We give some characterizations of weak-open compact images of metric spaces.

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1. Introduction and definitions

To find internal characterizations of certain images of metric spaces is one of central problems in general topology. Arhangel’skii [1] showed that a space is an open compact image of a metric space if and only if it has a development consisting of point-finite open covers, and some characterizations for certain quotient compact images of metric spaces are obtained in [3, 5, 8]. Recently, Xia [12] introduced the concept of weak-open mappings. By using it, certain g-first countable spaces are characterized as images of metric spaces under various weak-open mappings. Furthermore, Li and Lin in [4] proved that a space is g-metrizable if and only if it is a weak-open σ-image of a metric space.

The purpose of this paper is to give some characterizations of weak-open compact images of metric spaces, which showed that a space is a weak-open compact image of a metric space if and only if it has a weak development consisting of point-finite cs-covers.

In this paper, all spaces are Hausdorff, all mappings are continuous and surjective. \( \mathbb{N} \) denotes the set of all natural numbers. \( \tau(X) \) denotes the topology on a space \( X \). For the usual product space \( \prod_{i \in \mathbb{N}} X_i \), \( \pi_i \) denotes the projection \( \prod_{i \in \mathbb{N}} X_i \) onto \( X_i \). For a sequence \( \{x_n\} \) in \( X \), denote \( \langle x_n \rangle = \{x_n : n \in \mathbb{N}\} \).

Definition 1.1 [1]. Let \( \mathcal{P} = \bigcup \{\mathcal{P}_x : x \in X\} \) be a collection of subsets of a space \( X \). \( \mathcal{P} \) is called a weak base for \( X \) if

1. for each \( x \in X \), \( \mathcal{P}_x \) is a network of \( x \) in \( X \),
2. if \( U, V \in \mathcal{P}_x \), then \( W \subset U \cap V \) for some \( W \in \mathcal{P}_x \),
3. \( G \subset X \) is open in \( X \) if and only if for each \( x \in G \), there exists \( P \in \mathcal{P}_x \) such that \( P \subset G \).

\( \mathcal{P}_x \) is called a weak neighborhood base of \( x \) in \( X \), every element of \( \mathcal{P}_x \) is called a weak neighborhood of \( x \) in \( X \).
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Definition 1.2. Let $f : X \to Y$ be a mapping.

1. $f$ is called a weak-open mapping [12], if there exists a weak base $\mathcal{B} = \cup \{B_y : y \in Y\}$ for $Y$, and for each $y \in Y$, there exists $x_y \in f^{-1}(y)$ satisfying the following condition: for each open neighborhood $U$ of $x_y$, $B_y \subset f(U)$ for some $B_y \in \mathcal{B}_y$.

2. $f$ is called a compact mapping, if $f^{-1}(y)$ is compact in $X$ for each $y \in Y$.

It is easy to check that a weak-open mapping is quotient.

Definition 1.3 [2]. Let $X$ be a space, and $P \subset X$. Then the following hold.

1. A sequence $\{x_n\}$ in $X$ is called eventually in $P$, if the $\{x_n\}$ converges to $x$, and there exists $m \in \mathbb{N}$ such that $\{x\} \cup \{x_n : n \geq m\} \subset P$.

2. $P$ is called a sequential neighborhood of $x$ in $X$, if whenever a sequence $\{x_n\}$ in $X$ converges to $x$, then $\{x_n\}$ is eventually in $P$.

3. $P$ is called sequential open in $X$, if $P$ is a sequential neighborhood at each of its points.

4. $X$ is called a sequential space, if any sequential open subset of $X$ is open in $X$.

Definition 1.4 [7]. Let $\mathcal{P}$ be a cover of a space $X$.

1. $\mathcal{P}$ is called a cs-cover for $X$, if every convergent sequence in $X$ is eventually in some element of $\mathcal{P}$.

2. $\mathcal{P}$ is called an sn-cover for $X$, if every element of $\mathcal{P}$ is a sequential neighborhood of some point in $X$, and for any $x \in X$, there exists a sequential neighborhood $P$ of $x$ in $X$ such that $P \in \mathcal{P}$.

