It is well known that the concept of Hyers-Ulam-Rassias stability was originated by Th. M. Rassias (1978) and the concept of Ulam-Gavruta-Rassias stability was originated by J. M. Rassias (1982–1989) and by P. Gavruta (1999). In this paper, we give results concerning these two stabilities.

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1. Introduction

In 1940, Ulam [13] proposed the Ulam stability problem of additive mappings. In the next year, Hyers [5] considered the case of approximately additive mappings $f : E \to E'$, where $E$ and $E'$ are Banach spaces and $f$ satisfies inequality $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)$ exists for all $x \in E$ and that $L$ is the unique additive mapping satisfying $\|f(x) - L(x)\| \leq \varepsilon$. In 1978, Rassias [14] generalized the result to an approximation involving a sum of powers of norms. In 1982–1989, Rassias [8–11] treated the Ulam-Gavruta-Rassias stability on linear and nonlinear mappings and generalized Hyers result to the following theorem.

**Theorem 1.1 (J. M. Rassias).** Let $f : E \to E'$ be a mapping, where $E$ is a real-normed space and $E'$ is a Banach space. Assume that there exist $\theta > 0$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \theta \|x\|^p \|y\|^q$$

(1.1)

for all $x, y \in E$, where $r = p + q \neq 1$. Then there exists a unique additive mapping $L : E \to E'$
such that
\[ \| f(x) - L(x) \| \leq \frac{\theta}{|2 - 2^r|} \| x \|^r \] (1.2)
for all \( x \in E \).

However, the case \( r = 1 \) in the above inequality is singular. A counterexample has been given by Găvruta [2]. The above-mentioned stability involving a product of different powers of norms is called Ulam-Gavruta-Rassias stability by Bouikhalene and Elqorachi [1], Ravi and ArunKumar [12], and Nakmahachalasint [6]. In recent years, some other authors [3, 4, 7] have investigated the stability of additive mapping in various forms.

In this paper, we propose an \( n \)-dimensional additive functional equation and investigate its Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias stabilities.

### 2. The functional equation and the solution

**Theorem 2.1.** Let \( n > 1 \) be an integer and let \( X, Y \) be real vector spaces. A mapping \( f : X \to Y \) satisfies the functional equation
\[
 nf \left( \sum_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} f(x_i) + \sum_{1 \leq i < j \leq n} f(x_i + x_j) \quad \forall x_1, x_2, \ldots, x_n \in X \quad (2.1)
\]
if and only if \( f \) satisfies the Cauchy functional equation
\[
 f(x + y) = f(x) + f(y) \quad \forall x, y \in X. \quad (2.2)
\]

**Proof.** We first suppose that a mapping \( f : X \to Y \) satisfies (2.2). By the additivity of the Cauchy functional equation, we have
\[
 \sum_{i=1}^{n} f(x_i) + \sum_{1 \leq i < j \leq n} f(x_i + x_j) = \sum_{i=1}^{n} f(x_i) + \sum_{1 \leq i < j \leq n} (f(x_i) + f(x_j))
\]
\[
 = n \sum_{i=1}^{n} f(x_i) = nf \left( \sum_{i=1}^{n} x_i \right) \quad (2.3)
\]
for all \( x_1, x_2, \ldots, x_n \in X \). Hence, \( f \) satisfies (2.1).

Now suppose that a mapping \( f : X \to Y \) satisfies (2.1). Putting \( x_1 = x_2 = \cdots = x_n = 0 \) in (2.1), we have \( nf(0) = n f(0) + \left( \binom{n}{2} f(0) \right) \), which leads to \( f(0) = 0 \). Putting \( x_1 = x, x_2 = y \) and, if \( n > 2, x_3 = x_4 = \cdots = x_n = 0 \) in (2.1), we get
\[
 nf(x + y) = f(x) + f(y) + (n - 2) f(x) + (n - 2) f(y) + f(x + y) \quad \forall x, y \in X, \quad (2.4)
\]
which simplifies to \( f(x + y) = f(x) + f(y) \) as desired. \( \square \)
The following theorem treats the Hyers-Ulam-Rassias stability of (2.1).

