ON THE POINTWISE MULTIPLICATION IN BESOV AND LIZORKIN-TRIEBEL SPACES

DOUADI DRIHEM AND MADANI MOUSSAI

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Under some sufficient conditions satisfied by $F$-space of Lizorkin and Triebel and $B$-space of Besov, we prove some embeddings of types $F \cdot B \hookrightarrow F$, $F \cdot F \hookrightarrow F$, and $B \cdot B \hookrightarrow B$.

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1. Introduction and preparations

In Besov spaces and Lizorkin-Triebel spaces, this paper is concerned with proving some embeddings of the form

$$ F \cdot B \hookrightarrow F, \quad F \cdot F \hookrightarrow F, \quad B \cdot B \hookrightarrow B, \tag{1.1} $$

where $F$ and $B$, with three indices, will denote the Lizorkin-Triebel space $F_{p,q}^{s}$ and the Besov space $B_{p,q}^{s}$, respectively. The different embeddings obtained here are under certain restrictions on the parameters.

In this introduction, we will recall the definition of some spaces and some necessary tools. In Sections 2 and 3, we give the first contribution of this work. The theorems of Section 2 will treat the case $F \cdot B \hookrightarrow F$ where the first theorem is a generalization of the results of Franke [4, Section 3.2, Theorem 1, Section 3.4, Corollary 1] and Marschall [7]. The second theorem is in the sense of Johnsen’s works (see [5]). Section 3 will contain a treatment of the embeddings of the types $F \cdot F \hookrightarrow F$ and $B \cdot B \hookrightarrow B$ which presents an improvement of [3].

In the sense of [5, Theorems 6.5, 6.11], some limit cases are considered in Section 4, which constitute the second contribution of this paper. Section 5 is an application of our results to the continuity of pseudodifferential operators on Lizorkin-Triebel spaces.

We will work on the Euclidean space $\mathbb{R}^{n}$. If $f \in \mathcal{S}$, the Fourier transform is defined by the formula

$$ \mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^{n}} f(x) e^{-ix \cdot \xi} \, dx \quad (\xi \in \mathbb{R}^{n}) \tag{1.2} $$
and $\mathcal{F}^{-1} f$ denotes the inverse Fourier transform of $f$; as usual $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended from $\mathcal{S}$ to $\mathcal{S}'$.

Consider a partition of unity
\[ \psi(\xi) + \sum_{j=1}^{\infty} \phi(2^{-j} \xi) = 1 \quad (\xi \in \mathbb{R}^n), \tag{1.3} \]
where $\phi, \psi \in C_\infty$ are positive functions such that $\text{supp} \phi \subset \{ \xi \in \mathbb{R}^n : 1 \leq |\xi| \leq 3 \}$ and $\text{supp} \psi \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 3 \}$. We define the convolution operators $Q_j$ and $\Delta_k$ by the following:
\[
Q_j f = \mathcal{F}^{-1}(\psi(2^{-j} \cdot)) \ast f \quad (j = 1, 2, \ldots), \\
\Delta_k f = \mathcal{F}^{-1}(\phi(2^{-k} \cdot)) \ast f \quad (k = 0, 1, \ldots), \tag{1.4}
\]
and we set $Q_0 = \Delta_0$. Thus we obtain the Littlewood-Paley decomposition $f = \sum_{j=0}^{\infty} \Delta_j f$ (convergence in $\mathcal{S}'$).

Let us now recall the definitions of $F^{s}_{p,q}$ and $B^{s}_{p,q}$, where the general references include [1, 9–13].

**Definition 1.1.** Let $\gamma > 0$, $-\infty < s < \infty$, $0 < p \leq \infty$ (resp., $0 < p \leq \infty$), and $0 < q \leq \infty$. The space $L^p_r(\ell^q_s)$ (resp., $\ell^q_s(L^r_p)$) is the set of the sequences $\{f_k\}_{k \in \mathbb{N}} \subset \mathcal{S}'$ such that $\text{supp} \hat{f}_k \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^k \}$ and
\[
\left\| \{ f_k \}_{k \in \mathbb{N}} \right\|_{L^p_r(\ell^q_s)} = \left\| \{ 2^{ks} f_k \}_{k \in \mathbb{N}} \right\|_{L_p(\ell^q_s)} < \infty, \tag{1.5}
\]
and ($\text{resp.} \left\| \{ f_k \}_{k \in \mathbb{N}} \right\|_{\ell^q_s(L^r_p)} = \left\| \{ 2^{ks} f_k \}_{k \in \mathbb{N}} \right\|_{\ell^q(L^r_p)} < \infty$).

**Definition 1.2.** (i) Let $0 < p < \infty$, $0 < q \leq \infty$, and $-\infty < s < \infty$, then
\[
F^s_{p,q} = \{ f \in \mathcal{S}' : \left\| \{ 2^{ks} \Delta_k f \}_{k \in \mathbb{N}} \right\|_{L_p(\ell^q_s)} < \infty \}. \tag{1.6}
\]
(ii) Let $0 < p, q \leq \infty$, and $-\infty < s < \infty$, then
\[
B^s_{p,q} = \{ f \in \mathcal{S}' : \left\| \{ 2^{ks} \Delta_k f \}_{k \in \mathbb{N}} \right\|_{\ell^q(L^r_p)} < \infty \}. \tag{1.7}
\]

**Remark 1.3.** We introduce the maximal function
\[
\Delta_k^{*,a} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\Delta_k f(x-y)|}{1 + (2^k |y|)^a}, \tag{1.8}
\]
for all $x \in \mathbb{R}^n$, $f \in \mathcal{S}'$, $a > 0$, and $k = 0, 1, \ldots$. Then, in Definition 1.2 (i) (resp., (ii)), we can replace $\Delta_k f$ by $\Delta_k^{*,a} f$ with $a > (n/\min(p,q))$ (resp., $a > n/p$), (cf. see [13, Theorem 2.3.2]).
The product \( f \cdot g \) is defined by
\[
f \cdot g = \lim_{j \to \infty} Q_j f \cdot Q_j g \quad (\forall f, g \in \mathcal{F}')
\] (1.9)
if the limit on the right-hand side exists in \( \mathcal{F}' \) (see [10, Section 4.2]), and we have
\[
\Delta_k(f \cdot g) = \sum_{j, \ell = 0}^{\infty} \Delta_k(\Delta_j f \cdot \Delta_\ell g)
\] (1.10)
where
\[
\Pi_{k,1}(f, g) = \Delta_k(\tilde{\Delta}_k f \cdot Q_{k+1} g), \quad \Pi_{k,2}(f, g) = \Delta_k(Q_{k+1} f \cdot \tilde{\Delta}_k g),
\] \[
\Pi_{k,3}(f, g) = \sum_{j = k}^{\infty} \Delta_k(\Delta_j f \cdot \bar{\Delta}_j g),
\] (1.11)
with \( \tilde{\Delta}_k = \sum_{j = k-2}^{k+4} \Delta_j \) and \( \bar{\Delta}_k = \sum_{j = k-1}^{k+1} \Delta_j \).

In the below proofs of the different cases of type (1.1), written as \( G_1 \cdot G_2 \rightarrow G_3 \), to see \( f \cdot g \) belongs to \( G_3 \), \((f \in G_1, g \in G_2)\), it suffices to an estimate of terms of the form \( \|\{\Pi_{k, i}(f, g)\}_{k \in \mathbb{N}}| L^p(\ell^q)\| \) and \( \|\{\Pi_{k, i}(f, g)\}_{k \in \mathbb{N}}| \ell^p_q(L^p)\| \), \( i \in \{1, 2, 3\} \).

