \begin{pmatrix} 0 \\ - \frac{q^2}{(1-q)^2} \Lambda_q f(x) \end{pmatrix},
\] (4.14)
The solution of this system is
\[
D_q V_1(x) = \frac{q^2}{(1-q)^2 w(x)} \Lambda_q U_2(x) \Lambda_q f(x),
\] (4.15)
\[
D_q V_2(x) = - \frac{q^2}{(1-q)^2 w(x)} \Lambda_q U_1(x) \Lambda_q f(x),
\]
where
\[
w(x) = \det \begin{pmatrix} \Lambda_q U_1(x) & \Lambda_q U_2(x) \\ D_q U_1(x) & D_q U_2(x) \end{pmatrix} = \Lambda_q U_1(x) D_q U_2(x) - \Lambda_q U_2(x) D_q U_1(x) \] (4.16)
is the \( q \)-Wronskian of the \( q \)-Sturm-Liouville problem.

Now since
\[
\Lambda_q^{-1} w(x) = U_1(x) \Lambda_q^{-1} D_q U_2(x) - U_2(x) \Lambda_q^{-1} D_q U_1(x),
\] (4.17)
using (1.5), and the fact that \( U_1 \) and \( U_2 \) are fundamental solution of the \( q \)-difference equation \( \Delta_q U(x) = z U(x) \), we obtain
\[
D_q \Lambda_q^{-1} w(x) = D_q \Lambda_q^{-1} D_q U_2(x) U_1(x) + D_q U_2(x) D_q U_1(x) \\
- D_q \Lambda_q^{-1} D_q U_1(x) U_2(x) - D_q U_1(x) D_q U_2(x) \\
= \frac{q}{(1-q)^2} [\Delta_q U_2(x) U_1(x) - \Delta_q U_1(x) U_2(x)] = 0.
\] (4.18)
Therefore

\[ w(q^n) = \text{constant}, \quad \forall n \in \mathbb{Z}. \tag{4.19} \]

Finally, for \( x \in \mathbb{R}^+ \) we get

\[ w(x) = w(0) = U_1(0) D_q U_2(0) - U_2(0) D_q U_1(0). \tag{4.20} \]

Therefore the functions \( V_1 \) and \( V_2 \) satisfy

\[ D_q V_1(x) = \frac{q^2}{(1-q)^2 w(0)} \Lambda_q U_2(x) \Lambda_q f(x), \tag{4.21} \]

\[ D_q V_2(x) = -\frac{q^2}{(1-q)^2 w(0)} \Lambda_q U_1(x) \Lambda_q f(x), \]

which gives

\[ V_1(x) = -\frac{q^2}{(1-q)^2 w(0)} \int_x^1 \Lambda_q U_2(y) \Lambda_q f(y) dy, \tag{4.22} \]

\[ V_2(x) = -\frac{q^2}{(1-q)^2 w(0)} \int_0^x \Lambda_q U_1(y) \Lambda_q f(y) dy. \]

The condition \( U \in \mathcal{B} \) requires

\[ U_1(0) = U_2(1) = 0, \tag{4.23} \]

which implies

\[ w(0) = -U_2(0) D_q U_1(0). \tag{4.24} \]

Using the fact that

\[ D_q \sin (x, q^2) = \frac{1}{1-q} \cos (x, q^2), \tag{4.25} \]

we obtain

\[ w(0) = \frac{1}{1-q} \sqrt{qz} \sinh (\sqrt{qz}, q^2). \tag{4.26} \]

This completes the proof. \( \square \)

5. Poincare inequality

Here, we give a \( q \)-analogue of the Poincare inequality.

**Theorem 5.1.** Given \( f \in \mathcal{D} \), then

\[ \int_0^1 [f(x)]^2 d_q x \leq \frac{(1-q)^2}{q \lambda_1^2} \int_0^1 [D_q f(x)]^2 d_q x. \tag{5.1} \]
Proof. For \( f \in \mathbb{D} \), we write

\[
f(x) = \sum_{n=1}^{\infty} \frac{1}{k_n} a_n \sin (\sqrt{q} \lambda_n x, q^2) , \quad \forall x \in [0, 1],
\]

where

\[
a_n = \int_0^1 f(x) \sin (\sqrt{q} \lambda_n x, q^2) \, dq x, \quad k_n = \| \sin (\sqrt{q} \lambda_n x, q^2) \|_{L^2_q}^2.
\]

Therefore

\[
\Delta_q f(x) = -\sum_{n=1}^{\infty} \frac{1}{k_n} a_n \lambda_n^2 \sin (\sqrt{q} \lambda_n x, q^2), \quad x \in [0, 1].
\]

This implies

\[
\int_0^1 \Delta_q f(x) f(x) \, dq x = -\sum_{n=1}^{\infty} a_n^2 \lambda_n^2, \quad \int_0^1 [f(x)]^2 \, dq x = \sum_{n=1}^{\infty} a_n^2.
\]

