We prove the existence of positive explosive solutions for the equation $\Delta u + \lambda(|x|)|\nabla u(x)| = \varphi(x, u(x))$ in the whole space $\mathbb{R}^N$ ($N \geq 3$), where $\lambda : [0, \infty) \to [0, \infty)$ is a continuous function and $\varphi : \mathbb{R}^N \times [0, \infty) \to [0, \infty)$ is required to satisfy some hypotheses detailed below. More precisely, we will give a necessary and sufficient condition for the existence of a positive solution that blows up at infinity.

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1. Introduction and the main result

Semilinear elliptic problems involving gradient term with boundary blowup interested many authors. Namely, Bandle and Giarrusso [1] developed existence and asymptotic behaviour results for large solutions of

$$\Delta u + |\nabla u(x)|^a = f(u)$$  \hspace{1cm} (1.1)

in a bounded domain.

In the case $f(u) = p(x) u^r$, $a > 0$, and $r > \max(1, a)$, Lair and Wood [7] dealt with the above equation in bounded domain and the whole space. They proved the existence of entire large solution under the condition $\int_0^\infty r \max_{|x|=r} p(x) dr < \infty$ when the domain is $\mathbb{R}^N$.

Recall that $u$ is a large solution on a bounded domain $\Omega$ in $\mathbb{R}^N$, if $u(x) \to +\infty$ as $\text{dist}(x, \partial \Omega) \to 0$, and $u$ is called an entire large solution if $u$ is defined on $\mathbb{R}^N$ and $\lim_{|x| \to +\infty} u(x) = +\infty$.

Ghergu et al. [3] considered more general equation

$$\Delta u + q(x)|\nabla u(x)|^a = p(x) f(u),$$  \hspace{1cm} (1.2)

where $0 \leq a \leq 2$, $p$ and $q$ are Hölder continuous functions on $(0, \infty)$. We note that the Keller-Osserman condition on $f$ (see [6, 8]) remains the key condition for the existence for their works.
2 Blowup solutions for semilinear problems

In the present paper, we are interested in the study of nonlinear elliptic problems with boundary blowup, of the type

\[ \Delta u + \lambda(|x|) |\nabla u(x)| = \varphi(x, u(x)), \quad \text{in } \mathbb{R}^N, \]

\[ u \geq 0, \quad u \neq 0, \quad \lim_{|x| \to +\infty} u(x) = +\infty, \]

\[(P)\]

where \( \lambda : [0, \infty) \to [0, \infty) \) is a continuous function and \( \varphi \) satisfies the following hypotheses.

(H1) \( \varphi : \mathbb{R}^N \times [0, \infty) \to [0, \infty) \) is measurable, continuous with respect to the second variable.

(H2) There exist nonnegative functions \( p, q, \) and \( f \) satisfying for each \( x \in \mathbb{R}^N \) and \( t \geq 0, \)

\[ p(|x|) f(t) \leq \varphi(x, t) \leq q(|x|) f(t), \]

\[(1.3)\]

where \( f \) is required to satisfy.

(H3) \( f \in \mathcal{C}^1([0, \infty)) \) such that \( f' \geq 0, f(0) = 0, f > 0 \) on \( (0, \infty), \)

\[ \int_1^\infty \frac{1}{f(\zeta)} d\zeta = +\infty, \]

\[(1.4)\]

and \( p, q \) are allowed to verify.

(H4) \( p, q : (0, \infty) \to [0, \infty) \) are continuous functions satisfying

\[ \int_0^1 s(1 - s)q(s) ds < +\infty. \]

\[(1.5)\]

Clearly, we see by (1.3) that the function \( p \) also satisfies (1.5).