Definition 1.5 [7]. Let $\{\mathcal{P}_n\}$ be a sequence of covers of a space $X$.

1. $\{\mathcal{P}_n\}$ is called a point-star network for $X$, if for each $x \in X$, $\langle st(x, \mathcal{P}_n) \rangle$ is a network of $x$ in $X$.

2. $\{\mathcal{P}_n\}$ is called a weak development for $X$, if for each $x \in X$, $\langle st(x, \mathcal{P}_n) \rangle$ is a weak neighborhood base for $X$.

2. Results

Theorem 2.1. The following are equivalent for a space $X$.

1. $X$ is a weak-open compact image of a metric space.

2. $X$ has a weak development consisting of point-finite cs-covers.

3. $X$ has a weak development consisting of point-finite sn-covers.

Proof. (1) ⇒ (2). Suppose that $f : M \to X$ is a weak-open compact mapping with $M$ a metric space. Let $\{\mathcal{U}_n\}$ be a sequence consisting of locally finite open covers of $M$ such that $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$ and $\langle st(K, \mathcal{U}_n) \rangle$ forms a neighborhood base of $K$ in $M$ for each compact subset $K$ of $M$ (see [7, Theorem 1.3.1]). For each $n \in \mathbb{N}$, put $\mathcal{P}_n = f(\mathcal{U}_n)$. Since $f$ is compact, then $\{\mathcal{P}_n\}$ is a point-finite cover sequence of $X$.

If $x \in V$ with $V$ open in $X$, then $f^{-1}(x) \subset f^{-1}(V)$. Since $f^{-1}(x)$ compact in $M$, then $st(f^{-1}(x), \mathcal{U}_n) \subset f^{-1}(V)$ for some $n \in \mathbb{N}$, and so $st(x, \mathcal{P}_n) \subset V$. Hence $\langle st(x, \mathcal{P}_n) \rangle$ forms a network of $x$ in $X$. Therefore, $\{\mathcal{P}_n\}$ is a point-star network for $X$.

We will prove that every $\mathcal{P}_k$ is a cs-cover for $X$. Since $f$ is weak-open, there exists a weak base $\mathcal{B} = \cup \{B_x : x \in X\}$ for $X$, and for each $x \in X$, there exists $m_x \in f^{-1}(x)$
satisfying the following condition: for each open neighborhood \( U \) of \( m_x \) in \( M \), \( B \subseteq f(U) \) for some \( B \in \mathcal{B}_x \).

For each \( x \in X \) and \( k \in \mathbb{N} \), let \( \{x_n\} \) be a sequence converging to a point \( x \in X \). Take \( U \in \tau_k \) with \( m_x \in U \). Then \( B \subseteq f(U) \) for some \( B \in \mathcal{B}_x \). Since \( B \) is a weak neighborhood of \( x \) in \( X \), then \( B \) is a sequential neighborhood of \( x \) in \( X \) by \cite[Corollary 1.6.18]{6}, so \( f(U) \in \mathcal{P}_k \) is also. Thus \( \{x_n\} \) is eventually in \( f(U) \). This implies that each \( \mathcal{P}_k \) is a \( cs \)-cover for \( X \). Since \( f(U) \) is a sequential neighborhood of \( x \) in \( X \), then \( st(x, \mathcal{P}_k) \) is also. Obviously, \( X \) is a sequential space. So \( \langle st(x, \mathcal{P}_k) \rangle \) is a weak neighborhood base of \( x \) in \( X \).

In words, \( \{\mathcal{P}_n\} \) is a weak development consisting of point-finite \( cs \)-covers for \( X \).