**Theorem 3.1.** Let $n > 1$ be an integer, let $X$ be a real vector space, and let $Y$ be a Banach space. Given real numbers $\delta, \theta \geq 0$ and $p \in (0, 1) \cup (1, \infty)$ with $\delta = 0$ when $p > 1$. If a mapping $f : X \to Y$ satisfies the inequality

$$
\left\| nf \left( \sum_{i=1}^{n} x_i \right) - \sum_{i=1}^{n} f(x_i) - \sum_{1 \leq i < j \leq n} f(x_i + x_j) \right\| \leq \delta + \theta \sum_{i=1}^{n} \|x_i\|^p \tag{3.1}
$$

for all $x_1, x_2, \ldots, x_n \in X$, then there exists a unique additive mapping $L : X \to Y$ that satisfies (2.1) and the inequality

$$
\left\| f(x) - L(x) \right\| \leq \frac{2\delta}{n} + \frac{2\theta}{(n-1)|2-2^p|} \|x\|^p \quad \forall x \in X. \tag{3.2}
$$

The mapping $L$ is given by

$$
L(x) = \begin{cases} 
\lim_{m \to \infty} 2^{-m} f(2^m x) & \text{if } 0 < p < 1 \\
\lim_{m \to \infty} 2^m f(2^{-m} x) & \text{if } p > 1
\end{cases} \quad \forall x \in X. \tag{3.3}
$$

**Proof.** Putting $x_1 = x_2 = \cdots = x_n = 0$ in (3.1), we have $\|nf(0) - nf(0) - \binom{n}{2} f(0)\| \leq \delta$. Thus, $\|f(0)\| \leq 2\delta/(n^2 - n)$. Setting $x_1 = x_2 = x$ and, if $n > 2$, $x_3 = x_4 = \cdots = x_n = 0$ in (3.1), we have

$$
\left\| nf(2x) - 2f(x) - (n-2)f(0) - f(2x) - 2(n-2)f(x) - \binom{n-2}{2} f(0) \right\| \leq \delta + 2\theta \|x\|^p, \tag{3.4}
$$

which simplifies to

$$
(n-1) \left\| f(2x) - 2f(x) - \frac{n-2}{2} f(0) \right\| \leq \delta + 2\theta \|x\|^p. \tag{3.5}
$$

Therefore,

$$
\left\| 2f(x) - f(2x) \right\| \leq \frac{n-2}{2} \left\| f(0) \right\| + \frac{\delta + 2\theta \|x\|^p}{n-1} \leq \frac{2\delta}{n} + \frac{2\theta}{n-1} \|x\|^p. \tag{3.6}
$$

We first consider the case where $0 < p < 1$. Rewrite the above inequality (3.6) as

$$
\left\| f(x) - 2^{-1} f(2x) \right\| \leq \frac{\delta}{n} + \frac{\theta}{n-1} \|x\|^p. \tag{3.7}
$$
For every positive integer \( m \),

\[
\|f(x) - 2^{-m} f(2^m x)\| = \left\| \sum_{i=0}^{m-1} (2^{-i} f(2^i x) - 2^{-(i+1)} f(2^{i+1} x)) \right\|
\leq \sum_{i=0}^{m-1} \|2^{-i} f(2^i x) - 2^{-(i+1)} f(2^{i+1} x)\|
= \sum_{i=0}^{m-1} 2^{-i} \|f(2^i x) - 2^{-1} f(2 \cdot 2^i x)\|. \tag{3.8}
\]

Substituting \( x \) with \( x, 2x, 2^2x, \ldots, 2^{m-1}x \) in (3.7), the above inequality becomes

\[
\|f(x) - 2^{-m} f(2^m x)\| \leq \frac{\delta}{n} \sum_{i=0}^{m-1} 2^{-i} + \frac{\theta}{n-1} \|x\|^p \sum_{i=0}^{m-1} 2^{i(p-1)}. \tag{3.9}
\]

Consider the sequence \( \{2^{-m} f(2^m x)\} \). For all positive integers \( k < l \), we have

\[
\|2^{-k} f(2^k x) - 2^{-l} f(2^l x)\| = 2^{-k} \|f(2^k x) - 2^{-(l-k)} f(2^{l-k} \cdot 2^k x)\|
\leq 2^{-k} \left( \frac{\delta}{n} \sum_{i=0}^{l-k-1} 2^{-i} + \frac{\theta}{n-1} \|2^k x\|^p \sum_{i=0}^{l-k-1} 2^{i(p-1)} \right)
\leq \frac{2^{-k} \delta}{n} \sum_{i=0}^{\infty} 2^{-i} + \frac{\theta}{n-1} 2^{-k(1-p)} \|x\|^p \sum_{i=0}^{\infty} 2^{i(p-1)}. \tag{3.10}
\]