Now we recall some lemmas which are useful for us.

**Lemma 1.4.** (i) Let \(-\infty < s_i < \infty, 0 < p_i < \infty\) (resp., \(0 < p_i \leq \infty\)), and \(0 < q_i \leq \infty\) (with \(i = 0, 1\)). If
\[
s_0 > s_1, \quad p_0 = p_1,
\] (1.12)
or
\[
s_0 \geq s_1, \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1} \quad (q_0 \leq q_1 \text{ for Besov space}),
\] (1.13)
then it holds
\[
F_{p_0, q_0}^{s_0} \hookrightarrow F_{p_1, q_1}^{s_1} \quad (\text{resp., } B_{p_0, q_0}^{s_0} \hookrightarrow B_{p_1, q_1}^{s_1}).
\] (1.14)

(ii) Let \(-\infty < s, s_i < \infty, 0 < p, p_i < \infty, \text{ and } 0 < q, q_i \leq \infty\) (with \(i = 0, 1\)) such that \(s_0 - n/p_0 = s - n/p = s_1 - n/p_1\). If
\[
s_0 > s > s_1, \quad q_0 \leq p \leq q_1,
\] (1.15)
or
\[
s_0 = s = s_1, \quad q_0 \leq \min(p, q), \quad q_1 \geq \max(p, q),
\] (1.16)
then it holds
\[
B_{p_0, q_0}^{s_0} \hookrightarrow F_{p, q}^{s} \hookrightarrow B_{p_1, q_1}^{s_1}.
\] (1.17)
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(iii) Let \(-\infty < s < \infty\), \(0 < p < \infty\) (resp., \(0 < p \leq \infty\)), and \(0 < q \leq \infty\). If

\[
s > \frac{n}{p},
\]

or

\[
s = \frac{n}{p}, \quad 0 < p \leq 1 \text{ (resp., } 0 < q \leq 1),
\]

then it holds

\[
F^s_{p,q} \subset L_\infty \quad (\text{resp., } B^s_{p,q} \subset L_\infty).
\]

Lemma 1.5. Let \(0 < \gamma < 1\) and \(0 < q \leq \infty\). Let \(\{\xi_k\}_{k \in \mathbb{N}}\) be a sequence of positive real numbers such that \(\|\{\xi_k\}_{k \in \mathbb{N}} \|_q = A < \infty\). Then the sequences \(\delta_k = \sum_{j=0}^{k} \gamma^{j-k} \xi_j\) and \(\eta_k = \sum_{j=k}^{\infty} \gamma^{j-k} \xi_j\) belong to \(\ell_q\), and the estimate

\[
\|\{\delta_k\}_{k \in \mathbb{N}} \|_q + \|\{\eta_k\}_{k \in \mathbb{N}} \|_q \leq cA
\]

(1.21)

holds. The constant \(c\) depends only on \(\gamma\) and \(q\).

Lemma 1.6. Let \(0 < p \leq \infty\) and \(\gamma > 0\). Let \(\{f_j\}_{j \in \mathbb{N}} \subset L_p\) be a sequence of functions such that \(\text{supp} \hat{f}_j \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \gamma 2^j\}\). Then the estimate

\[
\|\Delta_k f_j \|_{L_p} \leq c 2^{(j-k)q} \|f_j \|_{L_p} \quad \left( k \leq j < \infty, \quad q = \max\left(0, \frac{n}{p} - n\right)\right)
\]

holds. The constant \(c\) depends only on \(n\), \(p\), and \(\gamma\).

Lemma 1.7. Let \(0 < p < 1\) and \(\gamma > 0\). Let \(\{f_j\}_{j \in \mathbb{N}} \subset L_p\) be a sequence of functions such that \(\text{supp} \hat{f}_j \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \gamma 2^j\}\). Then the estimate

\[
\left\|\sum_{j=0}^{\infty} f_j \right\|_{B^q_{p,\infty}} \leq c \left\|\{2^j f_j\}_{j \in \mathbb{N}} \right\|_{L_p(\ell_\infty)} \quad \left( q = \frac{n}{p} - n\right)
\]

(1.23)

holds. The constant \(c\) depends only on \(n\), \(p\), and \(\gamma\).

Lemma 1.8. Let \(0 < p \leq q \leq \infty\) and \(\gamma > 0\). Then there exists a constant \(c = c(n,p,q) > 0\) such that for all \(f \in L_p\) with \(\text{supp} \hat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \gamma\}\), one has

\[
\|f \|_{L_q} \leq cy^{n(1/p - 1/q)} \|f \|_{L_p}.
\]

(1.24)

For Lemma 1.4, we can see [11, Sections 2.3 and 2.8] and [12, Section 2.7]. Lemma 1.5 follows from Young’s inequality in \(\ell_q\). The proof of Lemma 1.6 is given in [4, Section 2.4, Theorem 1(iii)] and Lemma 1.7 in [7, Lemma 3]. For the proof of Lemma 1.8, we can see [14, Proposition 2.13], \(1 \leq p \leq q \leq \infty\), it is the classical inequality of Bernstein.
2. Multiplication of mixed type

The following results give an extension of the sufficient hypotheses used in [5, Theorem 6.1].

**Theorem 2.1.** Let \(0 < p, p_1, p_2 < \infty, 0 < q, q_2 \leq \infty, -\infty < s < \infty, \) and \(r > 0\) be such that

\[
-r + \max\left(0, \frac{n}{p_1} + \frac{n}{p_2} - n\right) < s < \min\left(\frac{n}{p_1}, r\right),
\]

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{r}{n}, \quad \frac{1}{q_2} \geq \frac{1}{p_2} - \frac{r}{n}, \quad r < \frac{n}{p_2} \quad (\text{resp., } r = \frac{n}{p_2}).
\]

Then it holds

\[
F_{p_1, q}^s \cdot B_{p_2, q_2}^r \hookrightarrow F_{p, q}^s \quad (\text{resp., } F_{p_1, q}^s \cdot (B_{p_2, \infty}^{n/p_2} \cap L_\infty) \hookrightarrow F_{p_1, q}^s).
\]

**Corollary 2.2.** Under the hypotheses of Theorem 2.1. If \(r < n/p_2\) (resp., \(r = n/p_2\)) then it holds

\[
F_{p_1, q}^s \cdot F_{p_2, q_2}^r \hookrightarrow F_{p, q}^s \quad (\text{resp., } F_{p_1, q}^s \cdot F_{p_2, q_2}^{n/p_2} \hookrightarrow F_{p_1, q}^s \text{ for } p_2 \leq 1).
\]

Furthermore, in particular, if \(1 < p_1 < \infty \) and \(a > n/p\), can be taken \(s = 0\) in (2.3).

**Proof.** Since \(F_{p_2, q_2}^r \hookrightarrow B_{p_2, t}^r\) with \(t = (1/p_2 - r/n)^{-1}\), we obtain the first embedding. However, the second embedding follows from \(F_{p_2, q_2}^{n/p_2} \hookrightarrow B_{p_2, \infty}^{n/p_1} \hookrightarrow L_\infty\). □

**Remark 2.3.** In Corollary 2.2, when \(r < n/p_2\) (resp., \(r = n/p_2\)), we obtain [10, Theorems 4.4.3/2(21) and 4.4.4/2(16) (resp., Theorems 4.4.3/2(22) and 4.4.4/2(17))]. The particular case \(s = 0\) presents a complement of [10, Theorem 4.4.4/4(i)].

