For the Riemann-Liouville transform $\mathcal{R}_\alpha$, $\alpha \in \mathbb{R}_+$, associated with singular partial differential operators, we define and study the Weyl transforms $W_\sigma$ connected with $\mathcal{R}_\alpha$, where $\sigma$ is a symbol in $S^m$, $m \in \mathbb{R}$. We give criteria in terms of $\sigma$ for boundedness and compactness of the transform $W_\sigma$.

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1. Introduction

In his book [14], Wong studies the properties of pseudodifferential operators arising in quantum mechanics, first envisaged by Weyl [13], as bounded linear operators on $L^2(\mathbb{R}^n)$ (the space of square integrable functions on $\mathbb{R}^n$ with respect to the Lebesgue measure). For this reason, M. W. Wong calls the operators treated in his book Weyl transforms.

Here, we consider the singular partial differential operators

$$\Delta_1 = \frac{\partial}{\partial x},$$
$$\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{2\alpha + 1}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}, \quad (r, x) \in ]0, +\infty[ \times \mathbb{R}, \; \alpha \geq 0. \quad (1.1)$$

We associate to $\Delta_1$ and $\Delta_2$ the Riemann-Liouville transform $\mathcal{R}_\alpha$ defined on $C_\alpha(\mathbb{R}^2)$ (the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable) by

$$\mathcal{R}_\alpha(f)(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^{1} f\left( rs\sqrt{1 - t^2}, x + rt \right) (1 - t^2)^{\alpha - 1/2} (1 - s^2)^{\alpha-1} \, dt \, ds & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^{1} f\left( r\sqrt{1 - t^2}, x + rt \right) \frac{dt}{\sqrt{1 - t^2}} & \text{if } \alpha = 0. \end{cases} \quad (1.2)$$

For more general integral transforms, we can see [2].
The transform $\mathcal{R}_\alpha$ generalizes the mean operator defined by

$$\mathcal{R}_\alpha(f)(r,x) = \frac{1}{2\pi} \int_0^{2\pi} f(r \sin \theta, x + r \cos \theta) d\theta. \quad (1.3)$$

The mean operator $\mathcal{R}_\alpha$ and its dual play an important role and have many applications, for example, in image processing of the so-called synthetic aperture radar (SAR) data [5, 6], or in the linearized inverse scattering problem in acoustics [3].

In [1], we have defined a convolution product and a Fourier transform $\mathcal{F}_\alpha$ associated with $\mathcal{R}_\alpha$, and we have established many harmonic analysis results (inversion formula, Paley-Wiener, and Plancherel theorems, etc.).

Using these results, we define and study, in this paper the Weyl transforms associated with $\mathcal{R}_\alpha$, we give criteria in terms of symbols to prove the boundedness and compactness of these transforms. To obtain these results, we have first defined the Fourier-Wigner transform associated with the operator $\mathcal{R}_\alpha$, and we have established for it an inversion formula.

More precisely, in Section 2, we recall some properties of harmonic analysis for the operator $\mathcal{R}_\alpha$. In Section 3, we define the Fourier-Wigner transform associated with $\mathcal{R}_\alpha$, study some of its properties, and prove an inversion formula.

In Section 4, we introduce the Weyl transform $W_\sigma$ associated with $\mathcal{R}_\alpha$, with $\sigma$ a symbol in class $S^m$, for $m \in \mathbb{R}$, and we give its connection with the Fourier-Wigner transform. We prove that for $\sigma$ sufficiently smooth, $W_\sigma$ is a compact operator from $L^2(d\nu)$, the space of square integrable functions on $[0, +\infty[\times\mathbb{R}$, with respect to the measure

$$d\nu(r,x) = \frac{1}{2^\alpha \Gamma(\alpha + 1)\sqrt{2\pi}} r^{2\alpha + 1} dr \otimes dx, \quad (1.4)$$

into itself.

In Section 5, we define $W_\sigma$ for $\sigma$ in a certain space $L^p(d\nu \otimes dy)$, with $p \in [1, 2]$, and we establish that $W_\sigma$ is again a compact operator.

In Section 6, we define $W_\sigma$ for $\sigma$ in another function space, and use this to prove in Section 7 that for $p > 2$, there exists a function $\sigma \in L^p(d\nu \otimes dy)$, with the property that the Weyl transform $W_\sigma$ is not bounded on $L^2(d\nu)$.

For more Weyl transforms, we can see [8, 15].

## 2. Riemann-Liouville transform associated with the operators $\Delta_1$ and $\Delta_2$

In this section, we recall some properties of the Riemann-Liouville transform that we use in the next sections. For more details, see [1].

For all $(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}$, the system

$$\Delta_1 u(r,x) = -i\lambda u(r,x),$$
$$\Delta_2 u(r,x) = -\mu^2 u(r,x), \quad (2.1)$$
$$u(0,0) = 1, \quad \frac{\partial u}{\partial r} (0,x) = 0, \quad \forall x \in \mathbb{R},$$

admits a unique solution given by

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha \left( r \sqrt{\mu^2 + \lambda^2} \right) \exp(-i\lambda x),$$  \hspace{1cm} (2.2)

where $j_\alpha$ is the modified Bessel function defined by

$$j_\alpha(s) = \frac{2\alpha}{\Gamma(\alpha + 1)} J_\alpha(s) \frac{s^{\alpha}}{\Gamma(\alpha + 1)} + \infty \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\alpha + k + 1)} \left( \frac{s}{2} \right)^{2k},$$  \hspace{1cm} (2.3)

and $J_\alpha$ is the Bessel function of first kind and index $\alpha$ (see [7, 12]).

Moreover, we have

$$\sup_{(r,x) \in \mathbb{R}^2} \mid \varphi_{\mu,\lambda}(r,x) \mid = 1 \text{ iff } (\mu, \lambda) \in \Gamma,$$  \hspace{1cm} (2.4)

where $\Gamma$ is the set defined by

$$\Gamma = \mathbb{R}^2 \cup \{ (i\mu, \lambda); (\mu, \lambda) \in \mathbb{R}^2, |\mu| \leq |\lambda| \}.$$  \hspace{1cm} (2.5)

**Proposition 2.1.** The eigenfunction $\varphi_{\mu,\lambda}$ given by (2.2) has the following Mehler integral representation:

$$\varphi_{\mu,\lambda}(r,x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^{1} \cos(\mu rs \sqrt{1-t^2}) e^{-i\lambda (x+rt)} (1-t^2)^{\alpha-1/2} (1-s^2)^{\alpha-1} \ dt \ ds & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^{1} \cos(r \mu \sqrt{1-t^2}) e^{-i\lambda (x+rt)} \ dt \sqrt{1-t^2} & \text{if } \alpha = 0. \end{cases}$$  \hspace{1cm} (2.6)

This result shows that

$$\varphi_{\mu,\lambda}(r,x) = \mathcal{R}_\alpha \left( \cos(\mu.) \exp(-i\lambda.) \right)(r,x),$$  \hspace{1cm} (2.7)

where $\mathcal{R}_\alpha$ is the Riemann-Liouville transform associated with the operators $\Delta_1$ and $\Delta_2$, given in the introduction.