In the sequel, we put

\[ h(r) = \int_0^r \frac{1}{K(t)} \left( \int_0^t K(s)q(s) ds \right) dt, \quad \text{for } r \in [0, \infty), \]

\[(1.6)\]

where \( K(t) := t^{N-1} \exp(\int_0^t \lambda(s) ds), \) for each \( t > 0, \) and we define the function \( F \) on \([1, \infty)\) by

\[ F(t) = \int_1^t \frac{1}{f(\zeta)} d\zeta. \]

\[(1.7)\]

From the hypotheses adopted on \( f, \) we note that the function \( F \) is a bijection from \([1, \infty)\) to \([0, \infty).\)

Our main result is the following.
Theorem 1.1. Assume that \((H_1)-(H_4)\) hold. Moreover, assume that
\[
\int_0^\infty \frac{1}{K(t)} \left( \int_0^t K(s)(q-p)(s)f \circ F^{-1}(2h(s))ds \right) dt < +\infty. \tag{1.8}
\]
Then problem \((P)\) has a positive entire solution if and only if
\[
\int_1^\infty \frac{1}{K(t)} \left( \int_0^t K(s)p(s)ds \right) dt = +\infty. \tag{1.9}
\]

Example 1.2. Let \(\alpha \geq 0\) and \(\beta \in [0,1]\). Assume that for \(t \geq 0\),
\[f(t) = (1 + t)^\beta \ln(1 + t)\]
and \(p(t) = 1/t^\alpha\). Then the following problem:
\[
\Delta u + \frac{1}{1 + |x|} |\nabla u(x)| = \frac{(1 + u(x))^{\beta}}{|x|^\alpha} \ln (1 + u(x)), \quad \text{in } \mathbb{R}^N,
\]
\[
u \geq 0, \quad u \neq 0, \quad \lim_{|x| \to +\infty} u(x) = +\infty
\]
has an explosive solution if and only if \(0 \leq \alpha < 2\).

Motivation for the present contribution stems from the one of Ghergu and Rădulescu [4] who considered the following problem:
\[
\Delta u + |\nabla u(x)| = p(x)f(u), \quad \text{in } \Omega, \tag{1.11}
\]
where \(\Omega\) is either a smooth bounded domain or the whole space and \(f\) is a nondecreasing function satisfying \(f \in C^0_{\text{loc}}(0, \infty)\), \(f(0) = 0\), \(f > 0\) on \((0, \infty)\), and \(\Lambda = \sup_{x \geq 1} f(x)/x < \infty\). The authors studied the existence and nonexistence of explosive solutions under the assumption that
\[
\int_0^\infty r \left( \max_{|x|=r} p(x) - \min_{|x|=r} p(x) \right) \Psi(r) dr < +\infty, \tag{1.12}
\]
where \(\Psi(r) = \exp(\Lambda/(N-2) \int_0^r \min_{|x|=r} p(x) dr)\). More precisely, they showed in the case of \(\Omega = \mathbb{R}^N\) that the above problem has positive solution if and only if
\[
\int_1^\infty e^{-t} t^{1-N} \left( \int_0^t e^{t s} s^{N-1} \min_{|x|=s} p(x) ds \right) dt = +\infty. \tag{1.13}
\]
We remark that the condition (1.4) adopted on \(f\) includes the sublinear case, \(\sup_{x \geq 1} f(x)/x < \infty\), studied by Ghergu and Rădulescu [4].
The outline of the paper is as follows. In Section 2, we will give some auxiliary results. The comparison result obtained in Section 2, Theorem 2.6, is used in Section 3 to prove the main result of this work.

2. Auxiliary results

In this section, we suppose that \((A, p)\) satisfies

\[(H_3)\ A \text{ is a nonnegative continuous function on } [0, \infty), \text{ positive and differentiable on } (0, \infty), \text{ and } p : (0, \infty) \to [0, \infty) \text{ is continuous function satisfying}
\]

\[
\int_0^1 A(s)p(s)ds < +\infty, \quad \int_0^1 \frac{1}{A(t)} \left( \int_0^t A(s)p(ds) \right) dt < +\infty. \quad (2.1)
\]

For any given continuous function \(\psi\) on \((0, \infty)\), we put

\[
h_\psi(r) = \int_0^r \frac{1}{A(t)} \left( \int_0^t A(s)\psi(s)ds \right) dt, \quad \text{for } r \in [0, \infty). \quad (2.2)
\]

We consider the following problem:

\[
\frac{1}{A} (Au)' = p(t)f(u), \quad \text{in } (0, \infty),
\]

\[
Au'(0) := \lim_{t \to 0^+} A(t)u'(t) = 0, \quad u(0) = \alpha \geq 1. \quad (2.3)
\]

We state the following.