To prove Theorem 2.1, we need the following lemma.

**Lemma 2.4.** Let \(0 < p < \infty\) and \(a > n/p\). Then there exists a constant \(c > 0\) such that

\[
\|\{Q_j^{*,a}g\}_{j \in \mathbb{N}}\|_{L_p(\ell_\infty)} \leq c\|g\|_{F_{p,2}^0},
\]

for any \(g \in F_{p,2}^0\).

**Proof.** First, we define the maximal function of \(Q_jg\), of Hardy-Littlewood type, by the formula

\[
MQ_jg(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |Q_jg(y)|\,dy,
\]

where \(B(x,r)\) is the ball centered at \(x\) of radius \(r\) and \(|B(x,r)|\) denotes its measure. Next, let \(t > 0\) satisfy \(n/a < t < p\). From [13, Theorem 1.3.1], we have

\[
Q_j^{*,a}g(x) \leq Q_j^{*,n/t}g(x) \leq c(M |Q_jg|^{t}(x))^{1/t} \quad (\forall x \in \mathbb{R}^n).
\]
Then we obtain
\[
\left\| \sup_{j \in \mathbb{N}} Q_j^{*, a_1} g \ | L_p \right\| \leq c \left\| \left( \sup_{j \in \mathbb{N}} M |Q_j g| \right)^{1/t} \right\|_{L_p} = c \left\| \sup_{j \in \mathbb{N}} M |Q_j g| ^{t} \right\|_{L_p/t}^{1/t} \\
\leq c' \left\| \sup_{j \in \mathbb{N}} |Q_j g| ^{t} \right\|_{L_p/t}^{1/t} .
\]

(2.7)

A proof of the last inequality may be found in [13, Theorem 2.2.2, page 89]. Now, it is easy to see that the last member of (2.7) is bounded by
\[
\left\| \sup_{j \in \mathbb{N}} |Q_j g| \right\|_{L_p} \leq c \left\| g \right\|_{F^0_{p,2}} .
\]

(2.8)

Inequality (2.8) follows from the equality between the local Hardy spaces \( h_p \) and \( F^0_{p,2} \), (cf. see [12, Section 2.2, page 37, and Theorem 2.5.8/1]).

\[\square\]

Proof of Theorem 2.1

Case 1 \((r < n/p_2)\). (i) Estimate of \({\Pi}_{k,1}(f,g)\)\(_{k \in \mathbb{N}}\). Since
\[
\left\| \Pi_{k,1}(f,g)(x) \right\| = \left| \int_{\mathbb{R}^n} \left( \mathcal{F}^{-1} \varphi(y) \right) (Q_{k+1} g \cdot \tilde{\Delta}_k f)(x - 2^{-k} y) dy \right| \\
\leq c Q_{k+1}^{*, a_1} g(x) \tilde{\Delta}_k^{*, a_2} f(x) \quad (\forall x \in \mathbb{R}^n),
\]

where \(Q_{k}^{*, a_1}\) and \(\tilde{\Delta}_k^{*, a_2}\) are defined as in Remark 1.3, we obtain
\[
\left\| \{2^{k_s} \Pi_{k,1}(f,g)\}_{k \in \mathbb{N}} | \ell_q \right\| \leq c \sup_{j \in \mathbb{N}} \left\| Q_j^{*, a_1} g \right\| \left\| \{2^{k_s} \tilde{\Delta}_k^{*, a_2} f\} \right\|_{k \in \mathbb{N}} | \ell_q \right\|,
\]

(2.10)

where \(a_1\) and \(a_2\) are real numbers at our disposal. We set \(1/b = 1/p_2 - r/n\). The left-hand side of (2.10), in \(L_p\)-norm, is bounded by
\[
c \left\| \sup_{j \in \mathbb{N}} Q_j^{*, a_1} g \ | L_b \right\| \left\| \{2^{k_s} \tilde{\Delta}_k^{*, a_2} f\} \right\|_{k \in \mathbb{N}} | L_{p_1} | \ell_q \right\|.
\]

(2.11)

Choose \(a_1 > n/b\) and \(a_2 > n/\min(p_1, q)\), then both Lemma 2.4 and the embedding \(B^r_{p_2, a_2} \hookrightarrow F^0_{b,2} \) yield that (2.11) is estimated as desired.

(ii) Estimate of \({\Pi}_{k,2}(f,g)\)\(_{k \in \mathbb{N}}\). Let \(u \in \mathbb{R}\) such that
\[
\max \left( 0, \frac{1}{p_1} - \frac{r}{n} \right) < \frac{1}{u} < \min \left( \frac{1}{p_1}, \frac{1}{p_1} - \frac{s}{n} \right).
\]

(2.12)

We set
\[
\frac{1}{v} = \frac{1}{p_2} + \frac{1}{u}, \quad \sigma = s - \frac{n}{p} + \frac{n}{v}, \quad \beta = s - \frac{n}{p_1} + \frac{n}{u}.
\]

(2.13)
We have

\[ \ell_p^a(L_v^q) \hookrightarrow L_p^q(\ell_p^a), \quad F_{p,1,q}^\beta \hookrightarrow B_{u,p_1}^\beta. \]  \tag{2.14}

For the first embedding of (2.14), we can see \cite[Section 2.3, Theorem 3]{4}. On the other hand, the Hölder inequality yields

\[ 2^{k\sigma} \|\Pi_{k,2}(f,g) | L_v \| \leq c 2^{k\sigma} \|\hat{\triangle} k g | L_{p_1} \| \left( 2^{k\beta} \sum_{j=0}^{k+1} 2^{-j\beta} \cdot 2^{j\beta} \|\Delta_j f | L_u \| \right). \]  \tag{2.15}

We set \( 1/\tilde{q}_2 = 1/p - 1/p_1 \). Applying, successively, the Hölder inequality again in \( \ell_p \)-norm and Lemma 1.5, we obtain the bound \( c \|g | B_{p_1,\tilde{q}_2}^r \| \|f | B_{u,p_1}^\beta \|. \) So (2.14) and \( B_{p_1,\tilde{q}_2}^r \hookrightarrow B_{p_2,\tilde{q}_2}^r \) give

\[ \|\{2^{k\beta} \Pi_{k,2}(f,g) \}_{k \in N} \| \|L_p(\ell_q)\| \leq c \|g | B_{p_1,\tilde{q}_2}^r \| \|f | F_{p_1,q}^\beta \|. \]  \tag{2.16}

(iii) Estimate of \( \{\Pi_{k,3}(f,g) \}_{k \in N} \). We first consider \( 1/p_1 + 1/p_2 < 1 \). Let \( u \in \mathbb{R} \) such that

\[ \max \left( 0, \frac{1}{p_1} - \frac{r}{n}, \frac{1}{p_1} - \frac{r+s}{n} \right) < \frac{1}{u} < \frac{1}{p_1}. \]  \tag{2.17}

We use the notations \( v, \sigma \), and \( \beta \) from (2.13). Lemma 1.6 provides

\[ 2^{k\sigma} \|\Pi_{k,3}(f,g) | L_v \| \leq c 2^{k(\beta+r)} \sum_{j=k}^{\infty} 2^{-j(\beta+r)} \cdot 2^{j(\beta+r)} \|\hat{\triangle} j g | L_{p_2} \| \|\Delta_j f | L_u \|. \]  \tag{2.18}

A similar argument as above yields

\[ \|\{2^{k\sigma} \Pi_{k,3}(f,g) \}_{k \in N} \| \|L_p(\ell_q)\| \leq c \|\{2^{j(\beta+r)} \|\hat{\triangle} j g | L_{p_2} \| \|\Delta_j f | L_u \| \}_{j \in N} \| \|L_p\|. \]  \tag{2.19}

We set \( 1/\tilde{q}_2 = 1/p - 1/p_1 \). By the Hölder inequality in \( \ell_p \)-norm, the right-hand side of (2.19) is bounded by \( c \|g | B_{p_2,\tilde{q}_2}^r \| \|f | B_{u,p_1}^\beta \|. \) Then we conclude the desired estimate by (2.14).