(2) \( \Rightarrow \) (3). By Theorem A in \cite{5}, \( X \) is a sequential space. It suffices to prove that if \( \mathcal{P} \) is a point-finite \( cs \)-cover for \( X \), then some subset of \( \mathcal{P} \) is an \( sn \)-cover for \( X \). For each \( x \in X \), denote \( (\mathcal{P})_x = \{P_i : i \leq k\} \), where \( (\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\} \). If each element of \( (\mathcal{P})_x \) is not a sequential neighborhood of \( x \) in \( X \), then for each \( i \leq k \), there exists a sequence \( \{x_{in}\} \) converging to \( x \) such that \( \{x_{in}\} \) is not eventually in \( P_i \). For each \( n \in \mathbb{N} \) and \( i \leq k \), put \( y_{i+\lfloor n/2 \rfloor} = x_{in} \), then \( \{y_m\} \) converges to \( x \) and is not eventually in each \( P_i \), a contradiction. Thus there exists \( P_x \in \mathcal{P} \) such that \( P_x \) is a sequential neighborhood of \( x \) in \( X \). Put \( \mathcal{F} = \{P_x : x \in X\} \), then \( \mathcal{F} \) is an \( sn \)-cover for \( X \).

(3) \( \Rightarrow \) (1). Suppose \( \{\mathcal{P}_n\} \) is a weak development consisting of point-finite \( sn \)-covers for \( X \). For each \( i \in \mathbb{N} \), let \( \mathcal{P}_i = \{P_\alpha : \alpha \in \Lambda_i\} \), endow \( \Lambda_i \) with the discrete topology, then \( \Lambda_i \) is a metric space. Put

\[
M = \left\{ \alpha = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Lambda_i : \langle P_\alpha \rangle \text{ forms a network at some point } x_\alpha \text{ in } X \right\}, \tag{2.1}
\]

and endow \( M \) with the subspace topology induced from the usual product topology of the collection \( \{\Lambda_i : i \in \mathbb{N}\} \) of metric spaces, then \( M \) is a metric space. Since \( X \) is Hausdorff, \( x_\alpha \) is unique in \( X \). For each \( \alpha \in M \), we define \( f : M \to X \) by \( f(\alpha) = x_\alpha \). For each \( x \in X \) and \( i \in \mathbb{N} \), there exists \( \alpha_i \in \Lambda_i \) such that \( x \in P_{\alpha_i} \). From \( \{\mathcal{P}_i\} \) being a point-star network for \( X \), \( \{P_{\alpha_i} : i \in \mathbb{N}\} \) is a network of \( x \) in \( X \). Put \( \alpha = (\alpha_i) \), then \( \alpha \in M \) and \( f(\alpha) = x \). Thus \( f \) is surjective. Suppose \( \alpha = (\alpha_i) \in M \) and \( f(\alpha) = x \in U \in \tau(X) \), then there exists \( n \in \mathbb{N} \) such that \( P_{\alpha_n} \subseteq U \). Put

\[
V = \{\beta \in M : \text{the } n\text{th coordinate of } \beta \text{ is } \alpha_n\}. \tag{2.2}
\]

Then \( \alpha \in V \in \tau(M) \), and \( f(V) \subseteq P_{\alpha_n} \subseteq U \). Hence \( f \) is continuous.

For each \( x \in X \) and \( i \in \mathbb{N} \), put

\[
B_i = \{\alpha_i \in \Lambda_i : x \in P_{\alpha_i}\}, \tag{2.3}
\]

then \( \prod_{i \in \mathbb{N}} B_i \) is compact in \( \prod_{i \in \mathbb{N}} \Lambda_i \). If \( \alpha = (\alpha_i) \in \prod_{i \in \mathbb{N}} B_i \), then \( \langle P_\alpha \rangle \) is a network of \( x \) in \( X \). So \( \alpha \in M \) and \( f(\alpha) = x \). Hence \( \prod_{i \in \mathbb{N}} B_i \subseteq f^{-1}(x) \). If \( \alpha = (\alpha_i) \in f^{-1}(x) \), then \( x \in \bigcap_{i \in \mathbb{N}} P_{\alpha_i} \), so \( \alpha \in \prod_{i \in \mathbb{N}} B_i \). Thus \( f^{-1}(x) \subseteq \prod_{i \in \mathbb{N}} B_i \). Therefore, \( f^{-1}(x) = \prod_{i \in \mathbb{N}} B_i \). This implies that \( f \) is a compact mapping.
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We will prove that $f$ is weak-open. For each $x \in X$, since every $\mathcal{P}_i$ is an $sn$-cover for $X$, then there exists $\alpha_i \in \Lambda_i$ such that $P_{\alpha_i}$ is a sequential neighborhood of $x$ in $X$. From $\{\mathcal{P}_i\}$ a point-star network for $X$, $\langle P_{\alpha_i} \rangle$ is a network of $x$ in $X$. Put $\beta_x = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Lambda_i$, then $\beta_x \in f^{-1}(x)$.