**Theorem 2.1.** Under the hypotheses \((H_3)\) and \((H_5)\), the problem (2.3) has a positive solution \(u \in C([0, \infty)) \cap C^1((0, \infty))\). Further, on \([0, \infty)\),

\[
\alpha + f(\alpha)h_p(r) \leq u(r) \leq F^{-1}(F(\alpha) + h_p(r)). \quad (2.4)
\]

**Proof.** Let \((u_k)_{k \geq 0}\) be the sequence of functions defined on \([0, \infty)\) by \(u_0(r) = \alpha\) and

\[
u_{k+1}(r) = \alpha + \int_0^r \frac{1}{A(t)} \left( \int_0^t A(s)p(s)f(u_k(s))ds \right) dt, \quad \forall k \in \mathbb{N}. \quad (2.5)
\]

Clearly, we have for each \(k \in \mathbb{N}\), \(t \to u_k(t)\) is a nondecreasing function on \([0, +\infty)\).

By induction, we prove that \((u_k)_{k \geq 0}\) is a nondecreasing sequence.

Since the function \(f\) is nondecreasing, we obtain by (2.5) that for each \(k \geq 0\),

\[
u_k(t) \leq f(u_k(t)) \frac{1}{A(t)} \int_0^t A(s)p(ds), \quad t \geq 0. \quad (2.6)
\]

That is,

\[
\frac{u_k(t)}{f(u_k(t))} \leq \frac{1}{A(t)} \int_0^t A(s)p(ds), \quad t \geq 0. \quad (2.7)
\]
Then
\[
\int_0^r \frac{u'_k(t)}{f(u_k(t))} \, dt \leq \int_0^r \frac{1}{A(t)} \left( \int_0^t A(s) p(s) \, ds \right) \, dt, \quad r \geq 0.
\] (2.8)

It follows that for each \( r \geq 0, \)
\[
F(u_k(r)) - F(\alpha) = \int_{\alpha}^{u_k(r)} \frac{1}{f(\xi)} \, d\xi \leq h_p(r).
\] (2.9)

So
\[
u_k(r) \leq F^{-1}(F(\alpha) + h_p(r)), \quad r \geq 0.
\] (2.10)

Then the sequence \((u_k)_{k \geq 0}\) converges and the function \( u = \sup_{k \in \mathbb{N}} u_k \) is finite and satisfies for each \( r \geq 0, \)
\[
\int_0^r \frac{1}{A(t)} \left( \int_0^t A(s) p(s) f(u(s)) \, ds \right) \, dt.
\] (2.11)

So, \( u \in \mathcal{C}([0, \infty)) \cap \mathcal{C}^1((0, \infty)). \) Thus \( u \) is a solution of the problem (2.3). Moreover, from the monotonicity of \( f \) and (2.10), we obtain (2.4).

Remark 2.2. The solution of problem (2.3) satisfying (2.4) is bounded if and only if
\[
\int_0^{+\infty} \frac{1}{A(t)} \left( \int_0^t A(s) p(s) \, ds \right) \, dt < +\infty.
\] (2.12)

Example 2.3. Let \( A(t) = t^\delta \) for \( t \in [0, \infty), \) where \( \delta \geq 0. \) Assume that for \( t > 0, \) \( p(t) = 1/t^\mu (1+t)^{-\nu}, \) with \( \mu < \min(2,1+\delta) \) and \( \nu \in \mathbb{R}. \) Let \( a, b \geq 0 \) such that \( a + b > 0, \) \( \beta \geq 0, \) and \( 0 \leq \alpha \leq 1, \) set \( f(t) = (at^a + b) \ln(1+t^\beta) \) for \( t \in [0, \infty), \) then the problem
\[
\frac{1}{A} (Au')' = \frac{1}{t^\mu (1+t)^{\nu-\mu}} f(u(t)), \quad \text{in } [0, \infty),
\] (2.13)
\[
Au'(0) = 0, \quad u(0) = u_0 \geq 1
\]
has a positive solution \( u \in \mathcal{C}([0, \infty)) \cap \mathcal{C}^1((0, \infty)). \) Moreover \( u \) is bounded if and only if \( \delta > 1 \) and \( \mu < 2 < \nu.\)