Using that

\[
\int_0^1 \Delta_q^{-1} D_q^2 f(x) f(x) \, dq x = q \int_0^1 D_q f(x) \Delta_q^{-1} f(x) \, dq x
\]

\[
= q [D_q f(1) f(1) - D_q f(0) f(0)] - q \int_0^1 [D_q f(x)]^2 \, dq x
\]

\[
= -q \int_0^1 [D_q f(x)]^2 \, dq x,
\]

we obtain

\[
\int_0^1 [D_q f(x)]^2 \, dq x = -\frac{q}{(1-q)^2} \int_0^1 \Delta_q f(x) f(x) \, dq x
\]

\[
= \frac{q}{(1-q)^2} \sum_{n=1}^{\infty} a_n^2 \lambda_n^2 = \frac{q \lambda_1^2}{(1-q)^2} \sum_{n=1}^{\infty} \left( \frac{\lambda_n}{\lambda_1} \right)^2 a_n^2.
\]

From the inequality

\[
\frac{\lambda_n}{\lambda_1} \geq 1, \quad \text{for every } n \geq 1,
\]

we conclude that

\[
\int_0^1 [f(x)]^2 \, dq x \leq \frac{(1-q)^2}{q \lambda_1^2} \int_0^1 [D_q f(x)]^2 \, dq x.
\]

This completes the proof. \(\square\)
As an application, consider the function

\[ f(x) = x(x - 1). \]  

(5.10)

After simple calculations, we obtain

\[ \left( \frac{\lambda_1}{1 - q} \right)^2 \leq \frac{1/[3]_q}{1/[5]_q - 2/[4]_q + 1/[3]_q}. \]  

(5.11)

6. Zeta function for the operator \( \Delta_q \)

Theorem 6.1. (1) If \( q^3 < (1 - q^2)^2 \), then the Zeta function for the operator \( \Delta_q \)

\[ \zeta_q(s) = \sum_{p=1}^{\infty} \left( \frac{\lambda_1}{k_p} \right)^s \]  

(6.1)

is analytic in the region \( \{ s \in \mathbb{C}, \Re(s) > 0 \} \).

(2) For every \( n \in \mathbb{N}^* \),

\[ \zeta_q(2n) = \left( -\frac{q\lambda_2^2}{(1 - q^2)^2} \right)^n \int_0^1 \cdots \int_0^1 k(x_1, x_2) \cdots k(x_{n-1}, x_n) k(x_n, x_1) \ dx_1 \cdots dx_n. \]  

(6.2)

Proof. In [8], it is proved that if

\[ q^{2\alpha+2} < (1 - q^2)^2, \]  

(6.3)

then the positive roots \( w_k^{(\alpha)}(q^2) \) of the Hahn-Exton \( q \)-Bessel function \( j^{(3)}_{\alpha}(x, q) \) satisfy

\[ \lim_{k \to \infty} q^k w_k^{(\alpha)}(q^2) = 1. \]  

(6.4)

Since

\[ \sin(x, q^2) = \frac{(q^2, q^2)}{(q, q^2)} \sqrt{x} j^{(3)}_{1/2}(x, q^2), \]  

(6.5)

we have

\[ \lambda_k = \frac{w_k^{(1/2)}(q^2)}{\sqrt{q}} \sim \frac{1}{\sqrt{q}} q^{-k}, \]  

(6.6)

which leads to the first result.

To prove the second result we use the Mercer theorem for the operator \( u_k \)

\[ \frac{q}{(1 - q^2)} k(x, y) = -\sum_{p=1}^{\infty} \frac{1}{k_p} \frac{1}{\lambda_p} \sin(\sqrt{q} \lambda_p x, q^2) \sin(\sqrt{q} \lambda_p y, q^2). \]  

(6.7)
and the orthogonality relations

\[ \frac{1}{k_p} \int_0^1 \sin(\sqrt{q}\lambda_p x, q^2) \sin(\sqrt{q}\lambda_m x, q^2) \, dq \, x = \delta_{pm}. \]  

(6.8)

This completes the proof. \(\square\)

Example 6.2.

\[ \zeta_q(2) = -\frac{q\lambda_1^2}{(1-q)^2} \int_0^1 k(x,x) \, dq \, x = \frac{q\lambda_1^2}{(1-q)^2} \left[ \frac{1}{[2]_q} - \frac{1}{[3]_q} \right], \]

(6.9)

In [7], it is proved that \(\sin((1-q)x, q^2) \to \sin(x)\), as \(q \to 1^-\). This implies

\[ \lim_{q \to 1^-} \frac{\lambda_n}{1-q} = n\pi. \]  

(6.10)

Thus, when \(q \to 1^-\), we obtain the well-known identities for the Euler Zeta function

\[ \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \]  

(6.11)

7. Spectral resolution

We can now find the spectral measure of \(\Delta_q\) from the resolvent, see the Stieltjes-Perron inversion formula [9].