We denote by

(i) $\mathcal{C}^\infty_{\ast,c}(\mathbb{R}^2)$ the subspace of $\mathcal{C}^\infty(\mathbb{R}^2)$ consisting of functions with compact support;

(ii) $d\nu(r,x)$ the measure defined on $[0, +\infty] \times \mathbb{R}$ by

$$d\nu(r,x) = c_\alpha r^{2\alpha+1} dr \otimes dx,$$  \hspace{1cm} (2.8)

with $c_\alpha = 1/\sqrt{2\pi} \Gamma(\alpha + 1)$;

(iii) $L^p(d\nu)$ the space of measurable functions $f$ on $[0, +\infty] \times \mathbb{R}$, satisfying

$$\|f\|_{p,\nu} = \left( \int_{\mathbb{R}} \int_0^{+\infty} |f(r,x)|^p d\nu(r,x) \right)^{1/p} < +\infty \text{ if } p \in [1, +\infty[,$$  \hspace{1cm} (2.9)

$$\|f\|_{\infty,\nu} = \text{ess sup}_{(r,x) \in [0, +\infty[ \times \mathbb{R}} |f(r,x)| < +\infty \text{ if } p = +\infty;$$
The convolution product associated with the Riemann-Liouville transform of $f$ is defined on $L^1(dy)$ by
\[
\int_{\Gamma} f(\mu, \lambda) d\gamma(\mu, \lambda) = c_\alpha \left\{ \int_{\mathbb{R}} \int_{0}^{+\infty} f(\mu, \lambda) (\mu^2 + \lambda^2)^\alpha \mu d\mu d\lambda + \int_{\mathbb{R}} \int_{0}^{+|\lambda|} f(i\mu, \lambda) (\lambda^2 - \mu^2)^\alpha \mu d\mu d\lambda \right\};
\] (2.10)

$L^p(dy)$, $p \in [1, +\infty]$, the space of measurable functions on $\Gamma$ satisfying
\[
\|f\|_{p,y} = \left( \int_{\Gamma} |f(\mu, \lambda)|^p d\gamma(\mu, \lambda) \right)^{1/p} < +\infty \quad \text{if } p \in [1, +\infty],
\]
\[
\|f\|_{\infty,y} = \text{ess sup}_{(\mu, \lambda) \in \Gamma} |f(\mu, \lambda)| < +\infty \quad \text{if } p = +\infty.
\] (2.11)

\textbf{Definition 2.2.} (i) The translation operator associated with Riemann-Liouville transform is defined on $L^1(dy)$, for all $(r, x), (s, y) \in [0, +\infty] \times \mathbb{R}$, by
\[
\mathcal{T}_{(r,x)} f(s, y) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi \Gamma(\alpha + 1/2)}} \int_{0}^{\alpha} f\left(\sqrt{r^2 + s^2 + 2rs \cos \theta}, x + y \right) \sin^{2\alpha} \theta d\theta.
\] (2.12)

(ii) The convolution product associated with the Riemann-Liouville transform of $f, g \in L^1(dy)$ is defined by
\[
\forall (r, x) \in [0, +\infty] \times \mathbb{R}, \quad f \ast g(r, x) = \int_{\mathbb{R}} \int_{0}^{+\infty} \mathcal{T}_{(r,-x)} \hat{f}(s, y) g(s, y) d\gamma(s, y),
\] (2.13)
where $\hat{f}(s, y) = f(s, -y)$.

We have the following properties.

(i) We have the following product formula:
\[
\mathcal{T}_{(r,x)} \varphi_{\mu,\lambda}(s, y) = \varphi_{\mu,\lambda}(r, x) \varphi_{\mu,\lambda}(s, y).
\] (2.14)

(ii) Let $f$ be in $L^1(dy)$. Then, for all $(s, y) \in [0, +\infty] \times \mathbb{R}$, we have
\[
\int_{\mathbb{R}} \int_{0}^{\infty} \mathcal{T}_{(s,y)} f(r, x) d\gamma(r, x) = \int_{\mathbb{R}} \int_{0}^{\infty} f(r, x) d\gamma(r, x).
\] (2.15)

(iii) If $f \in L^p(dy)$, $1 \leq p \leq +\infty$, then for all $(s, y) \in [0, +\infty] \times \mathbb{R}$, the function $\mathcal{T}_{(s,y)} f$ belongs to $L^p(dy)$, and we have
\[
\|\mathcal{T}_{(s,y)} f\|_{p,y} \leq \|f\|_{p,y}.
\] (2.16)

(iv) For $f, g \in L^1(dy)$, $f \ast g$ belongs to $L^1(dy)$, and the convolution product is commutative and associative.

(v) For $f \in L^1(dy), g \in L^p(dy)$, $1 < p \leq +\infty$, the function $f \ast g \in L^p(dy)$ and
\[
\|f \ast g\|_{p,y} \leq \|f\|_{1,y} \|g\|_{p,y}.
\] (2.17)
(vi) For \( f, g \in L_{*c}(\mathbb{R}^2) \), such that \( \text{supp} \ f \subset [−a_1, a_1] \times [−a_2, a_2] \) and \( \text{supp} \ g \subset [−b_1, b_1] \times [−b_2, b_2] \), the function \( f * g \) belongs to \( L_{*c}(\mathbb{R}^2) \) and

\[
\text{supp}(f * g) \subset [−(a_1 + b_1), a_1 + b_1] \times [−(a_2 + b_2), a_2 + b_2].
\]  

(2.18)

Definition 2.3. The Fourier transform associated with the Riemann–Liouville operator is defined on \( L^1(dy) \), by

\[
\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_a(f)(\mu, \lambda) = \int_{\mathbb{R}} \int_{0}^{+\infty} f(r, x)\varphi_{\mu, \lambda}(r, x)dv(r, x),
\]

where \( \Gamma \) is the set defined by the relation (2.5).

We have the following properties.

(i) Let \( f \) be in \( L^1(dy) \). For all \( (r, x) \in [0, +\infty[ \times \mathbb{R} \), we have

\[
\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_a(f)\big((r, x)\big)(\mu, \lambda) = \varphi_{\mu, \lambda}(r, x)\mathcal{F}_a(f)(\mu, \lambda).
\]

(ii) For \( f, g \in L^1(dy) \), we have

\[
\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_a(f \ast g)(\mu, \lambda) = \mathcal{F}_a(f)(\mu, \lambda)\mathcal{F}_a(g)(\mu, \lambda).
\]

(iii) For \( f \in L^1(dy) \), we have

\[
\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_a(f)(\mu, \lambda) = \mathcal{B} \circ \widetilde{\mathcal{F}}_a(f)(\mu, \lambda),
\]

where, for every \((\mu, \lambda) \in \mathbb{R}^2\),

\[
\widetilde{\mathcal{F}}_a(f)(\mu, \lambda) = \int_{\mathbb{R}} \int_{0}^{+\infty} f(r, x)j_a(r\mu)\exp(−i\lambda x)dv(r, x),
\]

\[
\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{B}f(\mu, \lambda) = f\left(\sqrt{\mu^2 + \lambda^2}, \lambda\right).
\]

(iv) For \( f \in L^1(dy) \) such that \( \mathcal{F}_a(f) \in L^1(dy) \), we have the inversion formula for \( \mathcal{F}_a \), for almost every \( (r, x) \in [0, +\infty[ \times \mathbb{R} \),

\[
f(r, x) = \int_{\Gamma} \mathcal{F}_a(f)(\mu, \lambda)\overline{\varphi_{\mu, \lambda}(r, x)}dv(\mu, \lambda).
\]

(2.25)

Proposition 2.4. Let \( f \) be in \( L^p(dy) \), with \( p \in [1, 2] \). Then, \( \mathcal{F}_a(f) \) belongs to \( L^{p'}(dy) \), with \( \frac{1}{p} + \frac{1}{p'} = 1 \), and \( \|\mathcal{F}_a(f)\|_{p', y} \leq \|f\|_{p, y} \).

Proof. The mapping \( \widetilde{\mathcal{F}}_a \) given by the relation (2.23) is an isometric isomorphism from \( L^2(dy) \) onto itself, then \( \|\widetilde{\mathcal{F}}_a(f)\|_{2, y} = \|f\|_{2, y} \).

On the other hand, we have \( \|\widetilde{\mathcal{F}}_a(f)\|_{\infty, y} \leq \|f\|_{1, y} \).

Thus, from these relations and the Riesz–Thorin theorem [10, 11], we deduce that for all \( f \in L^p(dy) \), with \( p \in [1, 2] \), the function \( \widetilde{\mathcal{F}}_a(f) \) belongs to \( L^{p'}(dy) \), with \( p' = p/(p - 1) \), and we have

\[
\|\widetilde{\mathcal{F}}_a(f)\|_{p', y} \leq \|f\|_{p, y}.
\]

(2.26)
We complete the proof by using the fact that
\[ \left\| F_\alpha(f) \right\|_{p',q} = \left\| \widetilde{F}_\alpha(f) \right\|_{p',q}, \] (2.27)
which is a consequence of the relation (2.22).