Corollary 2.4. Assume \((H_3)\) and \((H_5)\) hold. Assume moreover that \((H_6)\) holds, for all \( c > 0, \) there exists \( k > 0 \) such that for all \( x, y \in [0, c], |f(x) - f(y)| \leq k|x - y|. \) Then the problem (2.3) has a unique positive solution \( u \in \mathcal{C}([0, \infty)) \cap \mathcal{C}^1((0, \infty)) \) satisfying (2.4).

Proof. Existence follows from Theorem 2.1.

Now, let us prove the uniqueness. Let \( u \) and \( v \) be positive solutions of the problem (2.3). Then for each \( a > 0 \) and \( r \in [0,a], \) we have
\[
|u(r) - v(r)| \leq \int_0^r \frac{1}{A(t)} \left( \int_0^t A(s) p(s) |f(u(s)) - f(v(s))| \, ds \right) \, dt.
\] (2.14)
Blowup solutions for semilinear problems

Since $u$ and $v$ are continuous, it follows that there exists $c > 0$ such that $u(r), v(r) \in [0, c]$ for each $r \in [0, a]$.

So, by hypothesis $(H_6)$ and Fubini theorem, we obtain that

$$|u(r) - v(r)| \leq k \int_0^r \left[ A(s) p(s) \left( \int_s^a \frac{1}{A(t)} dt \right) \right] |u(s) - v(s)| \, ds. \quad (2.15)$$

By Gronwall’s lemma, we deduce that $u(r) = v(r)$ on $[0, a]$. This completes the proof. □

**Corollary 2.5.** Let $\lambda : [0, \infty) \to [0, \infty)$ be a continuous function and suppose that $f$ and $(A, p)$ satisfy, respectively, $(H_3)$ and $(H_5)$. Then the problem

$$\frac{1}{A} (Au')' + \lambda(u)(u')^2 = p(t)f(u), \quad \text{in } (0, \infty),$$

$$Au'(0) = 0, \quad u(0) = \alpha \geq 1 \quad (2.16)$$

has a positive solution $u \in C([0, \infty)) \cap C^1((0, \infty))$.

**Proof.** Let $\rho : [0, \infty) \to [0, \infty)$ be the function defined by $\rho(t) = \int_0^t \exp(\int_0^s \lambda(\zeta) d\zeta) d\zeta$. It is clear that $\rho$ is a bijection from $[0, \infty)$ to itself. Put $v = \rho(u)$. Then $v$ satisfies the following problem:

$$\frac{1}{A} (Av')' = p(t)g(v), \quad \text{in } (0, \infty),$$

$$Av'(0) = 0, \quad v(0) = \rho(\alpha) \geq 1 \quad (2.17)$$

where the function $g$ is defined on $[0, \infty)$ by $g \circ \rho = \rho' f$. Clearly, $g$ satisfies $(H_3)$. Hence by Theorem 2.1, the above problem has a solution $v$ belonging to $C([0, \infty)) \cap C^1((0, \infty))$. Therefore, $u = \rho^{-1}(v)$ is a solution of the problem (2.16). This completes the proof. □

Now, we will give a comparison result. For this aim, we suppose in what follows that

(i) $(A, p)$ and $(B, q)$ satisfy $(H_5)$, $p \leq q$, and $B/A$ is nondecreasing function,

(ii) $f$ and $g$ satisfy $(H_3)$ with $0 \leq g \leq f$.

For each $c \geq 1$, we define on $[0, +\infty)$ the function

$$m_c(r) := G^{-1}(G(c) + h_q(r)), \quad (2.18)$$

where $h_q$ is the function defined by (2.2) and $G^{-1}$ is the inverse of the function $G(t) = \int_1^t 1/g(\zeta) d\zeta$.

