The main purpose of this paper is to characterize the classes of matrices \(( \text{ces}[(p),(q)],c^\sigma)\) and \((\text{ces}[(p),(q)],l^\sigma_\infty)\), where \(c^\sigma\) is the space of all bounded sequences all of whose \(\sigma\)-means are equal, \(l^\sigma_\infty\) is the space of \(\sigma\)- bounded sequences, and \(\text{ces}[(p),(q)]\) is the generalized Cesàro sequence space.

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1. Introduction

Let \(\omega\) be the space of all sequences, real or complex, and let \(l_\infty\) and \(c\), respectively, be the Banach spaces of bounded and convergent sequences \(x = (x_n)\) with norm \(\|x\| = \sup_{k \geq 0} |x_k|\). Let \(\sigma\) be a mapping of the set of positive integers into itself. A continuous linear functional \(\phi\) on \(l_\infty\) is said to be an invariant mean or a \(\sigma\)-mean if and only if (i) \(\phi(x) \geq 0\), when the sequence \(x = (x_n)\) has \(x_n \geq 0\) for each \(n\); (ii) \(\phi(e) = 1\), where \(e = (1,1,1,\ldots)\); and (iii) \(\phi((x_\sigma(n))) = \phi(x)\), \(x \in l_\infty\).

For certain kinds of mappings, every \(\sigma\)-mean extends the limit functional \(\phi\) on \(c\) in the sense that \(\phi(x) = \lim x\) for \(x \in c\) (see [2, 15]). Consequently, \(c \subset c^\sigma\), where \(c^\sigma\) is the set of bounded sequences, all of whose invariant means are equal (see [1, 9, 10]). When \(\sigma\) is translation, the \(\sigma\)-means are classical Banach limits on \(l_\infty\) (see [2]) and \(c^\sigma\) is the set of almost convergent sequences \(\hat{c}\) (see [7]). Almost convergence for double sequences was introduced and studied by Móricz and Rhoades [8] and further by Mursaleen and Savaş [13], Mursaleen and Edely [12], and Mursaleen [11].

If \(x = (x_n)\), write \(Tx = (Tx_n) = (x_{\sigma(n)})\), then

\[
c^\sigma = \left\{ x \in l_\infty : \lim_{m \to \infty} t_{m,n}(x) = L, \text{ uniformly in } n, L = \sigma - \lim x \right\},
\]

(1.1)

where

\[
t_{m,n}(x) = \frac{1}{m+1} \sum_{j=0}^{m} T^j x_n \text{ with } T^j x_n = x_{\sigma^j(n)}, \quad t_{-1,n}(x) = 0.
\]

(1.2)

We define \(l^\sigma_\infty\) the space of \(\sigma\)- bounded sequences (Ahmad et al. [2]) in the following way.
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Let \( x_n = z_0 + z_1 + z_2 + \cdots + z_n \) and

\[
P_\infty = \left\{ z \in \omega : \sup_{m,n} |\psi_{m,n}(z)| < \infty \right\},
\]

where

\[
\psi_{m,n}(z) = t_{m,n}(x) - t_{m-1,n}(x)
= \frac{1}{m(m+1)} \sum_{j=1}^{m} \sum_{i=h_{j-1}+1}^{h_j} z_i, \quad h_j = \sigma^j(n).
\]

(1.4)

If \( \sigma(n) = (n+1) \), then \( l_\infty^\sigma \) is the set of almost bounded sequences \( \hat{l}_\infty \) (see [14]).

Let \( A = (a_{nk}) \) be an infinite matrix of complex numbers \( a_{nk} (n,k = 1,2,\ldots) \) and \( X, Y \) two subsets of \( \omega \). We say that the matrix \( A \) defines a matrix transformation from \( X \) into \( Y \) if for every sequence \( x = (x_k) \in X \) the sequence \( A(x) = (A_n(x)) \in Y \), where \( A_n(x) = \sum_k a_{nk}x_k \) converges for each \( n \). We denote the class of matrix transformations from \( X \) into \( Y \) by \( (X,Y) \).