We now study case \( 1/p_1 + 1/p_2 > 1 \). Let \( u \in \mathbb{R} \) such that

\[ \max \left( 0, 1 - \frac{1}{p_2} - \frac{1}{p_1} - \frac{r}{n} \right) < \frac{1}{u} < \frac{1}{p_1}. \]  \tag{2.20}

We employ the notations \( v \) and \( \sigma \) from (2.13). By Lemma 1.6, we obtain

\[ 2^{k\sigma} \|\Pi_{k,3}(f,g) | L_v \| \leq c 2^{k\mu} \sum_{j=k}^{\infty} 2^{-j\mu} \cdot 2^{j(\mu+\rho)} \|\hat{\triangle} j g | L_{p_2} \| \|\Delta_j f | L_u \|, \]  \tag{2.21}

where \( \rho = s - n/p_1 + n/u \) and \( \mu = s + r - n/p_1 - n/p_2 + n > 0 \), therefore,

\[ \|\{2^{k\sigma} \Pi_{k,3}(f,g) \}_{k \in N} \| \|L_p(\ell_q)\| \leq c \|\{2^{j(\mu+\rho)} \|\hat{\triangle} j g | L_{p_2} \| \|\Delta_j f | L_u \| \}_{j \in N} \| \|L_p\|. \]  \tag{2.22}
On the right-hand side of (2.22), we employ the Hölder inequality in $\ell_p$-norm (with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{\tilde{q}_2}$), $F^s_{p_1,q_1} \hookrightarrow B^0_{\mu,p_1}$, and $B^r_{p_2,q_2} \hookrightarrow B^r_{p_2,\tilde{q}_2}$ successively. Since $\sigma > s$ and $\nu < p$, we can finish the proof of this case by applying, in the left-hand side of (2.22), embeddings (2.14).

Case 2 ($r = n/p_2$). We only estimate $\{\Pi_{k,1}(f,g)\}_{k \in \mathbb{N}}$. It is sufficient to see that

$$2^{ks} |\Pi_{k,1}(f,g)| \leq c \|g \|_{L_\infty} (2^{ks} \Delta^*_k f)$$  \hspace{1cm} (2.23)

with $a > n/\min(p_1,q)$ and to take the $L_{p_1}(\ell_q)$-norm.

**Theorem 2.5.** Let $0 < p, p_1 < \infty, 0 < p_2, q \leq \infty, -\infty < s < \infty$, and $r > 0$ be such that

$$-r + \max \left(\frac{n}{p_1}, \frac{n}{p_2} - n\right) < s < r.$$  \hspace{1cm} (2.24)

If either of the following assertions is satisfied:

(i) $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$,

(ii) $\max(\frac{1}{p_1}, s/n) + \max(0,1/p_2 - r/n) < 1/p < 1/p_1 + 1/p_2$,

then it holds

$$F^s_{p_1,q_1} \cdot B^r_{p_2,\infty} \hookrightarrow F^s_{p,q}.$$  \hspace{1cm} (2.25)

**Corollary 2.6.** Let $p, p_1, q, r, s$ be as in Theorem 2.5 and $0 < p_2 < \infty$. If (i) or (ii) of Theorem 2.5 is satisfied, then the embedding $F^s_{p_1,q_1} \cdot F^r_{p_2,\infty} \hookrightarrow F^s_{p,q}$ holds.

The proof of Corollary 2.6 is immediate because $F^r_{p_2,\infty} \hookrightarrow B^0_{p_2,\infty}$.

**Remark 2.7.** We note that Theorem 2.5(i) when $p_2 = \infty$ is given in [4, Section 3.2, Theorem 1]. Also, we note that Corollary 2.6 is given in both [5, Theorem 6.1 with $r = p$ in formula (6.6)] and [10, Theorems 4.4.3/1(7) and 4.4.4/1(7)].

**Proof of Theorem 2.5(i).** Noting Remark 2.7, we only need to treat the part $0 < p_2 < \infty$.

(i) Estimate of $\{\Pi_{k,1}(f,g)\}_{k \in \mathbb{N}}$. From (2.9) and Lemma 2.4, we have

$$\|\{2^{ks}\Pi_{k,1}(f,g)\}_{k \in \mathbb{N}} \|_{L_p(\ell_q)} \leq c \|g \|_{F^0_{p_1,2}} \|f \|_{F^s_{p_1,q_1}}.$$  \hspace{1cm} (2.26)

By embeddings $B^r_{p_2,\infty} \hookrightarrow B^0_{p_2,\min(p_1,2)} \hookrightarrow F^0_{p_2,2}$, we obtain that the last term of (2.26) is bounded by the desired quantity.

(ii) Estimate of $\{\Pi_{k,2}(f,g)\}_{k \in \mathbb{N}}$. The Hölder inequality provides

$$\|\Pi_{k,2}(f,g) \|_{L_p} \leq c \left(2^{-kr} \sum_{j=0}^{k+1} 2^{-j(s-r)} \cdot 2^{-jr}\right) \|g \|_{B^r_{p_2,\infty}} \|f \|_{B^s_{p,\infty}}.$$  \hspace{1cm} (2.27)

The hypothesis $s < r$ yields

$$\|\{2^{ks}\Pi_{k,2}(f,g)\}_{k \in \mathbb{N}} \|_{\ell_{\min(p,q)}(L_p)} \leq c \|g \|_{B^r_{p_2,\infty}} \|f \|_{B^s_{p,\infty}}.$$  \hspace{1cm} (2.28)
Using embeddings
\[ \ell_{\min(p,q)}(L^p_f) \hookrightarrow L^p_f(e_q^s), \quad F_{p_1,q}^s \hookrightarrow B_{p_1,\infty}^s, \]
we obtain the desired result.

(iii) Estimate of \( \{\Pi_{k,3}(f,g)\} \). We set \( q = s + r - \max(0, n/p - n) \). Using Lemma 1.6, we obtain
\[ 2^k \| \Pi_{k,3}(f,g) \|_{L^p} \leq c 2^{-k r} \cdot 2^k \sum_{j=k}^\infty 2^{-j r} \| \Delta_j g \|_{L^p} |(2^j \| \Delta_j f \|_{L^p})| \]
\[ \leq c 2^{-k r} \| g \|_{B_{p_2,\infty}^s} \| f \|_{B_{p_1,\infty}^s}. \]  \hfill (2.30)

In this inequality, we take \( \ell_{\min(p,q)} \)-norm and we conclude the desired estimate using (2.29).