Theorem 7.1. Let \(\eta_n = -\lambda_n^2\) be a point from the spectrum of \(\Delta_q\). If \(u \in H_{q,1}\) and \(v \in H_{q,2}\), then the spectral measure \(E\) of \(\{\eta_n\}\) is given by

\[ \langle E\{\eta_n\} \, u, v \rangle = \frac{2}{(1-q)^2} \left[ \cosh(\sqrt{q}\eta_n, q^2) \sinh(x, q^2) \right]_{x=\sqrt{q}\eta_n} \langle u, \sin(\sqrt{q}\lambda_n x, q^2) \rangle \cdot \langle v, \sin(\sqrt{q}\lambda_n x, q^2) \rangle. \]  

(7.2)

Proof. In order to calculate \(E\{\eta_n\}\), we choose the interval \((a,b)\) so that it contains only \(\eta_n = -\lambda_n^2\) as a point from the spectrum. Then

\[ \langle E\{\eta_n\} \, u, v \rangle = \langle E(a,b) \, u, v \rangle \]

\[ = \lim_{\varepsilon \to 0} \frac{1}{2i\pi} \int_a^b \left[ \langle (s-i\varepsilon - \Delta_q)^{-1} u, v \rangle - \langle (s+i\varepsilon - \Delta_q)^{-1} u, v \rangle \right] \, ds \]  

(7.3)

\[ = \frac{1}{2i\pi} \int_{\{\eta_n\}} \langle (s-\Delta_q)^{-1} u, v \rangle \, ds. \]
Now observe that
\[
\langle (s - \Delta_q)^{-1} u, v \rangle = \int_0^1 \cdots \int_0^1 G_s(x, y) \Lambda_q u(y) v(x) d_q y d_q x. \tag{7.4}
\]
If \( z = \eta_n \), then \( \sinh(\sqrt{qz}, q^2) = 0 \), which implies that the fundamental solutions \( U_1 \) and \( U_2 \) of the \( q \)-difference equation
\[
\Delta_q U = z U \tag{7.5}
\]
are proportional as follows:
\[
U_2(x) = - \cosh(\sqrt{\eta_n}, q^2) U_1(x), \quad \forall x \in [0, 1]. \tag{7.6}
\]
Therefore
\[
\langle E\{\eta_n\} u, v \rangle = \int_0^1 \cdots \int_0^1 [\text{Res}_{s=\eta_n} G_s(x, y)] \Lambda_q u(y) v(x) d_q y d_q x
\]
\[
= \frac{q^2}{(1 - q)} \cosh(\sqrt{\eta_n}, q^2) \left[ \text{Res}_{s=\eta_n} \frac{1}{\sqrt{q^2} \sinh(\sqrt{q^2}, q^2)} \right]
\times \langle \Lambda_q u, \sinh(\sqrt{q\eta_n} q x, q^2) \rangle \cdot \langle v, \sin(\sqrt{q\eta_n} x, q^2) \rangle. \tag{7.7}
\]
If \( f(1) = 0 \), then
\[
\int_0^1 \Lambda_q f(x) d_q x = \frac{1}{q} \int_0^1 f(x) d_q x, \tag{7.8}
\]
which implies
\[
\langle \Lambda_q u, \sinh(\sqrt{q\eta_n} q x, q^2) \rangle = \frac{1}{q} \langle u, \sinh(\sqrt{q\eta_n} x, q^2) \rangle. \tag{7.9}
\]
Finally we have
\[
\langle E\{\eta_n\} u, v \rangle = \frac{2}{(1 - q)} \cosh(\sqrt{\eta_n}, q^2) \left\{ \frac{\sinh(\sqrt{q\lambda_n} x, q^2)}{\sin(\sqrt{q\lambda_n} x, q^2)} \right\} \cdot \langle u, \sin(\sqrt{q\lambda_n} x, q^2) \rangle \cdot \langle v, \sin(\sqrt{q\lambda_n} x, q^2) \rangle. \tag{7.10}
\]
This completes the proof. \( \square \)

**Corollary 7.2.**
\[
Kn = \left\| \sin(\sqrt{q\lambda_n} x, q^2) \right\|_{q^2}^2 = \frac{(1 - q)}{2} \frac{(d/dx) \sinh(x, q^2) \big|_{x = \sqrt{\eta_n}}}{\cosh(\sqrt{\eta_n}, q^2)}. \tag{7.11}
\]

**Proof.** For
\[
u = v = \sin(\sqrt{q\lambda_n} x, q^2), \tag{7.12}
\]
the result follows immediately from the following equality:

\[ E\{\eta_n\} \sin (\sqrt{q}\lambda_n x, q^2) = \sin (\sqrt{q}\lambda_n x, q^2). \]  

(7.13)

This finishes the proof. \(\square\)

**Remark 7.3.** In [5], it is proved that

\[ \int_0^1 x^2 J^{(3)}_{\alpha}(aqx, q^2) \, dqx = -\frac{1-q}{2} q^{a-1} J^{(3)}_{\alpha+1}(aq, q^2) \frac{d}{dx} J^{(3)}_{\alpha}(x, q^2) \bigg|_{x=a}, \]  

(7.14)

where \(a \neq 0\) is a real zero of \(J^{(3)}_{\alpha}(x, q^2)\). This formula can be employed to evaluate \(k_n\) by another method.

**References**


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