The main purpose of this paper is to characterize the classes \( \text{ces}((p),(q)),c^\sigma \) and \( \text{ces}((p),(q)),l_\infty^\sigma \) and deduce some known and unknown interesting results as corollaries.

The classes \( \text{ces}((p),(q)),c^\sigma \) and \( \text{ces}((p),(q)),l_\infty^\sigma \) are due to Khan and Rahman [4].

If \( \{q_n\} \) is a sequence of positive real numbers, then for \( p = (p_r) \) with \( \inf p_r > 0 \), we define the space \( \text{ces}((p),(q)) \) by

\[
\text{ces}((p),(q)) = \left\{ x \in \omega : \sum_{r=0}^{\infty} \left( \frac{1}{Q^{2r}} \sum_{r} q_k |x_k| \right)^{p_r} < \infty \right\},
\]

(1.5)

where \( Q_{2r} = q_{2r} + q_{2r+1} + \cdots + q_{2r+1} \) and \( \sum_r \) denotes a sum over the range \( 2^r \leq k < 2^{r+1} \).

Remark 1.1. If \( q_n = 1 \) for all \( n \), then \( \text{ces}((p),(q)) \) reduces to \( \text{ces}(p) \) studied by Lim [6]. Also, if \( p_n = p \) for all \( n \) and \( q_n = 1 \) for all \( n \), then \( \text{ces}((p),(q)) \) reduces to \( \text{ces}_p \) studied by Lim [5].

For any bounded sequence \( p \), the space \( \text{ces}((p),(q)) \) is a paranormed space with the paranorm given by (see [4])

\[
g(x) = \left( \sum_{r=0}^{\infty} \left( \frac{1}{Q^{2r}} \sum_{r} q_k |x_k| \right)^{p_r} \right)^{1/M}
\]

(1.6)

if \( H = \sup_r p_r < \infty \) and \( M = \max(1,H) \).
2. Sequence-to-sequence transformations

In this section, we characterize the classes \((\text{ces}[ (p), (q) ], c^\sigma )\) and \((\text{ces}[ (p), (q) ], l^\infty )\).

We write \(a(n,k)\) to denote the elements \(a_{nk}\) of the matrix \(A\), and for all integers \(n,m \geq 1\), we write

\[
t_{mn}(Ax) = \frac{Ax_n + TAx_n + \cdots + T^mA x_n}{m+1}
= \sum_k t(n,k,m)x_k,
\]

(2.1)

where \(t(n,k,m) = 1/(m+1) \sum_{j=0}^{m} a(\sigma^j(n),k)\).

We also define the spaces of \(\sigma\)-convergent series and \(\sigma\)-bounded series, respectively, as follows:

\[
c^\sigma_s = \left\{ x : \sum_{i=1}^{\infty} \frac{1}{i+1} \sum_{j=0}^{i} x_{\sigma^j(n)} \text{ is convergent uniformly in } n, \text{ as } m \to \infty \right\},
\]

(2.2)

\[
b^\sigma_s = \left\{ x : \sup_{n,m} \sum_{i=1}^{m} \frac{1}{i+1} \sum_{j=0}^{i} x_{\sigma^j(n)} < \infty \right\}.
\]

If we take \(\sigma(n) = n+1\), \(c^\sigma_s\) and \(b^\sigma_s\) reduce to \(\hat{c}_s\) and \(\hat{b}_s\), as defined below:

\[
\hat{c}_s = \left\{ x : \sum_{i=1}^{\infty} \frac{1}{i+1} \sum_{j=0}^{i} x_{j+n} \text{ is convergent uniformly in } n, \text{ as } m \to \infty \right\},
\]

(2.3)

\[
\hat{b}_s = \left\{ x : \sup_{n,m} \sum_{i=1}^{m} \frac{1}{i+1} \sum_{j=0}^{i} x_{j+n} < \infty \right\}.
\]

Now we prove the following theorem.