We denote by (see [1, 9])
(i) \( \mathcal{S}_*(\mathbb{R}^2) \) the space of infinitely differentiable functions on \( \mathbb{R}^2 \) rapidly decreasing together with all their derivatives, even with respect to the first variable;
(ii) \( \mathcal{S}_*(\Gamma) \) the space of functions \( f : \Gamma \to \mathbb{C} \) infinitely differentiable, even with respect to the first variable and rapidly decreasing together with all their derivatives, that is, for all \( k_1, k_2, k_3 \in \mathbb{N}, \)
\[ \sup_{(\mu, \lambda) \in \Gamma} \left(1 + |\mu|^2 + |\lambda|^2\right)^{k_1} \left| \left( \frac{\partial}{\partial \mu} \right)^{k_2} \left( \frac{\partial}{\partial \lambda} \right)^{k_3} f(\mu, \lambda) \right| < +\infty, \] (2.28)
where
\[ \frac{\partial f}{\partial \mu}(\mu, \lambda) = \begin{cases} \frac{\partial}{\partial r} (f(r, \lambda)) & \text{if } \mu = r \in \mathbb{R}, \\ \frac{1}{i} \frac{\partial}{\partial t} (f(it, \lambda)) & \text{if } \mu = it, |t| \leq |\lambda|. \end{cases} \] (2.29)

Each of these spaces is equipped with its usual topology.

Remark 2.5. From [1], the Fourier transform \( F_\alpha \) is an isomorphism from \( \mathcal{S}_*(\mathbb{R}^2) \) onto \( \mathcal{S}_*(\Gamma) \). The inverse mapping is given by
\[ \forall (r, x) \in \mathbb{R}^2, \quad F_\alpha^{-1}(f)(r, x) = \iint_{\Gamma} f(\mu, \lambda) \overline{\phi}_{\mu, \lambda}(r, x) d\gamma(\mu, \lambda). \] (2.30)

3. Fourier-Wigner transform associated with Riemann-Liouville operator

Definition 3.1. The Fourier-Wigner transform associated with the Riemann-Liouville operator is the mapping \( V \) defined on \( \mathcal{S}_*(\mathbb{R}^2) \times \mathcal{S}_*(\mathbb{R}^2) \), for all \( ((r, x), (\mu, \lambda)) \in \mathbb{R}^2 \times \Gamma \), by
\[ V(f, g)((r, x), (\mu, \lambda)) = \int_{\mathbb{R}} \int_{0}^{\infty} f(s, y) \phi_{\mu, \lambda}(s, y) \overline{T}(r, x) g(s, y) dy (s, y). \] (3.1)

Remark 3.2. The transform \( V \) can also be written in the forms
(i) \( V(f, g)((r, x), (\mu, \lambda)) = F_\alpha(f, \overline{T}(r, x) g)(\mu, \lambda); \)
(ii) \( V(f, g)((r, x), (\mu, \lambda)) = \hat{g} \ast (\phi_{\mu, \lambda} f)(r, -x), \)
where \( \hat{g}(s, y) = g(s, -y) \) and \( \ast \) is the convolution product given in Definition 2.2.

We denote by
(i) \( \mathcal{S}_*(\mathbb{R}^2 \times \mathbb{R}^2) \) the space of infinitely differentiable functions \( f((r, x), (s, y)) \) on \( \mathbb{R}^2 \times \mathbb{R}^2 \), even with respect to the variables \( r \) and \( s \), and rapidly decreasing together with all their derivatives;
(ii) $\mathcal{S}_*(\mathbb{R}^2 \times \Gamma)$ the space of infinitely differentiable functions $f((r,x),(\mu,\lambda))$ on $\mathbb{R}^2 \times \Gamma$, even with respect to the variables $r$ and $\mu$, and rapidly decreasing together with all their derivatives;

(iii) $L^p(d\nu \otimes d\gamma)$, $1 \leq p \leq +\infty$, the space of measurable functions on $([0, +\infty[ \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R})$, verifying for $p \in [1, +\infty[$;

\[
\|f\|_{p,\nu \otimes \gamma} = \left(\int_{\mathbb{R}^2} \int_0^{+\infty} |f((r,x),(s,y))|^p d\nu(r,x)d\gamma(s,y)\right)^{1/p} < +\infty,
\]

for $p = +\infty$,

\[
\|f\|_{\infty,\nu \otimes \gamma} = \text{ess sup}_{(r,x),(s,y) \in [0, +\infty[ \times \mathbb{R}} |f((r,x),(s,y))| < +\infty;
\]

(iv) $L^p(d\nu \otimes d\gamma)$, $1 \leq p \leq +\infty$, the space similarly defined (with $d\nu(r,x)d\gamma(\mu,\lambda)$ in the integrand).

**Proposition 3.3.** (i) The Fourier-Wigner transform $V$ is a bilinear, continuous mapping from $\mathcal{S}_*(\mathbb{R}^2) \times \mathcal{S}_*(\mathbb{R}^2)$ into $\mathcal{S}_*(\mathbb{R}^2 \times \Gamma)$.

(ii) For $p \in ]1,2]$,

\[
\|V(f,g)\|_{p',\nu \otimes \gamma} \leq \|f\|_{p,\nu} \|g\|_{p',\gamma}.
\]

The transform $V$ can be extended to a continuous bilinear operator, denoted also by $V$, from $L^p(d\nu) \times L^{p'}(d\gamma)$ into $L^{p'}(d\nu \otimes d\gamma)$, where $p' = p/(p-1)$ is the conjugate exponent of $p$.

**Proof.** (i) Let $f, g \in \mathcal{S}_*(\mathbb{R}^2)$, and let $F$ be the function defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by

\[
F((r,x),(s,y)) = f(s,y)\bar{\mathcal{F}}_{(r,x)}g(s,y).
\]

Then, we have for all $(s,y), (\mu,\lambda) \in \mathbb{R}^2$,

\[
\tilde{\mathcal{F}}_a \otimes I(F)((\mu,\lambda),(s,y)) = j_a(s\mu)\exp(i\lambda y)f(s,y)\tilde{\mathcal{F}}_a(g)(\mu,\lambda),
\]

where $I$ is the identity operator. Since $\tilde{\mathcal{F}}_a$ is an isomorphism from $\mathcal{S}_*(\mathbb{R}^2)$ onto itself, we deduce that the function $\tilde{\mathcal{F}}_a \otimes I(F)$ belongs to the space $\mathcal{S}_*(\mathbb{R}^2 \times \mathbb{R}^2)$ and consequently, $F \in \mathcal{S}_*(\mathbb{R}^2 \times \mathbb{R}^2)$. Then, (i) follows from the relation

\[
V(f,g)((r,x),(\mu,\lambda)) = I \otimes \mathcal{F}_a(F)((r,x),(\mu,\lambda)),
\]

and the fact that $\mathcal{F}_a$ is an isomorphism from $\mathcal{S}_*(\mathbb{R}^2)$ into $\mathcal{S}_*(\Gamma)$.

(ii) We get the result from Remark 3.2(i), Proposition 2.4, Minkowski’s inequality for integrals (see [4, page 186]), and from the relation (2.16).

**Theorem 3.4.** For all $f, g \in \mathcal{S}_*(\mathbb{R}^2)$, $(\mu,\lambda) \in \Gamma$ and $(r,x) \in \mathbb{R}^2$,

\[
\mathcal{F}_a \otimes \mathcal{F}_a^{-1}(V(f,g))((\mu,\lambda),(r,x)) = \mathcal{F}_{\mu,\lambda}(r,x)f(r,x)\mathcal{F}_a(g)(\mu,\lambda).
\]
8 Weyl transforms

Proof. This theorem follows from the relations (2.20) and (3.7).

Using the previous theorem and the relation (2.25), we get the following result.