Proof of Theorem 2.5 (ii). (1) Estimate of \( \{\Pi_{k,1}(f,g)\} \). We set \( 1/u = 1/p - 1/p_1 \). As in (2.26), we have the bound \( c \| g \|_{F_{u,2}^0} \| f \|_{F_{p_1,q}^s} \) which, by the embeddings \( B_{p_2,\infty}^r \hookrightarrow B_{p_2,u}^{n/p_2-n/u} \rightarrow F_{u,2}^0 \), is estimated as desired.

(2) Estimate of \( \{\Pi_{k,2}(f,g)\} \). In part, for technical reasons, we prove this in three separate cases:

Case 1 \( (s < 0) \). By Lemma 1.6, the Hölder inequality, and Lemma 1.8, we have
\[ \| \Pi_{k,2}(f,g) \|_{L^p} \leq c 2^{(n/p_1+n/p_2-n/p-r-s)k} \| g \|_{B_{p_2,\infty}^r} \| f \|_{B_{p_1,\infty}^s}. \]  \hfill (2.31)

Since \( n/p_1 + n/p_2 - n/p - r < 0 \), we obtain an inequality of type (2.28) and finish the proof of this case using (2.29).

Case 2 \( (0 \leq s < n/p_1) \). We set \( 1/b = 1/p_2 + 1/p_1 - s/n \). We continue with the following subcases.

Subcase 2.1 \( (r \leq n/p_2 \text{ and } p \leq b \text{ (or } s \leq n/p_2 < r \text{ and } p \leq b) \). As in Case 1, we have
\[ \| \Pi_{k,2}(f,g) \|_{L^p} \leq c y_k 2^{-k r} \| g \|_{B_{p_2,\infty}^r} \| f \|_{B_{p_1,\infty}^s}, \]  \hfill (2.32)

where
\[ y_k = \begin{cases} k + 2 & \text{if } p = b, \\ (1 - 2^{n/b-(n/p)-s})^{-1} & \text{if } p < b. \end{cases} \]  \hfill (2.33)

Now since \( \{2^{(s-r)}y_k\}_{k \in \mathbb{N}} \in \ell_{\min(p,q)} \), we conclude the desired conclusion using (2.28) and (2.29).

Subcase 2.2 \( (r \leq n/p_2 \text{ and } p > b \text{ (or } s \leq n/p_2 < r \text{ and } p > b) \). Let \( u > 0 \) satisfy
\[ \max \left( 0, \frac{1}{p} - \frac{1}{p_2} \right) < \frac{1}{u} < \frac{1}{p_1} - \frac{s}{n}. \]  \hfill (2.34)
We employ the notations $\nu$, $\sigma$, and $\beta$ from (2.13). We have
\[
\left\| \Pi_{k,2}(f,g) |_{L^r} \right\| \leq c \left\| g \right\|_{B^r_{p_2,\infty}} \left\| f \right\|_{B^\beta_{u,\infty}} \left( 2^{-k(\beta+r)} \right).
\] (2.35)

Since $\{2^{-k(\beta+r-\sigma)}\}_{k \in \mathbb{N}} \in \ell_\rho$, we can finish the proof of this case using (2.14).

**Subcase 2.3 ($n/p_2 < s < r$).** We have only case $p < b$ needs to be verified. As in (2.32), we immediately obtain the result.

**Case 3 ($s \geq n/p_1$).** We have the following subcases.

**Subcase 3.1 ($p < p_2$).** We set $1/v = 1/p - 1/p_2$. Observe that
\[
2^{k\varepsilon} \left\| \Pi_{k,2}(f,g) |_{L^p} \right\| \leq c 2^{k\varepsilon} \left\| \tilde{\Delta}_k g |_{L^{p_2}} \right\| \left( 2^{k(s-r)} \sum_{j=0}^{k+1} 2^j (n/p_1 - n/v) \varepsilon \right).
\] (2.36)

Then, we calculate $\ell_{\min(p,q)}$-norm and conclude the desired estimate by the fact that $\{2^{k(s-r)}\}_{k \in \mathbb{N}} \in \ell_{\min(p,q)}$.

**Subcase 3.2 ($s > n/p_1$ and $p \geq p_2$).** It suffices to apply both embedding $B^s_{p_1,\infty} \hookrightarrow B^{n/p_1}_{p_1,1}$ and (2.29) to
\[
\left\| \Pi_{k,2}(f,g) |_{L^p} \right\| \leq c \left\| \tilde{\Delta}_k g |_{L^{p_2}} \right\| Q_{k+1} | f |_{L^{\infty}} \leq c 2^{k(n/p_1-r-n/p)} \left\| g \right\|_{B^{s}_{p_2,\infty}} \left\| f \right\|_{B^{n/p_1}_{p_1,1}}.
\] (2.38)

**Subcase 3.3 ($s = n/p_1$ and $p \geq p_2$).** We choose $\alpha > 0$ such that $\varepsilon = \alpha - n/p + n/p_1 + n/p_2 - r < 0$, then it suffices to apply (2.29) to
\[
2^{kn/p_1} \left\| \Pi_{k,2}(f,g) |_{L^p} \right\| \leq c 2^{k\varepsilon} \left\| g \right\|_{B^{s}_{p_2,\infty}} \left\| f \right\|_{B^{n/p_1-a}_{p_1,\infty}}.
\] (2.39)

(3) **Estimate of** $\{\Pi_{k,2}(f,g)\}_{k \in \mathbb{N}}$. The proof of this case is obtained similarly to the proof of Theorem 2.1 just by replacing (2.17) and (2.20) with
\[
\max \left( 0, \frac{1}{p} - \frac{1}{p_2}, \frac{1}{p_1} - \frac{r + s}{n} \right) < \frac{1}{u} < \frac{1}{p_1},
\] (2.40)

respectively.

3. **Multiplication of types $F \cdot B$ and $B \cdot B$**

The next theorem presents a continuation of [3], [5, Theorem 6.1], [6], and [7, Section 5].
Theorem 3.1. Let $0 < p_1 < \infty$ (resp., $0 < p_1 \leq \infty$), $1 \leq p_2 \leq \infty$, $0 < q \leq \infty$, and $n/p_1 - n < s < \min(n/p_1, n/p_2)$. Then it holds

$$F_{p_1,q}^s \cdot (B_{p_2,\infty}^{n/p_2} \cap L_\infty) \hookrightarrow F_{p_1,q}^s \quad \text{(resp., } B_{p_1,\infty}^s \cdot (B_{p_2,\infty}^{n/p_2} \cap L_\infty) \hookrightarrow B_{p_1,q}^s).$$  

(3.1)

Remark 3.2. We note that (3.1), in the $F$-case, was proved by Franke [4, Section 3.4, Corollary 1] but only in the particular case

$$p_2 = \begin{cases} p_1 & \text{if } 0 < p_1 < 2, \\ p_1(p_1 - 1)^{-1} & \text{if } 2 \leq p_1 < \infty. \end{cases}$$  

(3.2)

Also this case yields

$$F_{p_1,q}^s \cdot (F_{p_2,\infty}^{n/p_2} \cap L_\infty) \hookrightarrow F_{p_1,q}^s.$$  

(3.3)

Remark 3.3. Theorem 3.1, when $1 \leq p_1 \leq p_2 < \infty$, was proved in [3].