**Theorem 2.6.** Assume that the assumptions (i) and (ii) are satisfied. Then for any $\beta \geq 1$ satisfying

$$\int_0^\infty \frac{1}{B(t)} \left( \int_0^t B(s)(q-p)(s)g(m_\beta(s)) \, ds \right) dt < +\infty, \quad (2.19)$$
there exists $\alpha > \beta$ such that problems

\[
\begin{align*}
\frac{1}{A}(Av')' &= p(t)f(v), \quad \text{in } [0, \infty), \\
Av'(0) &= 0, \quad v(0) = \alpha > 1,
\end{align*}
\]

(2.20)

\[
\begin{align*}
\frac{1}{B}(Bw')' &= q(t)g(w), \quad \text{in } [0, \infty), \\
Aw'(0) &= 0, \quad w(0) = \beta \geq 1
\end{align*}
\]

have positive continuous solutions satisfying

\[
v \geq w, \quad \text{in } [0, \infty).
\]

(2.21)

**Proof.** By Theorem 2.1, for any $\alpha > \beta \geq 1$, problems (2.20) have positive solutions $v$ and $w$ satisfying the integral equations

\[
\begin{align*}
v(r) &= \alpha + \int_{0}^{r} \frac{1}{A(t)} \left( \int_{0}^{t} A(s) p(s) f(v(s)) ds \right) dt, \quad r \geq 0, \\
w(r) &= \beta + \int_{0}^{r} \frac{1}{B(t)} \left( \int_{0}^{t} B(s) q(s) g(w(s)) ds \right) dt, \quad r \geq 0.
\end{align*}
\]

(2.22)

Let $\alpha > \beta \geq 1$. We intend to show that if the constant $\alpha$ is sufficiently large, more precisely

\[
\alpha > \beta + \left( \int_{0}^{\infty} \left( \int_{0}^{t} \frac{B(s)}{B(t)} (q-p)(s) g(m_{\beta}(s)) ds \right) dt \right),
\]

(2.23)

then we have

\[
v(r) \geq w(r), \quad r \geq 0.
\]

(2.24)

Using (ii) and the fact that $B/A$ and $f$ are nondecreasing functions on $[0, \infty)$, we obtain

\[
\begin{align*}
w(r) &= \beta + \int_{0}^{r} \left( \int_{0}^{t} \frac{B(s)}{B(t)} q(s) g(w(s)) ds \right) dt \\
&\leq \beta + \int_{0}^{r} \frac{1}{B(t)} \left( \int_{0}^{t} B(s)(q-p)(s) g(w(s)) ds \right) dt \\
&\quad + \int_{0}^{r} \frac{1}{A(t)} \left( \int_{0}^{t} A(s) p(s) f(w(s)) ds \right) dt.
\end{align*}
\]

(2.25)

On the other hand, by (2.4), we have

\[
w(r) \leq G^{-1} \left( G(\beta) + \int_{0}^{r} \frac{1}{A(t)} \left( \int_{0}^{t} A(s) q(s) ds \right) dt \right) = m_{\beta}(r).
\]

(2.26)
By (2.19) and (2.23), we obtain
\[ w(r) - \int_0^r \left( \int_0^t \frac{A(s)}{A(t)} p(s) f(w(s)) \, ds \right) \, dt \]
\[ \leq \beta + \int_0^r \left( \int_0^t \frac{B(s)}{B(t)} (q - p)(s) g(m_{\beta}(s)) \, ds \right) \, dt \]
(2.27)
\[ < \alpha = v(r) - \int_0^r \frac{1}{A(t)} \left( \int_0^t A(s) \, p(s) f(v(s)) \, ds \right) \, dt. \]

Then using a standard comparison theorem [9, Theorem VI, page 17], we obtain (2.21). □

3. Proof of the main result

Proof of Theorem 1.1. Recall that for each \( t > 0 \), \( K(t) := t^{N-1} \exp(\int_0^t \lambda(s) \, ds) \).

Necessity. We will proceed by contradiction. Suppose that (1.9) fails and let \( u \) be an entire large solution of problem \((P)\). Let
\[ v(x) := \int_{1}^{u(x) + 1} \frac{1}{f(\zeta)} \, d\zeta. \]  
(3.1)

Define the spherical mean of \( v \) by
\[ \varpi(r) := \frac{1}{w_N r^{N-1}} \int_{|x|=r} v(x) \, d\sigma_x, \]  
(3.2)
where \( w_N \) denotes the surface of the unit sphere in \( \mathbb{R}^N \).