Corollary 3.5. For \( f, g \in \mathcal{S}(\mathbb{R}^2) \),

(i) for all \((\mu, \lambda) \in \Gamma\),

\[
\int_{\mathbb{R}} \int_{0}^{\infty} \mathcal{F}_a \otimes \mathcal{F}_a^{-1}(V(f, g))((\mu, \lambda), (r, x)) \, dv(r, x) = \mathcal{F}_a(f)(\mu, \lambda) \mathcal{F}_a(g)(\mu, \lambda); \tag{3.9}
\]

(ii) for all \((r, x) \in [0, +\infty[^2 \times \mathbb{R}\),

\[
\int_{\Gamma} \mathcal{F}_a \otimes \mathcal{F}_a^{-1}(V(f, g))((\mu, \lambda), (r, x)) \, dy(\mu, \lambda) = f(r, x)g(r, x). \tag{3.10}
\]

Theorem 3.6. Let \( f, g \in L^1(dv) \cap L^2(dv) \), such that \( c = \int_{\mathbb{R}} \int_{0}^{\infty} g(r, x) \, dv(r, x) \neq 0 \). Then,

\[
\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_a(f)(\mu, \lambda) = \frac{1}{c} \int_{\mathbb{R}} \int_{0}^{\infty} V(f, g)((r, x), (\mu, \lambda)) \, dv(r, x). \tag{3.11}
\]

Proof. From the relation (3.1), we have for all \((\mu, \lambda) \in \Gamma\),

\[
\int_{\mathbb{R}} \int_{0}^{\infty} V(f, g)((r, x), (\mu, \lambda)) \, dv(r, x)
\]

\[
= \int_{\mathbb{R}} \int_{0}^{\infty} \left( \int_{\mathbb{R}} \int_{0}^{\infty} f(s, y) \varphi_{\mu, \lambda}(s, y) \mathcal{F}_{(r, x)}(g(s, y)) \, dv(s, y) \right) \, dv(r, x). \tag{3.12}
\]

Then, the result follows from the relation (2.15), Definition 2.3, the fact that

\[
\forall (r, x) \in [0, +\infty[^2 \times \mathbb{R}, \forall (\mu, \lambda) \in \Gamma, \quad |\varphi_{\mu, \lambda}(r, x)| \leq 1, \tag{3.13}
\]

and Fubini’s theorem.

Corollary 3.7. With the hypothesis of Theorem 3.6, if \( \mathcal{F}_a(f) \in L^1(dy) \), the following inversion formula for the Fourier-Wigner transform \( V \) holds:

\[
f(r, x) = \frac{1}{c} \int_{\Gamma} \mathcal{F}_{\mu, \lambda}(r, x) \left[ \int_{\mathbb{R}} \int_{0}^{\infty} V(f, g)((s, y), (\mu, \lambda)) \, dv(s, y) \right] \, dy(\mu, \lambda), \tag{3.14}
\]

for almost every \((r, x) \in \mathbb{R}^2\).

4. Weyl transform associated with Riemann-Liouville operator

In this section, we introduce and study the Weyl transform and give its connection with the Fourier-Wigner transform. To do this, we must define the class of pseudodifferential operators [14].

Definition 4.1. Let \( m \in \mathbb{R} \). Define \( S^m \) to be the set of symbols, consisting of all infinitely differentiable functions \( \sigma((r, x), (\mu, \lambda)) \) on \( \mathbb{R}^2 \times \Gamma \), even with respect to the variables \( r \) and \( \mu \), such that for all \( k_1, k_2, k_3, k_4 \in \mathbb{N} \), there exists a positive constant \( C = C(k_1, k_2, k_3, k_4, m) \).
satisfying
\[
\left| \left( \frac{\partial}{\partial r} \right)^{k_1} \left( \frac{\partial}{\partial x} \right)^{k_2} \left( \frac{\partial}{\partial \mu} \right)^{k_3} \left( \frac{\partial}{\partial \lambda} \right)^{k_4} \sigma((r,x),(\mu,\lambda)) \right| \leq C(1 + \mu^2 + 2\lambda^2)^{m-(k_1+k_4)}. \quad (4.1)
\]

Definition 4.2. For \( \sigma \in S^m, m \in \mathbb{R} \), define the operator \( H_\sigma \) on \( \mathcal{S}_*(\mathbb{R}^2) \times \mathcal{S}_*(\mathbb{R}^2) \), for all \((r,x) \in \mathbb{R}^2\),
\[
H_\sigma(f,g)(r,x) = \int_{\Gamma} \int_{\mathbb{R}} \int_{0}^{\infty} \sigma((s,y),(\mu,\lambda)) \varphi_{\mu,\lambda}(r,x)
\times V(f,g)((s,y),(\mu,\lambda))d\nu(s,y)dy(\mu,\lambda), \quad (4.2)
\]
\[
\mathbb{H}_\sigma(f,g) = H_\sigma(f,g)(0,0). \quad (4.3)
\]

Proposition 4.3. Let \( \sigma \) be the symbol given by
\[
\forall (r,x) \in \mathbb{R}^2, \forall (\mu,\lambda) \in \Gamma, \quad \sigma((r,x),(\mu,\lambda)) = - (\mu^2 + \lambda^2). \quad (4.4)
\]
Then for \( f, g \in \mathcal{S}_*(\mathbb{R}^2) \),
\[
\forall (r,x) \in \mathbb{R}^2, \quad H_\sigma(f,g)(r,x) = c \ell_a f(r,-x), \quad (4.5)
\]
where
\[
c = \int_{\mathbb{R}} \int_{0}^{\infty} g(r,x)d\nu(r,x), \quad \ell_a = \frac{\partial^2}{\partial r^2} + \frac{2\alpha + 1}{r} \frac{\partial}{\partial r}. \quad (4.6)
\]

Proof. From relations (3.1), (4.2) and Fubini’s theorem we get, for all \((r,x) \in \mathbb{R}^2\),
\[
H_\sigma(f,g)(r,x) = \int_{\Gamma} -(\mu^2 + \lambda^2) \varphi_{\mu,\lambda}(r,x)\left\{ \int_{\mathbb{R}} \int_{0}^{\infty} f(t,z)\varphi_{\mu,\lambda}(t,z) \times \left[ \int_{\mathbb{R}} \int_{0}^{\infty} \overline{\mathcal{G}}((t,z),g(s,y))d\nu(s,y) \right]d\nu(t,z) \right\}dy(\mu,\lambda). \quad (4.7)
\]
Now, by relation (2.15), it follows that
\[
H_\sigma(f,g)(r,x) = c \int_{\Gamma} -(\mu^2 + \lambda^2) \overline{\mathcal{F}}_a(f)(\mu,\lambda)\varphi_{\mu,\lambda}(r,x)d\nu(\mu,\lambda). \quad (4.8)
\]
The result follows from relation (2.25) and the fact that
\[
\forall (\mu,\lambda) \in \Gamma, \quad -(\mu^2 + \lambda^2) \overline{\mathcal{F}}_a(f)(\mu,\lambda) = \overline{\mathcal{F}}_a(\ell_a f)(\mu,\lambda). \quad (4.9)
\]

Definition 4.4. Let \( \sigma \in S^m, m < -(\alpha + 3/2) \). The Weyl transform associated with the Riemann–Liouville operator is the mapping \( W_\sigma \) defined on \( \mathcal{S}_*(\mathbb{R}^2) \), for all \((r,x) \in \mathbb{R}^2\), by
\[
W_\sigma(f)(r,x) = \int_{\Gamma} \int_{\mathbb{R}} \int_{0}^{\infty} \varphi_{\mu,\lambda}(r,x)\sigma((s,y),(\mu,\lambda)) \overline{\mathcal{G}}((r,x),f(s,y))d\nu(s,y)d\nu(\mu,\lambda). \quad (4.10)
\]
Theorem 4.5. Let $\sigma \in \mathcal{S}_\star(\mathbb{R}^2 \times \Gamma)$. The Weyl transform $W_\sigma$ is a continuous mapping from $\mathcal{S}_\star(\mathbb{R}^2)$ into itself.