Proof of Theorem 3.1. The estimates of $\{\Pi_{k,1}(f, g)\}_{k \in \mathbb{N}}$ and $\{\Pi_{k,3}(f, g)\}_{k \in \mathbb{N}}$ are similar to Theorem 2.1, see also [3]. For $\{\Pi_{k,2}(f, g)\}_{k \in \mathbb{N}}$ we take, in (2.16), $r = n/p_2$, and $q_2 = \tilde{q}_2 = \infty$, we obtain (3.1) in the $F$-case. In the $B$-case, we will employ the notations $u, v, \sigma$, and $\beta$ from (2.12) and (2.13) with the modifications $r = n/p_2$ and $\sigma = s - n/p_1 + n/v$. One has

$$2^{ks}||\Pi_{k,2}(f, g) \cdot L_v|| \leq c||g||B_{p_2,\infty}^{n/p_2}|| \left(2^{k\beta} \sum_{j=0}^{k+1} 2^{-j\beta} \cdot 2^{j\beta}||\Delta_j f \cdot L_u||\right).$$  

(3.4)

Since $\beta < 0$, the last inequality, in the $\ell_q$-norm, is bounded by the expression $c||g||B_{p_2,\infty}^{n/p_2}|| \| f \| B_{u,q}^\beta$. At the end, it suffices to use

$$\ell_q^s(L^p_{u}) \hookrightarrow \ell_q^s(L^p_{p_1}), \quad B_{p_1,q}^s \hookrightarrow B_{u,q}^\beta.$$  

(3.5) □

4. Some limit cases

We will prove results of independent interest concerning the limit case for the parameters $s + r$, see [5, Theorems 6.5 and 6.11].

Theorem 4.1. Let $0 < p_i, q_i, p_i \leq \infty$ (i = 1, 2), $-\infty < s < \infty$, and $r > 0$ such that

$$\frac{1}{q_1} + \frac{1}{q_2} \geq 1, \quad s + r = \frac{n}{p_1} + \frac{n}{p_2} - n > 0.$$  

(4.1)
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If either of the following assertions is satisfied:

(i) \[ r < \frac{n}{p}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{r}{n}, \quad s < \min \left( \frac{n}{p_1}, r \right), \]

\[
\frac{1}{q_2} \geq \frac{1}{p_2} - \frac{r}{n}, \quad q = \infty, \quad p_2 \neq \infty,
\]

(ii) \[
\max \left( \frac{1}{p_1}, \frac{s}{n} \right) + \max \left( 0, \frac{1}{p_2} - \frac{r}{n} \right) < \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}
\]

and either of the following cases is satisfied:

(1) \( s < r, \quad q_1 \leq q, \)

(2) \( s = r, \quad \max(q_1, q_2) \leq q, \)

then it holds

\[ B^s_{p_1, q_1} \cdot B^r_{p_2, q_2} \hookrightarrow B^s_{p, q}. \]

Remark 4.2. In Theorem 4.1(i), when \( r = n/p_2, \) we have

\[ B^s_{p_1, q_1} \cdot \left( B^{n/p_2}_{p_2, \infty} \cap L_\infty \right) \hookrightarrow B^s_{p_1, q_1}. \]

Theorem 4.3. Let \( 0 < p, \quad p_1, p_2 < \infty, \quad 0 < q \leq \infty, \quad -\infty < s < \infty, \) and \( r > 0 \) such that

\[ s + r = \frac{n}{p_1} + \frac{n}{p_2} - n > 0. \]

If either of the following assertions is satisfied:

(i) \[ r < \frac{n}{p_2}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{r}{n}, \quad s < \min \left( \frac{n}{p_1}, r \right), \]

(ii) \[
\max \left( \frac{1}{p_1}, \frac{s}{n} \right) + \max \left( 0, \frac{1}{p_2} - \frac{r}{n} \right) < \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}, \quad s \leq r,
\]

then it holds

\[ F^s_{p_1, q} \cdot F^r_{p_2, \infty} \hookrightarrow B^s_{p, \infty}. \]

Remark 4.4. In Theorem 4.3(i), when \( r = n/p_2, \) we have

\[ F^s_{p_1, q} \cdot \left( F^{n/p_2}_{p_2, \infty} \cap L_\infty \right) \hookrightarrow B^s_{p_1, \infty}. \]
Theorem 4.5. Let $0 < p, p_1 < \infty, 0 < p_2, q_1, q_2 \leq \infty, -\infty < s < \infty, \text{and} \ r > 0$ such that

$$q_1 \geq p_1, \quad \frac{1}{q_1} + \frac{1}{q_2} \geq 1, \quad s < r, \quad s + r = \max \left(0, \frac{n}{p_1} + \frac{n}{p_2} - n\right). \quad (4.11)$$

If either of the following assertions is satisfied:

(i)

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad (4.12)$$

(ii)

$$\max \left(\frac{1}{p_1}, \frac{s}{n}\right) + \max \left(0, \frac{1}{p_2} - \frac{r}{n}\right) < \frac{1}{p} < \frac{1}{p_1} + \frac{1}{p_2}, \quad (4.13)$$

then it holds

$$F_{p_1,q_1}^s \cdot B_{p_2,q_2}^r \hookrightarrow F_{p,q_1}^s. \quad (4.14)$$

Proof of Theorem 4.1 (i). (i) Estimate of \{\Pi_{k,1}(f,g)\}_{k \in \mathbb{N}}. We set $1/u = 1/p_2 - r/n$. The Hölder inequality and Lemma 2.4 give

$$2^{ks}||\Pi_{k,1}(f,g)|L_p|| \leq c ||g|F_{u,2}^0||(2^{ks}|\Delta_k f|L_p)||. \quad (4.15)$$

The embedding $B_{p_2,q_2}^r \hookrightarrow F_{u,2}^0$ together with the $\ell_\infty$-norm of (4.15) and $\ell_{q_1} \hookrightarrow \ell_\infty$ give the desired estimate.

(ii) Estimate of \{\Pi_{k,2}(f,g)\}_{k \in \mathbb{N}}. Using the notations $u, v, \sigma$, and $\beta$ from (2.12) and (2.13), we have, as in (2.15),

$$2^{ks}||\Pi_{k,2}(f,g)|L_L|| \leq c ||g|B_{p_1,\infty}^r||f|B_{u,\infty}^\beta||, \quad (4.16)$$

and the conclusion is obtained by (3.5).