Since \( u \) is a positive entire large solution of \((P)\), it follows by (1.4) that \( v \) is positive and
\[ \lim_{|x| \to \infty} v(x) = +\infty. \]

By [2, Section 1, Proposition 6], we obtain
\[ \Delta \varpi = \varpi' + \frac{N-1}{r} \varpi' = \Delta v. \]  
(3.3)

So
\[ \Delta v + \lambda(|x|) \nabla v \leq \frac{1}{w_N r^{N-1}} \int_{|x|=r} \Delta v(x) + \lambda(|x|) \left| \nabla v(x) \right|^2 \, d\sigma_x. \]  
(3.4)

By computation, we have on the ball
\[ \Delta v(x) + \lambda(|x|) \left| \nabla v(x) \right| = \frac{1}{f(u(x) + 1)} \Delta u(x) + \left( \frac{1}{f} \right)' (u(x) + 1) \left| \nabla u(x) \right|^2 \]
(3.5)
\[ + \frac{1}{f(u(x) + 1)} \lambda(|x|) \left| \nabla u(x) \right|. \]
Using the fact that \( f' \geq 0 \), we obtain
\[
\Delta v + \lambda(|x|) \nabla v \leq \frac{1}{wN r^{N-1}} \int_{|x|=r} \frac{1}{f(u(x) + 1)} \left( \Delta u(x) + \lambda(|x|) |\nabla u(x)| \right) d\sigma_x
\]
\[
\leq \frac{1}{wN r^{N-1}} \int_{|x|=r} \frac{1}{f(u(x) + 1)} p(|x|) f(u(x)) d\sigma_x \leq p(r).
\]
That is,
\[
v'' + \frac{N-1}{r} v' + \lambda(r)v' \leq p(r).
\]
Then
\[
\left( r^{N-1} \exp \left( \int_0^r \lambda(s) ds \right) v' \right)' \leq r^{N-1} \exp \left( \int_0^r \lambda(s) ds \right) p(r).
\]
Integrating (3.8) yields for each \( r \geq r_0 > 0 \), \( \bar{v}(r) \leq \bar{v}(r_0) + \int_0^r 1/K(t) \left( \int_0^t K(s) p(s) ds \right) dt \). Thus \( \bar{v} \) is bounded, contradiction. It follows that \( (P) \) has no positive large solution.

** Sufficiency. ** Suppose that (1.9) holds. We will use the comparison result given by Theorem 2.6 for \( A(t) = B(t) = K(t) = t^{N-1} \exp(\int_0^t \lambda(s) ds), p, q, \) and \( f \) satisfying, respectively, (H4) and (H3).

Let \( \beta \geq 1 \). Put for \( r \geq 0 \),
\[
m_\beta(r) := F^{-1}(F(\beta) + h(r)),
\]
where \( h \) is the function defined by (1.6).

First, we claim that
\[
\int_0^\infty \frac{1}{K(t)} \left( \int_0^t K(s)(q-p)(s) f(m_\beta(s)) ds \right) dt < +\infty.
\]
In fact, by (1.3) and (1.9), there exists \( 0 < r_0 < +\infty \) such that
\[
F(\beta) < \int_0^{r_0} \frac{1}{K(t)} \left( \int_0^t K(s)q(s) ds \right) dt = h(r_0).
\]
Then
\[
\int_0^{r_0} \frac{1}{K(t)} \left( \int_0^t K(s)(q-p)(s) f(m_\beta(s)) ds \right) dt
\leq \int_0^{r_0} \frac{1}{K(t)} \left( \int_0^t K(s)(q-p)(s) f \circ F^{-1}(2h(r_0)) ds \right) dt
\leq f \circ F^{-1}(2h(r_0)) \int_0^{r_0} \frac{1}{K(t)} \left( \int_0^t K(s)q(s) ds \right) dt < +\infty.
\]
On the other hand, by (1.8), we obtain
\[
\int_{t_0}^{\infty} \frac{1}{K(t)} \left( \int_0^t K(s)(q-p)(s) f(m(s)) \, ds \right) dt < \int_{t_0}^{\infty} \frac{1}{K(t)} \left( \int_0^t K(s)(q-p)(s) f \circ F^{-1}(2h(s)) \, ds \right) dt < +\infty.
\] (3.13)