Proof. Let $f \in \mathcal{S}_\star(\mathbb{R}^2)$, since $\tilde{\mathcal{F}}_\sigma$ is an isomorphism from $\mathcal{S}_\star(\mathbb{R}^2)$ onto itself, and

$$\forall (\mu, \lambda) \in \mathbb{R}^2, \quad \tilde{\mathcal{F}}_\sigma(\mathcal{T}_{(x,y)} f)(\mu, \lambda) = j_a(r_\mu) \exp(i\lambda x) \tilde{\mathcal{F}}_\sigma(f)(\mu, \lambda), \quad (4.11)$$

we deduce that for all $(r, x) \in [0, +\infty[ \times \mathbb{R}$, the function $(s, y) \mapsto \mathcal{T}_{(r,x)} f(s, y)$ belongs to $\mathcal{S}_\star(\mathbb{R}^2)$. Then, by the inversion formula for $\tilde{\mathcal{F}}_\sigma$, we get, for all $(s, y) \in \mathbb{R}^2$;

$$\mathcal{T}_{(r,x)} f(s, y) = \int_\mathbb{R} \int_0^{+\infty} j_a(r_\mu) \exp(i\lambda x) \tilde{\mathcal{F}}_\sigma(f)(\mu, \lambda) j_a(s_\mu) \exp(i\lambda y) dv(\mu, \lambda). \quad (4.12)$$

By Definition 4.4 and Fubini’s theorem, we obtain, for all $(r, x) \in \mathbb{R}^2$,

$$W_\sigma(f)(r, x)$$

$$= \int_{\Gamma} \varphi_{\mu, \lambda}(r, x) \left[ \int_\mathbb{R} \int_0^{+\infty} \tilde{\mathcal{F}}_\sigma(f)(t, z) j_a(r t) \exp(ixz) \times \left\{ \int_\mathbb{R} \int_0^{+\infty} \sigma((s, y), (\mu, \lambda)) j_a(s t) \exp(iyz) dv(s, y) \right\} dv(t, z) \right] dy(\mu, \lambda)$$

$$= \int_{\Gamma} \varphi_{\mu, \lambda}(r, x) \left[ \int_\mathbb{R} \int_0^{+\infty} \tilde{\mathcal{F}}_\sigma(f)(t, z) j_a(r t) \exp(ixz) \times \tilde{\mathcal{F}}_\sigma^{-1}(\sigma((\cdot, \cdot), (\mu, \lambda)))(t, z) dv(t, z) \right] dy(\mu, \lambda). \quad (4.13)$$

Now, the function

$$((t, z), (\mu, \lambda)) \mapsto \tilde{\mathcal{F}}_\sigma^{-1}(\sigma((\cdot, \cdot), (\mu, \lambda)))(t, z) \quad (4.14)$$

belongs to $\mathcal{S}_\star(\mathbb{R}^2 \times \Gamma)$.

On the other hand, the mapping $f \mapsto G_f$, given for all $((t, z), (\mu, \lambda)) \in \mathbb{R}^2 \times \Gamma$ by

$$G_f((t, z), (\mu, \lambda)) = \tilde{\mathcal{F}}_\sigma(f)(t, z) \tilde{\mathcal{F}}_\sigma^{-1}(\sigma((\cdot, \cdot), (\mu, \lambda)))(t, z), \quad (4.15)$$

is continuous from $\mathcal{S}_\star(\mathbb{R}^2)$ into $\mathcal{S}_\star(\mathbb{R}^2 \times \Gamma)$, and for all $(r, x) \in \mathbb{R}^2$, we have

$$W_\sigma(f)(r, x) = \int_{\Gamma} \left( \int_\mathbb{R} \int_0^{+\infty} G_f((t, z), (\mu, \lambda)) j_a(r t) \exp(ixz) \varphi_{\mu, \lambda}(r, -x) dv(t, z) \right) dv(\mu, \lambda)$$

$$= \tilde{\mathcal{F}}_\sigma^{-1} \otimes \tilde{\mathcal{F}}_\sigma^{-1}(G_f)((r, x), (r, -x)). \quad (4.16)$$

Since $\tilde{\mathcal{F}}_\sigma^{-1}$ is an isomorphism from $\mathcal{S}_\star(\Gamma)$ onto $\mathcal{S}_\star(\mathbb{R}^2)$, we deduce that $\tilde{\mathcal{F}}_\sigma^{-1} \otimes \tilde{\mathcal{F}}_\sigma^{-1}$ is an isomorphism from $\mathcal{S}_\star(\mathbb{R}^2 \times \Gamma)$ onto $\mathcal{S}_\star(\mathbb{R}^2 \times \mathbb{R}^2)$. \qed
Lemma 4.6. Let $\sigma \in \mathcal{S}_*(\mathbb{R}^2 \times \Gamma)$. Then, the function $k$ defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by

$$k((r,x),(s,y)) = \int_{\Gamma} \varphi_{\mu,\lambda}(r,x) \mathcal{T}_{(r,-x)}(\sigma((\cdot,\cdot),(\mu,\lambda)))(s,y) d\nu(\mu,\lambda)$$

(4.17)

belongs to $\mathcal{S}_*(\mathbb{R}^2 \times \mathbb{R}^2)$.

Proof. The function $k$ can be written in the form

$$k((r,x),(s,y)) = \mathcal{T}_{(r,-x)}(I \otimes \mathcal{T}_{\alpha}^{-1}(\sigma)((\cdot,\cdot),(r,-x)))(s,y).$$

(4.18)

Since the Fourier transform $\mathcal{T}_{\alpha}$ is an isomorphism from $\mathcal{S}_*(\mathbb{R}^2)$ onto $\mathcal{S}_*(\Gamma)$, we deduce that the function $I \otimes \mathcal{T}_{\alpha}^{-1}(\sigma)$ belongs to $\mathcal{S}_*(\mathbb{R}^2 \times \mathbb{R}^2)$.

Then, the lemma follows from the fact that for all $g \in \mathcal{S}_*(\mathbb{R}^2 \times \mathbb{R}^2)$, the function

$$((r,x),(s,y)) \mapsto \mathcal{T}_{(r,-x)}(g((\cdot,\cdot),(r,-x)))(s,y)$$

(4.19)

belongs to $\mathcal{S}_*(\mathbb{R}^2 \times \mathbb{R}^2)$. \hfill \square

Theorem 4.7. Let $\sigma \in \mathcal{S}_*(\mathbb{R}^2 \times \Gamma)$.

(i) For all $f \in \mathcal{S}_*(\mathbb{R}^2)$,

$$\forall (r,x) \in \mathbb{R}^2, \quad W_\sigma(f)(r,x) = \int_{\mathbb{R}} \int_{0}^{\infty} k((r,x),(s,y)) f(s,y) d\nu(s,y).$$

(4.20)

(ii) For $f \in \mathcal{S}_*(\mathbb{R}^2)$ and $p, p' \in [1, +\infty]$ such that $1/p + 1/p' = 1$,

$$\|W_\sigma(f)\|_{p',\nu} \leq \|k\|_{p',\nu} \|f\|_{p,\nu}.$$  

(4.21)

(iii) For $p \in [1, +\infty[$, the operator $W_\sigma$ can be extended to a bounded operator from $L^p(d\nu)$ into $L^{p'}(d\nu)$.

In particular

$$W_\sigma : L^2(d\nu) \longrightarrow L^2(d\nu)$$

(4.22)

is a Hilbert-Schmidt operator, and consequently it is compact.