(iii) Estimate of \{\Pi_{k,3}(f,g)\}_{k \in \mathbb{N}}. We set $1/b = 1/p_1 + 1/p_2$. By Lemma 1.6 and the Hölder inequality, we obtain

$$||\Pi_{k,3}(f,g)|L_L|| \leq c 2^{-k(s+r)} \sum_{j=k}^{\infty} (2^{js}|\Delta_j g|L_{p_2}||)(2^{js}|\Delta_j f|L_{p_2}||). \quad (4.17)$$

Using $\ell_d \hookrightarrow \ell_1$ (with $1/d = 1/q_1 + 1/q_2$) we employ the Hölder inequality again to conclude that the last term of (4.17) is bounded by $c ||g|B_{p_1,q_1}^r||f|B_{p_1,q_1}^s||$. We finish the proof of this case by applying the embedding $\ell_\infty^{s+r}(L^r_b) \hookrightarrow \ell_\infty(L^s_f)$. \hfill \Box

Proof of Theorem 4.1 (ii). For \{\Pi_{k,1}(f,g)\}_{k \in \mathbb{N}} and \{\Pi_{k,2}(f,g)\}_{k \in \mathbb{N}} we can use the same methods in Theorem 2.5, see also [10, Sections 4.4.3 and 4.4.4].
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Estimate of \{\Pi_{k,3}(f, g)\}_{k \in \mathbb{N}}. By Lemmas 1.6, 1.8 and the Hölder inequality, we have

\[ 2^{k_3} \| \Pi_{k,3}(f, g) \|_{L_p} \leq c 2^{k(n/p_1 + n/p_2 - n/p - r)} \sum_{j=k}^{\infty} (2^{jj} \| \overline{A}_j g \|_{L_{p_2}}) (2^{jj} \| \Delta_j f \|_{L_{p_1}}). \tag{4.18} \]

Since \( n/p_1 + n/p_2 - n/p - r < 0 \), we conclude the desired estimate using \( \ell_d \hookrightarrow \ell_1 \) (with \( 1/d = 1/q_1 + 1/q_2 \)). \( \square \)

Proof of Theorem 4.3(i). (i) Estimate of \{\Pi_{k,1}(f, g)\}_{k \in \mathbb{N}}. We set \( 1/u = 1/p_2 - r/n \), (i.e., \( 1/p = 1/p_1 - 1/u \)). As in (2.9), the choice of \( a_1 > n/u \) and \( a_2 > n/p_1 \) leads to

\[ \| \{2^{k_3} \Pi_{k,1}(f, g)\}_{k \in \mathbb{N}} \|_{L_p} \leq c \| \sup_{j \in \mathbb{N}} Q_j^{*a_2} g \|_{L_u} \| \{2^{k_3} \overline{A}_k^{*a_2} f\}_{k \in \mathbb{N}} \|_{L_p(\ell_u)} \]

\[ \leq c \| g \|_{F_{u,2}^0} \| f \|_{B_{p,1}^{*a_2}}. \]

(4.19)

We conclude the desired estimate by applying both \( F_{p_1, \infty}^{\varepsilon} \hookrightarrow F_{u,2}^0 \) and \( F_{p_1, \infty}^{\varepsilon} \hookrightarrow B_{p_1, \infty}^{\varepsilon} \).

(ii) Estimate of \{\Pi_{k,2}(f, g)\}_{k \in \mathbb{N}}. Using the notations \( u, v, \sigma, \) and \( \beta \) from (2.12) and (2.13), we have

\[ 2^{k_3} \| \Pi_{k,2}(f, g) \|_{L_p} \leq c \sup_{\ell \in \mathbb{N}} (2^{k_3} \| \overline{A}_j^{*a_2} g \|_{L_{p_1}}) \left(2^{k_3} \sum_{j=0}^{k_{j+1}} 2^{-j_2} (2^{2j} \| \Delta_j^{*a_2} f\|_{L_p}) \right). \tag{4.20} \]

Since \( \beta < 0 \), then

\[ \| \{2^{k_3} \Pi_{k,3}(f, g)\}_{k \in \mathbb{N}} \|_{L_p} \leq c \sup_{\ell \in \mathbb{N}} (2^{k_3} \| \overline{A}_j^{*a_2} g \|_{L_{p_1}}) \| \{2^{k_3} \overline{A}_j^{*a_2} f\}_{j \in \mathbb{N}} \|_{L_p}. \]

(4.21)

We choose \( a_1 > n/p_2 \) and \( a_2 > n/u \). We obtain the desired result by applying the Hölder inequality, the embeddings \( L_v(\ell_{\infty}^s) \hookrightarrow L_v^2(\ell_{\infty}^s) \hookrightarrow L_v^2(\ell_{\infty}^s) \) and \( F_{p_1, \infty}^{*a_2} \hookrightarrow F_{u, \infty}^{*a_2} \).

(iii) Estimate of \{\Pi_{k,3}(f, g)\}_{k \in \mathbb{N}}. We set \( 1/u = 1/p_2 + 1/p_1 \). We begin by the inequality

\[ \| \sum_{k=0}^{\infty} \Pi_{k,3}(f, g) \|_{B_{p, \infty}^{\varepsilon}} \leq c \| \sum_{j=0}^{\infty} Q_j (\overline{A}_j g \cdot \Delta_j f) \|_{B_{u, \infty}^{\varepsilon}}. \tag{4.22} \]

We can write

\[ \| Q_j (\overline{A}_j g \cdot \Delta_j f) \| \leq c \sup_{j \in \mathbb{N}} (\| \overline{A}_j^{*a_1} g \cdot \Delta_j^{*a_2} f\|_{L_{p_1}}). \]

(4.23)

We choose \( a_1 > n/p_2 \) and \( a_2 > n/\min(p_1, q) \). Then Lemma 1.7 gives the correct bound for (4.22). \( \square \)

The same method works for the proofs of Theorems 4.3(ii) and 4.5. We omit the details.
Remark 4.6. Theorems 4.1(i) and 4.3(i), when \(1 \leq p \leq \infty\), were proved by Johnsen in [5, Theorems 6.11 and 6.5], respectively.

5. Application

We consider \(S^{0}_{1,0}(E)\) (\(E\) a Banach space), the class of symbols \((x, \xi) \rightarrow a(x, \xi)\) satisfying

\[
\|\partial_{\xi}^{\beta}a(\cdot, \xi) \|_{E} \leq c_{\beta}(1 + |\xi|)^{-|\beta|} \quad (\forall \beta \in \mathbb{N}^{n}),
\]

and we define the pseudodifferential operator by the formula

\[
\text{Op}_{a} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^{n}} e^{ix \cdot \xi} a(x, \xi) \hat{f}(\xi) d\xi \quad (\forall f \in \mathcal{S}, \forall x \in \mathbb{R}^{n}).
\]

As mentioned in the introduction, the theorems of this section present an application of the previous results in this paper.

**Theorem 5.1.** Let \(1 \leq p, p_{1}, p_{2}, q, q_{2} \leq \infty\), \(-\infty < s < \infty\), and \(r > 0\). Under the hypotheses of Theorem 2.1 (with \(p_{2} \neq \infty\) and \(r < n/p_{2}\)) or Theorem 2.5, the operator \(\text{Op}_{a}\) is bounded from \(F_{p_{2}, q}^{s}\) to \(F_{p, q}^{s}\), for all \(a \in S^{0}_{1,0}(B_{p_{2}, q_{2}}^{r})\).

The proof of Theorem 5.1 is based on the following almost-orthogonality lemma.

**Lemma 5.2.** Let \(\gamma > 1\) and let \(p, p_{1}, p_{2}, q, q_{2}, r, s\) be the same as in Theorem 2.1 (with \(p_{2} \neq \infty\) and \(r < n/p_{2}\)) or Theorem 2.5. For all sequences \(\{m_{j}\}_{j \in \mathbb{N}} \subset B_{p_{2}, q_{2}}^{r}\) and all sequences \(\{f_{j}\}_{j \in \mathbb{N}}\) of functions such that \(\text{supp} \hat{f}_{j} \subset \{\xi \in \mathbb{R}^{n} : \gamma^{-1}2^{j} \leq |\xi| \leq \gamma 2^{j}\}\), the estimate

\[
\left\| \sum_{j=0}^{\infty} m_{j} \cdot f_{j} \right\|_{F_{p_{1}, q_{1}}^{s}} \leq c\left\|\{f_{j}\}_{j \in \mathbb{N}} \right\|_{L_{p_{1}}^{r}(\ell_{q_{1}}^{s})}
\]

holds with \(c = c' \sup_{j \geq 0} \|m_{j}\|_{B_{p_{2}, q_{2}}^{r}}\).