This yields (3.10).
Thus by Theorem 2.6, there exists \( \alpha > \beta \) such that the problems
\[
\frac{1}{K}(Kv')' = p(t)f(v), \quad \text{in } [0, \infty),
\]
\( Kv'(0) = 0, \quad v(0) = \alpha > 1, \) \quad (3.14)
\[
\frac{1}{K}(Kw')' = q(t)f(w), \quad \text{in } [0, \infty),
\]
\( Kw'(0) = 0, \quad w(0) = \beta \geq 1 \)

have positive solutions satisfying \( v \geq w \) in \([0, \infty)\).

Now, for all \( k \geq 0 \), we consider the problem
\[
\Delta u_k + \lambda(|x|) |\nabla u_k(x)| = \varphi(x, u_k(x)), \quad \text{in } B(0,k),
\]
\( u_k(x) = v(k), \quad \text{on } \partial B(0,k). \)

It is clear that \( w \) and \( v \) are positive sub- and supersolutions of \((P_k)\). Then the problem \((P_k)\) has at least a positive solution \( u_k \) and
\[
w(|x|) \leq u_k(x) \leq v(|x|), \quad \text{in } B(0,k), \quad \forall k \geq 1. \] (3.15)

Now, by [5, Theorem 14.3], the sequence \((\nabla u_k)_k\) is bounded on every compact set in \( \mathbb{R}^N \). Consequently, the sequence \((u_k)_k\) is bounded and equicontinuous on each compact of \( \mathbb{R}^N \). Therefore, by Ascoli-Arzelà theorem, the sequence \((u_k)_k\) has a uniformly convergent, subsequence \((u^1_k)_k\) in \( \mathcal{C}(B(0,1)) \). Setting \( u^1 = \lim_{k \to +\infty} u^1_k \). Then \( (\varphi(\cdot, u^1_k))_k \) converges uniformly to \( \varphi(\cdot, u^1) \) and so \( (\Delta u^1_k + \lambda(|x|) |\nabla u^1_k(x)|)_k \) converges uniformly to \( \varphi(\cdot, u^1) \) on \( B(0,1) \).

Then, using the fact that \( (\Delta + \lambda \nabla) \) is a closed operator, we conclude that \( u^1 \) satisfies \((P)\) in \( B(0,1) \).

Similarly, the sequence \((u^1_k)_k\) has a uniformly convergent sequence \((u^2_k)_k\) on \( B(0,2) \) and let \( u^2 = \lim_{k \to +\infty} u^2_k \). Using the same arguments as above, we claim that \( u^2 \) satisfies \((P)\) in \( B(0,2) \). Further, we have \( u^2 = u^1 \) on \( B(0,1) \).

Repeating this procedure, we construct a sequence \((u^n)_n\) satisfying \((P)\) in \( B(0,n) \) and \( u^{n+1} = u^n \) on \( B(0,n) \), for all \( n \). The sequence \((u^n)_n\) converges in \( L^\infty_{\text{loc}}(\mathbb{R}^N) \) to the function \( u \) given by \( u(x) = u^n(x) \) on \( B(0,n) \).
Using (3.15), we obtain $w \leq u^n \leq v$ in $B(0, n)$, for all $n \geq 1$. Letting $n$ to $+\infty$, it follows that $w \leq u \leq v$ in $\mathbb{R}^N$ and $u$ satisfies the equation
\[
\Delta u + \lambda(|x|) |\nabla u(x)| = \varphi(x, u(x)), \quad \text{in } \mathbb{R}^N.
\] (3.16)

By (1.9) and Remark 2.2, we obtain $\lim_{|x| \to \infty} w(x) = +\infty$.

Consequently, $u$ is a positive entire large solution of problem $(P)$. \qed

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References


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