Proof. (i) Let $f$ be in $\mathcal{S}_*(\mathbb{R}^2)$. From Definition 4.4, for all $(\mu,\lambda) \in \mathbb{R}^2$, we have

$$W_\sigma(f)(r,x) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \phi_{\mu,\lambda}(r,x) \sigma((s,y),(\mu,\lambda)) \mathcal{T}_{(r,x)} f(s,y) d\nu(s,y) \right) d\nu(\mu,\lambda)$$

$$= \int_{\mathbb{R}} \phi_{\mu,\lambda}(r,x) \left( \int_{\mathbb{R}} \sigma((s,y),(\mu,\lambda)) \mathcal{T}_{(r,x)} f(s,y) d\nu(s,y) \right) d\nu(\mu,\lambda).$$

(4.23)

Using Fubini’s theorem, and the equality

$$\int_{\mathbb{R}} \int_{0}^{\infty} \sigma((s,y),(\mu,\lambda)) \mathcal{T}_{(r,x)} f(s,y) d\nu(s,y)$$

$$= \int_{\mathbb{R}} \int_{0}^{\infty} f(s,y) \mathcal{T}_{(r,-x)}(\sigma((\cdot,\cdot),(\mu,\lambda)))(s,y) d\nu(s,y),$$

(4.24)
we get

\[
W_\sigma(f)(r,x) = \int_\mathbb{R} \int_0^\infty f(s,y) \left\{ \int_\Gamma \phi_{\mu,\lambda}(r,x) \mathcal{T}_{(r,x)}(\sigma((\cdot, \cdot), (\mu, \lambda)))(s,y) d\nu(\mu, \lambda) \right\} d\nu(s,y) = \int_\mathbb{R} \int_0^\infty f(s,y) k((r,x), (s,y)) d\nu(s,y). \tag{4.25}
\]

(ii) follows from (i), Hölder’s inequality, and Lemma 4.6.

(iii) From (ii) and the fact that the space \( \mathcal{S}_*(\mathbb{R}^2) \) is dense in \( L^p(d\nu) \), \( p \in [1, +\infty[ \), we deduce that \( W_\sigma \) can be extended to a continuous mapping from \( L^p(d\nu) \) into \( L^p(d\nu) \).

By Lemma 4.6, the kernel \( k \) belongs to \( L^2(d\nu \otimes d\nu) \), hence \( W_\sigma \) is a Hilbert-Schmidt operator. In particular, it is compact. □

**Theorem 4.8.** Let \( \sigma \in S^m \), \( m < -(\alpha + 3/2) \). For all \( f, g \in \mathcal{S}_*(\mathbb{R}^2) \), we have

\[
\mathbb{H}_\sigma(f, g) = \left\langle \frac{W_\sigma(g)}{f} \right\rangle, \tag{4.26}
\]

where \( \langle \cdot / \cdot \rangle \) is the inner product of \( L^2(d\nu) \).

**Proof.** From Definition (3.1) and relations (4.2), (4.3), we get

\[
\mathbb{H}_\sigma(f, g) = \int_\Gamma \left\{ \int_\mathbb{R} \int_0^\infty \sigma((r,x), (\mu, \lambda)) \left( \int_\mathbb{R} \int_0^\infty f(s,y) \phi_{\mu,\lambda}(s,y) \times \mathcal{T}_{(r,x)}g(s,y) d\nu(s,y) \right) d\nu(r,x) \right\} d\nu(\mu, \lambda). \tag{4.27}
\]

Using Fubini’s theorem, we obtain

\[
\mathbb{H}_\sigma(f, g) = \int_\mathbb{R} \int_0^\infty f(s,y) \left\{ \int_\mathbb{R} \phi_{\mu,\lambda}(s,y) \left( \int_\mathbb{R} \int_0^\infty \sigma((r,x), (\mu, \lambda)) \times \mathcal{T}_{(r,x)}g(s,y) d\nu(r,x) \right) d\nu(\mu, \lambda) \right\} d\nu(s,y). \tag{4.28}
\]

The theorem follows from Definition 4.4 and the fact that for all \( ((r,x), (s,y)) \in [0, +\infty[ \times \mathbb{R} \),

\[
\mathcal{T}_{(r,x)}g(s,y) = \mathcal{T}_{(s,y)}g(r,x). \tag{4.29}
\]

□

5. **Weyl transform associated with symbol in** \( L^p(d\nu \otimes d\gamma) \), \( 1 \leq p \leq 2 \)

In this section, we will see that relation (4.26) allows us to prove that the Weyl transform with symbol in \( L^p(d\nu \otimes d\gamma) \), \( 1 \leq p \leq 2 \), is a compact operator.
We denote by $\mathcal{B}(L^2(d\nu))$ the $\mathbb{C}^*$-algebra of bounded operators $\psi$ from $L^2(d\nu)$ into itself, equipped with the norm

$$
\| \psi \|_* = \sup_{\|f\|_{2,\nu} = 1} \| \psi(f) \|_{2,\nu}.
$$

(5.1)

**Theorem 5.1.** For $p \in [1,2]$, there exists a unique bounded operator $Q$ from $L^p(d\nu \otimes d\gamma)$ into $\mathcal{B}(L^2(d\nu))$: $\sigma \mapsto Q_\sigma$, such that for all $f,g \in \mathcal{F}_*(\mathbb{R}^2)$,

$$
\left\langle \frac{Q_\sigma(g)}{f} \right\rangle = \left\langle \frac{W_\sigma(g)}{f} \right\rangle = \mathbb{H}_\sigma(f,g)
$$

(5.3)

\begin{align*}
&= \int_\Gamma \left( \int_\mathbb{R} \int_0^\infty \sigma((r,x),(\mu,\lambda)) V(f,g)((r,x),(\mu,\lambda)) d\nu(r,x) \right) d\gamma(\mu,\lambda).
\end{align*}

(5.2)

On the other hand, from Proposition 3.3(ii) and Cauchy-Shwartz inequality, we have

$$
\left| \left\langle \frac{Q_\sigma(g)}{f} \right\rangle \right| \leq \|\sigma\|_{2,\nu \otimes \gamma} \|f\|_{2,\nu} \|g\|_{2,\nu}.
$$

(5.4)

This implies that $Q_\sigma \in \mathcal{B}(L^2(d\nu))$ and

$$
\|Q_\sigma\|_* \leq \|\sigma\|_{2,\nu \otimes \gamma}.
$$

(5.5)

We complete the proof by using the fact that the space $\mathcal{F}_*(\mathbb{R}^2 \times \Gamma)$ is dense in $L^2(d\nu \otimes d\gamma)$.

(i) The case $p = 2$.

Let $\sigma \in \mathcal{F}_*(\mathbb{R}^2 \times \Gamma)$. For $g \in \mathcal{F}_*(\mathbb{R}^2)$, we put $Q_\sigma(g) = W_\sigma(g)$.

From Theorem 4.8, we obtain

$$
\left\langle \frac{Q_\sigma(g)}{f} \right\rangle = \left\langle \frac{W_\sigma(g)}{f} \right\rangle = \mathbb{H}_\sigma(f,g)
$$

(5.3)

\begin{align*}
&= \int_\Gamma \left( \int_\mathbb{R} \int_0^\infty \sigma((r,x),(\mu,\lambda)) V(f,g)((r,x),(\mu,\lambda)) d\nu(r,x) \right) d\gamma(\mu,\lambda).
\end{align*}

(5.2)

On the other hand, from Proposition 3.3(ii) and Cauchy-Shwartz inequality, we have

$$
\left| \left\langle \frac{Q_\sigma(g)}{f} \right\rangle \right| \leq \|\sigma\|_{2,\nu \otimes \gamma} \|f\|_{2,\nu} \|g\|_{2,\nu}.
$$

(5.4)

This implies that $Q_\sigma \in \mathcal{B}(L^2(d\nu))$ and

$$
\|Q_\sigma\|_* \leq \|\sigma\|_{2,\nu \otimes \gamma}.
$$

(5.5)

We complete the proof by using the fact that the space $\mathcal{F}_*(\mathbb{R}^2 \times \Gamma)$ is dense in $L^2(d\nu \otimes d\gamma)$.

(ii) The case $p = 1$ can be obtained by the same way.

(iii) Using the cases $p = 1$, $p = 2$, and the Riesz-Thorin theorem [10, 11], we complete the proof for all $p \in [1,2]$. \qed

**Remark 5.2.** In the following, the operator $Q_\sigma$ will be denoted by $W_\sigma$.

**Theorem 5.3.** For $\sigma \in L^p(d\nu \otimes d\gamma)$, $1 \leq p \leq 2$, the operator $W_\sigma$ from $L^2(d\nu)$ into itself is a compact operator.