Let $\{U_{m\beta_x}\}$ be a decreasing neighborhood base of $\beta_x$ in $M$, and put

$$\mathcal{B}_x = \{ f(U_{m\beta_x}) : m \in \mathbb{N} \},$$

$$\mathcal{B} = \bigcup \{ \mathcal{B}_x : x \in X \},$$

then $\mathcal{B}$ satisfies (1), (2) in Definition 1.1. Suppose $G$ is open in $X$. For each $x \in G$, from $\beta_x \in f^{-1}(x)$, $f^{-1}(G)$ is an open neighborhood of $\beta_x$ in $M$. Thus $U_{m\beta_x} \subset f^{-1}(G)$ for some $m \in \mathbb{N}$, so $f(U_{m\beta_x}) \subset G$ and $f(U_{m\beta_x}) \in \mathcal{B}_x$. On the other hand, suppose $G \subset X$ and for $x \in G$, there exists $B \in \mathcal{B}_x$ such that $B \subset G$. Let $B = f(U_{m\beta_x})$ for some $m \in \mathbb{N}$, and let $\{x_n\}$ be a sequence converging to $x$ in $X$. Since $P_{\alpha_i}$ is a sequential neighborhood of $x$ in $X$ for each $i \in \mathbb{N}$, then $\{x_n\}$ is eventually in $P_{\alpha_i}$. For each $n \in \mathbb{N}$, if $x_n \in P_{\alpha_i}$, let $\alpha_{in} = \alpha_i$; if $x_n \notin P_{\alpha_i}$, pick $\alpha_{in} \in \Lambda_i$ such that $x_n \in P_{\alpha_{in}}$. Thus there exists $n_i \in \mathbb{N}$ such that $\alpha_{in} = \alpha_i$ for all $n > n_i$. So $\{\alpha_{in}\}$ converges to $\alpha_i$. For each $n \in \mathbb{N}$, put

$$\beta_n = (\alpha_{in}) \in \prod_{i \in \mathbb{N}} \Lambda_i,$$

then $f(\beta_n) = x_n$ and $\{\beta_n\}$ converges to $\beta_x$. Since $U_{m\beta_x}$ is an open neighborhood $\beta_x$ in $M$, then $\{\beta_n\}$ is eventually in $U_{m\beta_x}$, so $\{x_n\}$ is eventually in $G$. Hence $G$ is a sequential neighborhood of $x$. So $G$ is sequential open in $X$. By $X$ being a sequential space, $G$ is open in $X$. This implies $\mathcal{B}$ is a weak base for $X$.

By the idea of $\mathcal{B}$, $f$ is weak-open.

We give examples illustrating Theorem 2.1 of this note.

**Example 2.2.** Let $X$ be the Arens space $S_2$ (see [6, Example 1.8.6]). It is not difficult to see that the space is a weak-open compact image of a metric space. But $X$ is not an open compact image of a metric space, because $X$ is not developable. Thus the following holds.

A weak-open compact image of a metric space is not always an open compact image of a metric space.

**Example 2.3.** Let $Y$ be the weak Cauchy space in [10, Example 2.14(3)]. By the construction, $Y$ is a quotient compact image of a metric space. But $Y$ is not Cauchy, $Y$ is not a weak-open compact image of a metric space by Theorem 2.1. Thus the following holds:

A quotient compact image of a metric space is not always a weak-open compact image of a metric space.

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References


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