**Theorem 2.1.** Let \(1 < p_r \leq \sup_r p_r < \infty\). Then \(A \in (\text{ces}[ (p), (q) ], c^\sigma )\) if and only if

(i) there exists an integer \(E > 1\) such that for all \(n\),

\[
U(E) = \sup_m \sum_{r=0}^{\infty} \left( Q_{2^r} \max_k \left( \frac{|t(n,k,m)|}{q_k} \right) \right)^{l_r} E^{-l_r} < \infty,
\]

(2.4)

where \(1/p_r + 1/t_r = 1, r = 0, 1, 2, \ldots\), and \(\max_k\) means maximum over \(2^r \leq k < 2^{r+1}\);

(ii) \(a_{nk} = (a_{nk})_{n=1}^{\infty} \in c^\sigma\) for each \(k\), that is, \(\lim_{m} t(n,k,m) = u_k\) uniformly in \(n\), for each \(k\).

In this case, \(\sigma\)-limit of \(Ax\) is \(\sum_{k=1}^{\infty} u_k x_k\).

**Proof.**

We assume that \(A \in (\text{ces}[ (p), (q) ], c^\sigma )\). Now \(\sum_{k=1}^{\infty} t(n,k,m)x_k\) exists for each \(m\) and \(n\) and \(x \in \text{ces}[ (p), (q) ]\), whence \(\{ t(n,k,m) \}_{k} \in c^\sigma\) for each \(m\) and \(n\), (see F. M. Khan and M. A. Khan [3] for Köthe-Toeplitz and continuous duals of \(\text{ces}[ (p), (q) ]\)).
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Therefore, it follows that each \( \{f_{m,n}\}_m \) defined by

\[
 f_{m,n}(x) = t_{m,n}(Ax)
\]  

(2.5)

is an element of \( \text{ces}^*[\langle p \rangle, \langle q \rangle] \). Since \( \text{ces}[\langle p \rangle, \langle q \rangle] \) is complete and further for each \( n \), \( \sup_{m} |t_{m,n}(Ax)| < \infty \) on \( \text{ces}[\langle p \rangle, \langle q \rangle] \). Now arguing with the uniform boundedness principle, we have condition (i). Since \( e_k \in \text{ces}[\langle p \rangle, \langle q \rangle] \), condition (ii) follows.

**Sufficiency.** Suppose that the conditions hold. Fix \( n \in \mathbb{N} \). For every integer \( s \geq 1 \), from (i) we have

\[
 \sum_{r=0}^{s} \left( Q_{2r} \max_{r} (q_{k}^{-1} | t(n,k,m) |) \right)^{t_{r}} E^{-t_{r}} \leq \sup_{m} \sum_{r=0}^{\infty} \left( Q_{2r} \max_{r} (q_{k}^{-1} | t(n,k,m) |) \right)^{t_{r}} E^{-t_{r}}.
\]  

(2.6)

Now letting \( s \to \infty \), we obtain

\[
 \lim_{m \to \infty} \sum_{r=0}^{\infty} \left( Q_{2r} \max_{r} (q_{k}^{-1} | t(n,k,m) |) \right)^{t_{r}} E^{-t_{r}} \leq \sup_{m} \sum_{r=0}^{\infty} \left( Q_{2r} \max_{r} (q_{k}^{-1} | t(n,k,m) |) \right)^{t_{r}} E^{-t_{r}}.
\]  

(2.7)

Therefore, from (ii) we have

\[
 \sum_{r=0}^{\infty} \left( Q_{2r} \max_{r} (q_{k}^{-1} | u_{k} |) \right)^{t_{r}} E^{-t_{r}} \leq \sup_{m} \sum_{r=0}^{\infty} \left( Q_{2r} \max_{r} (q_{k}^{-1} | t(n,k,m) |) \right)^{t_{r}} E^{-t_{r}} < \infty.
\]  