**Proof.** Observe that \(\Delta_{k} f_{j} \neq 0\) and \(Q_{k+1} f_{j} \neq 0\) if \(k - N \leq j \leq k + N + 2\) and \(j \leq k + N + 2\), respectively, where \(N = \lfloor \log_{2} \gamma \rfloor\); (here \([x]\) denotes the greatest integer less than or equal to \(x\)). Then it suffices to apply Theorem 2.1 (and/or Theorem 2.5) to the following decomposition:

\[
\Delta_{k}\left( \sum_{j=0}^{\infty} m_{j} \cdot f_{j} \right) = \sum_{\ell = -N}^{N+2} \Pi_{k, 1}(m_{k+\ell}, f_{k+\ell}) + \sum_{j=0}^{k+N+2} \Pi_{k, 2}(m_{j}, f_{j}) + \sum_{\ell = -N}^{N+2} \Pi_{k, 3}(m_{+\ell}, f_{+\ell}),
\]

where \(\Pi_{k, 3}(m_{+\ell}, f_{+\ell}) = \sum_{j=k}^{\infty} \Delta_{k}(\Delta_{j} m_{j+\ell} \cdot \Delta_{j} f_{j+\ell})\) (see also (2.7)). \(\square\)

**Proof of Theorem 5.1.** We begin by writing

\[
a(x, \xi) = (2\pi)^{-n} \int_{\mathbb{R}^{n}} (1 + |u|^{2})^{-(n+1)/2} a_{u}(x, \xi) du + \lambda(x, \xi),
\]

with
where $\lambda(x, \xi) = 0$ for $|\xi| \geq 3$,

\[
\left| \partial_\xi^\beta \lambda(\cdot, \xi) \right| |_{B^r_{p.q}} \leq c_\beta (1 + |\xi|)^{-|\beta|} \quad (\forall \beta \in \mathbb{N}^n),
\]

\[
a_u(x, \xi) = \sum_{j=0}^{\infty} m_{j,u}(x) \theta_u(2^{-j} \xi),
\]

(5.6)

\[
\sup_{j \in \mathbb{N}, u \in \mathbb{R}^n} \left| m_{j,u} \right| |_{B^r_{p.q}} \leq c,
\]

\[
\theta_u(\xi) = (2\pi)^{-n} (1 + |u|^2)^{(n+1-L)/2} e^{iu \cdot \xi} \theta(\xi),
\]

$\theta$ is a $C^\infty$ function with $\text{supp} \theta \subset \{ \xi \in \mathbb{R}^n : 1 \leq |\xi| \leq 3 \}$, and

\[
\left| \theta_u^{(\beta)} \right| |_{L_\infty} \leq c \quad (\forall \beta \in \mathbb{N}^n, |\beta| \leq L - n - 1).
\]

(5.7)

For the decomposition (5.5), we refer the reader to [2] or [8]. Now, by Lemma 5.2, we have

\[
\left| Op_a f \right| |_{F^{s}_{p,q}} \leq c \left| \left\{ 2^j f_{u,j} \right\} \right| |_{L_{p_1}(\ell_q)} \leq c'_f |f| |_{F^{s}_{p_1,q}},
\]

(5.8)

where $\hat{F}(f_{u,j})(\xi) = \theta_u(2^{-j} \xi) \hat{f}^{\cdot} (\xi)$ and $c'$ is independent of $u$. Next, we can write

\[
Op_\lambda f(x) = \int_{\mathbb{R}^n} (1 + |u|^2)^{-(n+1)/2} b_u(x) f(x + u) du,
\]

(5.9)

where

\[
b_u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iu \cdot \xi} (I - \Delta)^{n/2} \lambda(x, \xi) d\xi.
\]

(5.10)

Theorems 2.1 and 2.5 immediately give

\[
\left| Op_\lambda f \right| |_{F^{s}_{p,q}} \leq \left( \sup_{u \in \mathbb{R}^n} \left| b_u \cdot f(\cdot + u) \right| |_{F^{s}_{p,q}} \right) \left| f \right| |_{F^{s}_{p_1,q}} \leq c \left| f \right| |_{F^{s}_{p_1,q}}.
\]

(5.11)

\[\square\]

**Theorem 5.3.** Let $1 \leq p, p_1, q, q_1 \leq \infty$, $r > 0$, and

\[
-r + \frac{n}{p_1} + \frac{n}{p} - n < s < \min \left( \frac{n}{p}, r \right).
\]

(5.12)

Suppose that $a \in S^0_{1,0}(B^r_{p_1,q_1})$ if $r > n/p_1$ and $a \in S^0_{1,0}(L_\infty) \cap S^0_{1,0}(B^{n/p_1}_{p_1,\infty})$ if $r = n/p_1$. Then the operator $Op_a$ is bounded on $F^{s}_{p,q}$ and $B^{s}_{p,q}$.
For the proof, we apply Theorem 3.1 and proceed as in Theorem 5.1, however, we need an almost-orthogonality estimate of the type in Lemma 5.2, that is, the following lemma.

**Lemma 5.4.** Let $\gamma > 1$, $0 < p, p_1, q, q_1 \leq \infty$, $r \geq n/p_1$, and $s$ be as in Theorem 5.3. For all sequences of functions $\{f_j\}_{j \in \mathbb{N}}$ such that $\text{supp} \, \hat{f}_j \subset \{\xi \in \mathbb{R}^n : \gamma^{-1}2^j \leq |\xi| \leq 2^j \}$ and all sequences $\{m_j\}_{j \in \mathbb{N}} \subset B'_{p_1, q_1}$ (or $\{m_j\}_{j \in \mathbb{N}} \subset B_{p_1, q_1}^{n/p} \cap L_\infty$), the estimates

$$
\left\| \sum_{j=0}^{\infty} m_j \cdot f_j \right\|_{\ell_s^{p_1, q_1}} \leq c \left\| \{ f_j \}_{j \in \mathbb{N}} \right\|_{L_r^{p_1} \ell_{q_1}^s},
$$

$$
\left\| \sum_{j=0}^{\infty} m_j \cdot f_j \right\|_{B_{p, q}^{s_1}} \leq c' \left\| \{ f_j \}_{j \in \mathbb{N}} \right\|_{\ell_q^s \ell_{p}^{s_1}},
$$

(5.13)

hold. The constants $c$ and $c'$ are of the form $c'' \sup_{j \in \mathbb{N}} (\| m_j \|_{L_\infty} + \| m_j \|_{B'_{p_1, q_1}})$.

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**References**


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Douadi Drihem: Department of Mathematics, Laboratory of Mathematics Pure and Applied, M‘Sila University, P.O. Box 166, M‘Sila 28000, Algeria
E-mail address: douadidr@yahoo.fr

Madani Moussai: Department of Mathematics, Laboratory of Mathematics Pure and Applied, M‘Sila University, P.O. Box 166, M‘Sila 28000, Algeria
E-mail address: mmoussai@yahoo.fr