**Proof.** Let $\sigma \in L^p(d\nu \otimes d\gamma)$, $1 \leq p \leq 2$, and let $(\sigma_k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{F}_*(\mathbb{R}^2 \times \Gamma)$, such that

$$
\|\sigma_k - \sigma\|_{p,\nu \otimes \gamma} \to 0.
$$

(5.6)

From relation (5.5), we have $\|W_{\sigma_k} - W_\sigma\|_* \leq \|\sigma_k - \sigma\|_{p,\nu \otimes \gamma}$. This implies that

$$
W_{\sigma_k} \to W_\sigma, \quad \text{in} \ \mathcal{B}(L^2(d\nu)).
$$

(5.7)
But from Theorem 4.7, we know that for all $k \in \mathbb{N}$, the operator $W_{\sigma_k}$ is compact, then the result of the theorem follows from the fact that the subspace $\mathcal{H}(L^2(d\nu))$ of $\mathcal{B}(L^2(d\nu))$ consisting of compact operators is a closed ideal of $\mathcal{B}(L^2(d\nu))$. \hfill \square

6. Weyl transform with symbol in $S'_*(\mathbb{R}^2 \times \Gamma)$

We denote by

(i) $S'_*(\mathbb{R}^2)$ the space of tempered distributions on $\mathbb{R}^2$, even with respect to the first variable. It is the topological dual of $S_*(\mathbb{R}^2)$;

(ii) $S'_*(\mathbb{R}^2 \times \Gamma)$ the space of tempered distributions on $\mathbb{R}^2 \times \Gamma$, even with respect to the first variables of $\mathbb{R}^2$ and $\Gamma$. It is the topological dual of $S_*(\mathbb{R}^2 \times \Gamma)$.

**Definition 6.1.** For $\sigma \in S'_*(\mathbb{R}^2 \times \Gamma)$ and $g \in S_*(\mathbb{R}^2)$, define the operator $W_{\sigma}(g)$ on $S_*(\mathbb{R}^2)$, by

$$
[W_{\sigma}(g)](f) = \sigma(V(f,g)), \quad f \in S_*(\mathbb{R}^2),
$$

(6.1)

where $V$ is the mapping given by (3.1).

**Remark 6.2.** From Proposition 3.3, it is clear that $W_{\sigma}(g)$ given by (6.1) belongs to $S'_*(\mathbb{R}^2)$.

For a slowly increasing function $h$ on $\mathbb{R}^2 \times \Gamma$, we denote by $\sigma_h$ the element of $S'_*(\mathbb{R}^2 \times \Gamma)$ defined by

$$
\sigma_h(F) = \iint_{\Gamma} \int_{\mathbb{R}} \int_0^\infty F((r,x),(\mu,\lambda)) h((r,x),(\mu,\lambda)) d\nu(r,x) d\gamma(\mu,\lambda).
$$

(6.2)

Then, we have the following.

**Proposition 6.3.** Let $\sigma_1 \in S'_*(\mathbb{R}^2 \times \Gamma)$, given by the function equal to 1. One has

$$
W_{\sigma_1}(g) = c\delta,
$$

(6.3)

where $c = \int_{\mathbb{R}} \int_0^\infty g(r,x) d\nu(r,x)$ and $\delta$ is the Dirac distribution at $(0,0)$.

**Proof.** By relation (6.1), we have for all $f$ in $S_*(\mathbb{R}^2)$,

$$
[W_{\sigma_1}(g)](f) = \sigma_1(V(f,g)),
$$

$$
= \iint_{\Gamma} \left( \int_{\mathbb{R}} \int_0^\infty V(f,g)((r,x),(\mu,\lambda)) d\nu(r,x) \right) d\gamma(\mu,\lambda),
$$

(6.4)

and by Theorem 3.6

$$
[W_{\sigma_1}(g)](f) = c \iint_{\Gamma} \mathcal{F}_a(f)(\mu,\lambda) d\gamma(\mu,\lambda).
$$

(6.5)

We complete the proof by using relation (2.25). \hfill \square
Remark 6.4. From Proposition 6.3, we deduce that there exists a function in $L^\infty(\mathbb{R}^2 \times \Gamma)$ given by

$$c = \int_{\mathbb{R}} \int_{0}^{\infty} g(r,x) d\nu(r,x) \neq 0,$$  \hfill (6.6)

the distribution $W_\sigma(g)$ is not given by a function of $L^2(d\nu)$.

7. Weyl transform with symbol in $L^p(d\nu \otimes d\gamma)$, $2 < p < \infty$

Theorem 7.1. Let $p \in ]2, +\infty[. There exists a function $\sigma \in L^p(d\nu \otimes d\gamma)$, such that the Weyl transform $W_\sigma$ defined by (6.1) is not a bounded linear operator on $L^2(d\nu)$.

We break down the proof into two lemmas, of which the theorem is an immediate consequence.

Lemma 7.2. Let $2 < p < \infty$. Suppose that for all $\sigma \in L^p(d\nu \otimes d\gamma)$, the Weyl transform $W_\sigma$ given by relation (6.1) is a bounded linear operator on $L^2(d\nu)$. Then, there exists a positive constant $M$ such that

$$\|W_\sigma\|_* \leq M \|\sigma\|_{p,\nu \otimes \gamma}, \quad \forall \sigma \in L^p(d\nu \otimes d\gamma).$$ \hfill (7.1)

Proof. Under the assumption of the lemma, there exists for each $\sigma \in L^p(d\nu \otimes d\gamma)$ a positive constant $C_\sigma$ such that

$$\|W_\sigma(g)\|_{2,\nu} \leq C_\sigma \|g\|_{2,\nu}, \quad \text{for } g \in L^2(d\nu).$$ \hfill (7.2)

Let $f, g \in \mathcal{F}_*(\mathbb{R}^2)$ such that $\|f\|_{2,\nu} = \|g\|_{2,\nu} = 1$, and let us define the operator

$$Q_{f,g} : L^p(d\nu \otimes d\gamma) \rightarrow \mathbb{C}$$ \hfill (7.3)

by

$$Q_{f,g}(\sigma) = \left\langle \frac{W_\sigma(g)}{f} \right\rangle.$$ \hfill (7.4)

Then,

$$\sup_{\|f\|_{2,\nu} = \|g\|_{2,\nu} = 1} \left| Q_{f,g}(\sigma) \right| \leq C_\sigma.$$ \hfill (7.5)

By the Banach-Steinhauss theorem, the operator $Q_{f,g}$ is bounded on $L^p(d\nu \otimes d\gamma)$, then there exists a positive constant $M$ such that

$$\|Q_{f,g}\|_* = \sup_{\|\sigma\|_{p,\nu \otimes \gamma} = 1} \left| Q_{f,g}(\sigma) \right| \leq M.$$ \hfill (7.6)

From this, we deduce that for all $f, g \in \mathcal{F}_*(\mathbb{R}^2)$, and $\sigma \in L^p(d\nu \otimes d\gamma)$, we have

$$\left| \left\langle \frac{W_\sigma(g)}{f} \right\rangle \right| \leq M \|\sigma\|_{p,\nu \otimes \gamma} \|f\|_{2,\nu} \|g\|_{2,\nu},$$ \hfill (7.7)

which implies (7.1). \qed
Lemma 7.3. For $2 < p < \infty$, there is no positive constant $M$ satisfying (7.1).

Proof. Suppose that there exists $M > 0$ such that relation (7.1) holds.

Let $p'$ be such that $1/p + 1/p' = 1$, then $p' \in [1, 2[$.