The right-hand side of the above inequality approaches 0 as \( k \to \infty \). Therefore, \( L(x) = \lim_{m \to \infty} 2^{-m} f(2^m x) \) is well defined. Taking the limit of (3.9) as \( m \to \infty \), we have

\[
\|f(x) - L(x)\| \leq \frac{\delta}{n} \sum_{i=0}^{\infty} 2^{-i} + \frac{\theta}{n-1} \|x\|^p \sum_{i=0}^{\infty} 2^{i(p-1)} = \frac{2\delta}{n} + \frac{2\theta}{(n-1)(2-2^p)} \|x\|^p \quad \forall x \in X. \tag{3.11}
\]

To show that \( L \) satisfies (2.1), replace each \( x_i \) in (3.1) with \( 2^m x_i \). This results in

\[
\left\|nf\left(\sum_{i=1}^{n} 2^m x_i\right) - \sum_{i=1}^{n} f(2^m x_i) - \sum_{1 \leq i < j \leq n} f(2^m x_i + 2^m x_j)\right\| \leq \left( \delta + \theta \sum_{i=1}^{n} \|2^m x_i\|^p \right). \tag{3.12}
\]

Dividing the above inequality by \( 2^m \) and taking the limit as \( m \to \infty \), we obtain

\[
\left\|nL\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} L(x_i) - \sum_{1 \leq i < j \leq n} f(x_i + x_j)\right\| \leq \lim_{m \to \infty} \left( \frac{\delta}{2^m} + \frac{\theta}{2^{m(1-p)}} \sum_{i=1}^{n} \|x_i\|^p \right) = 0, \tag{3.13}
\]

which verifies that \( L \) indeed satisfies (2.1).
To prove the uniqueness of \( L \), suppose there is a mapping \( L' : X \to Y \) such that \( L' \) satisfies (2.1) and (3.2). The additivity of \( L \) and \( L' \) is asserted by Theorem 2.1; hence,

\[
\|L(x) - L'(x)\| = 2^{-m}\|L(2^m x) - L'(2^m x)\| \\
\leq 2^{-m}\left(\|L(2^m x) - f(2^m x)\| + \|L'(2^m x) - f(2^m x)\|\right) \\
\leq 2^{-m} \cdot 2 \left(\frac{2\delta}{n} + \frac{2\theta}{(n-1)(2-2^p)} \|2^m x\|^p\right) \to 0. \tag{3.14}
\]

Thus, \( L(x) = L'(x) \) for all \( x \in X \).

For the case \( p > 1, \delta = 0 \) and (3.7) must be replaced by

\[
\|f(x) - 2f(2^{-1}x)\| \leq \frac{2\theta}{n-1} \|2^{-1}x\|^p. \tag{3.15}
\]

The rest of the proof can be done in the same fashion as that of the case \( 0 < p < 1 \). \( \Box \)

4. Ulam-Gavruta-Rassias stability

The following theorem treats the Ulam-Gavruta-Rassias stability of (2.1).

**Theorem 4.1.** Let \( n > 1 \) be an integer, let \( X \) be a real vector space, and let \( Y \) be a Banach space. Given real numbers \( \delta, \theta \geq 0 \) and \( p \in (0,1) \cup (1,\infty) \) with \( \delta = 0 \) when \( p > 1 \). If a mapping \( f : X \to Y \) satisfies the inequality

\[
\left\|n f \left( \sum_{i=1}^{n} x_i \right) - \sum_{i=1}^{n} f(x_i) - \sum_{1 \leq i < j \leq n} f(x_i + x_j) \right\| \leq \delta + \theta \sum_{1 \leq i < j \leq n} \|x_i\|^{p/2}\|x_j\|^{p/2} \tag{4.1}
\]

for all \( x_1, x_2, \ldots, x_n \in X \), then there exists a unique additive mapping \( L : X \to Y \) that satisfies (2.1) and the inequality

\[
\|f(x) - L(x)\| \leq \frac{2\delta}{n} + \frac{\theta}{(n-1)(2-2^p)} \|x\|^p \quad \forall x \in X. \tag{4.2}
\]

The mapping \( L \) is given by (3.3).

**Proof.** We make the same substitution as in the proof of Theorem 3.1 and obtain instead of (3.5) the following inequality:

\[
(n-1) \left\|f(2x) - 2f(x) - \frac{n-2}{2} f(0) \right\| \leq \delta + \theta \|x\|^p \quad \forall x \in X. \tag{4.3}
\]

The rest of the proof, apart from a multiplicative factor of 2 appears before \( \theta \), can be carried over from that of Theorem 3.1. \( \Box \)

It should be remarked that in the case where \( n = 2 \), functional equation (2.1) reduces to the Cauchy functional equation, and the Ulam-Gavruta-Rassias stability of this problem has been treated by J. M. Rassias, and the result has been restated in Theorem 1.1.
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