We consider for $f, g \in \mathcal{F}_\ast(\mathbb{R}^2)$, the function $V(f, g)$ given by the relation (3.1). We have

$$\|V(f, g)\|_{p', \gamma \otimes y} = \sup_{\|\sigma\|_{p, \nu y} = 1} \left| \left\langle \frac{W_\sigma(g)}{f} \right\rangle \right| \leq \sup_{\|\sigma\|_{p, \nu y} = 1} \|W_\sigma(g)\|_{2, \nu} \|f\|_{2, \nu}, \quad (7.8)$$

and consequently

$$\|V(f, g)\|_{p', \gamma \otimes y} \leq M \|f\|_{2, \nu} \|g\|_{2, \nu}. \quad (7.9)$$

Now, let $f, g \in L^2(d\nu)$, we choose sequences $(f_k)_{k \in \mathbb{N}}$ and $(g_k)_{k \in \mathbb{N}}$ in $\mathcal{F}_\ast(\mathbb{R}^2)$, approximating $f$ and $g$ in the $\|\cdot\|_{2, \nu}$-norm.

From (7.9), we get

$$\|V(f_k, g_k)\|_{p', \gamma \otimes y} \leq M \|f_k\|_{2, \nu} \|g_k\|_{2, \nu} \quad (7.10)$$

which implies that $(V(f_k, g_k))_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^{p'}(d\nu \otimes d\gamma)$. Then, it converges to some function $F$ in $L^{p'}(d\nu \otimes d\gamma)$.

Now, using Proposition 3.3, we deduce that $F = V(f, g)$, and

$$\forall f, g \in L^2(d\nu), \quad \|V(f, g)\|_{p', \gamma \otimes y} \leq M \|f\|_{2, \nu} \|g\|_{2, \nu}. \quad (7.11)$$

We will exhibit an example where the relation (7.11) leads to a contradiction. Let $f$ be defined on $\mathbb{R}^2$, even with respect to the first variable, and supported in $[-1, 1] \times [-1, 1]$.

Then, for all $((r, x), (\mu, \lambda)) \in \mathbb{R}^2 \times \Gamma$,

$$|V(f, f)((r, x), (\mu, \lambda))| \leq |f| * |\hat{f}|((r, -x)), \quad (7.12)$$

where $*$ is the convolution product given by Definition 2.2. From (2.18), we deduce that for all $(\mu, \lambda) \in \Gamma$, the function $(r, x) \mapsto V(f, f)((r, x), (\mu, \lambda))$ is supported in $[-2, 2] \times [-2, 2]$.

On the other hand, by Hölder’s inequality, we have

$$\left( \int_{\Gamma} \int_{-2}^{2} \int_{0}^{2} V(f, f)((r, x), (\mu, \lambda)) d\nu(r, x) \right)^{1/p'} \leq \left( \int_{-2}^{2} \int_{0}^{2} d\nu(r, x) \right)^{1/p} \left( \int_{\Gamma} \int_{-2}^{2} \int_{0}^{+\infty} |V(f, f)((r, x), (\mu, \lambda))|^{p'} d\nu(r, x) d\gamma(\mu, \lambda) \right)^{1/p'}$$

$$= \left( \int_{-2}^{2} \int_{0}^{2} d\nu(r, x) \right)^{1/p} \|V(f, f)\|_{p', \gamma \otimes y} \leq M \left( \int_{-2}^{2} \int_{0}^{2} d\nu(r, x) \right)^{1/p} \|f\|_{2, \nu}^{2}, \quad (7.13)$$
The last inequality follows from (7.9). Now, Theorem 3.6 implies that the function

\[(\mu, \lambda) \mapsto -\int_0^{+\infty} V(f, f)((r, x), (\mu, \lambda))d\nu(r, x) = c\mathcal{F}_a(f)(\mu, \lambda) \tag{7.14}\]

belongs to \(L^p(d\gamma)\), here \(c = \int_0^{+\infty} f(r, x)d\nu(r, x)\).

If we pick \(c = \int_0^{+\infty} f(r, x)d\nu(r, x) \neq 0\), and the last inequality, we deduce that the function \(\mathcal{F}_a(f)\) belongs to \(L^p(d\gamma)\), and

\[\|\mathcal{F}_a(f)\|_{p', \gamma} \leq \frac{M}{|\gamma|} \left(\int_{-\infty}^{\infty} \int_0^{+\infty} d\nu(r, x)\right)^{1/p} \|f\|_{2, \nu}^2. \tag{7.15}\]

In the following, we consider the particular function \(f\) given by

\[f(r, x) = |r|^\beta \mathbf{1}_{[-1,1]}(r)\mathbf{1}_{[-1,1]}(x), \tag{7.16}\]

where \(\mathbf{1}_{[-1,1]}\) is the characteristic function of the interval \([-1, 1]\).

This function belongs to \(L^1(d\nu) \cap L^2(d\nu)\), for \(\beta > -(\alpha + 1)\), and we have

\[\mathcal{F}_a(f)(\mu, \lambda) = \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)\sqrt{2\pi}} \frac{\sin \lambda}{\lambda} \int_0^1 r^{\beta+2\alpha+1} j_\alpha(r\mu)dr, \tag{7.17}\]

so

\[\|\mathcal{F}_a(f)\|_{p', \nu} = \frac{2^{p'}}{(2^{\alpha+1}\Gamma(\alpha+1)\sqrt{2\pi})^{p'+1}} \int_{\mathbb{R}} \left| \left| \int_0^{+\infty} r^{p'} d\lambda \right|_{\nu} \right|^p \mu^{2\alpha+1}d\mu. \tag{7.18}\]

However

\[\int_0^1 r^{\beta+2\alpha+1} j_\alpha(r\mu)dr = \frac{1}{\mu^{\beta+2\alpha+2}} \int_0^\mu r^{\beta+2\alpha+1} j_\alpha(r)dr. \tag{7.19}\]

Using the asymptotic expansion of \(j_\alpha\) (see [7, 12]), given by

\[j_\alpha(r) = \frac{2^{\alpha+1/2}\Gamma(\alpha+1)}{\sqrt{\pi} r^{\alpha+1/2}} \left[ \cos \left( r - \alpha \frac{\pi}{2} - \frac{\pi}{4} \right) + O\left( \frac{1}{r} \right) \right], \quad \text{as } (r \to +\infty), \tag{7.20}\]

we deduce that for \(-(\alpha + 1) < \beta < -(\alpha + 1/2)\), the integral

\[a = \int_0^{+\infty} r^{\beta+2\alpha+1} j_\alpha(r)dr \tag{7.21}\]
exists and is finite. This involves that

$$\int_{0}^{1} r^{\beta+2\alpha+1} j_{\alpha}(r\mu) dr \sim \frac{a}{\mu^{\beta+2\alpha+2}}, \quad \text{as } (\mu \to +\infty). \quad (7.22)$$

Then, there exist $A, B > 0$ such that for

$$\mu > A, \quad \left| \int_{0}^{1} r^{\beta+2\alpha+1} j_{\alpha}(r\mu) dr \right| \geq \frac{B}{\mu^{\beta+2\alpha+2}}. \quad (7.23)$$

Replacing in relation (7.18), we get

$$\|\bar{\mathcal{F}}_{\alpha}(f)\|_{p',\gamma} \geq \frac{(2B)^{p'}}{(2^{\alpha}(\alpha+1)\sqrt{2\pi})^{p' + 1}} \int_{\mathbb{R}} \left| \frac{\sin \lambda}{\lambda} \right|^{p'} d\lambda \int_{A}^{+\infty} \frac{d\mu}{\mu^{p'(2\alpha+\beta+2) - 2\alpha - 1}}. \quad (7.24)$$

Thus, for $\beta < -(2\alpha + 2) + (2\alpha + 2/p')$,

$$\|\bar{\mathcal{F}}_{\alpha}(f)\|_{p',\gamma} = \|\bar{\mathcal{F}}_{\alpha}(f)\|_{p',\nu} = +\infty. \quad (7.25)$$

This shows that relation (7.15) is false if we pick

$$\beta \in \left[ -(\alpha + 1), \min \left( -\left( \alpha + \frac{1}{2} \right), -(2\alpha + 2) + \frac{2\alpha + 2}{p'} \right) \right]. \quad (7.26)$$

References


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<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>April 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>July 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>October 1, 2009</td>
</tr>
</tbody>
</table>

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