(2.8)

Hence \( (u_{k})_k \) and \( \{t(n,k,m)\}_k \in \text{ces}^*[\langle p \rangle, \langle q \rangle] \), therefore the series \( \sum_{k=1}^{\infty} t(n,k,m)x_k \) and \( \sum_{k=1}^{\infty} u_{k}x_k \) converge for each \( m \) and \( n \) and \( x \in \text{ces}[\langle p \rangle, \langle q \rangle] \). For given \( \epsilon > 0 \) and \( x \in \text{ces}[\langle p \rangle, \langle q \rangle] \), choose \( s \) such that

\[
 \left( \sum_{r=s+1}^{\infty} \left( \frac{1}{Q_{2r}} \sum_{r} q_{k} | x_k | \right)^{p_{r}} \right)^{1/M} < \epsilon.
\]  

(2.9)

Since (ii) holds, there exists \( m_0 \) such that

\[
 \left| \sum_{k=1}^{s} t(n,k,m) - u_k \right| < \epsilon \quad \forall \ m > m_0.
\]  

(2.10)

Since (i) holds, it follows that

\[
 \left| \sum_{k=s+1}^{\infty} t(n,k,m) - u_k \right| \text{ is arbitrary small.}
\]  

(2.11)
Therefore,
\[
\lim_{m \to \infty} \sum_{k=1}^{\infty} t(n,k,m)x_k = \sum_{k=1}^{\infty} u_k x_k, \quad \text{uniformly in } n. \tag{2.12}
\]

This completes the proof. \qed

Remark 2.1. For different choices of \( p, q, \) and \( \sigma \), we can deduce many corollaries from the above theorem to characterize the matrix classes, for example, \((\text{ces}(p), c^\sigma), (\text{ces}_p, c^\sigma), (\text{ces}_p(q), c^\sigma), (\text{ces}[(p), (q)], \tilde{c}), \) and so forth. The class \((\text{ces}(p), \tilde{c})\) was characterized by F. M. Khan and M. A. Khan [3] which we can obtain directly from our theorem by taking \( q_n = 1 \) for all \( n \) and \( \sigma(n) = n + 1 \).

We write (see [2])
\[
x_0 = z_0 + z_1 + \cdots + z_n,
\]
\[
\psi_{m,n}(Az) = \sum_k \alpha(n,k,m)z_k, \tag{2.13}
\]
where
\[
\alpha(n,k,m) = \frac{1}{m(m+1)} \sum_{j=1}^{m} j \left[ \sum_{i=h_{j-1}+1}^{h_j} a_{ik} \right], \quad h_j = \sigma^j(n). \tag{2.14}
\]

Now we prove the following theorem.

**Theorem 2.2.** Let \( 1 < p_r \leq \sup_r p_r < \infty \). Then \( A \in (\text{ces}[(p), (q)], l^\alpha_\infty) \) if and only if
\[
\sup_{m,n} \sum_{r=0}^{\infty} \left( Q_{2r} \max_r (q_r^{-1} |\alpha(n,k,m)|) \right)^{t_r} E^{-t_r} < \infty, \tag{2.15}
\]
where \( E \) is an integer greater than \( 1 \) and \( 1/p_r + 1/t_r = 1, r = 0, 1, 2, \ldots \).

**Proof**

**Necessity.** Suppose that \( A \in (\text{ces}[(p), (q)], l^\alpha_\infty) \). Now \( \sum_{k=1}^{\infty} \alpha(n,k,m)z_k \) exists for each \( m \) and \( n \) and \( z \in \text{ces}[(p), (q)] \), whence \( \{ \alpha(n,k,m) \}_k \in \text{ces}^*[(p), (q)] \) for each \( m \) and \( n \). Therefore, it follows that \( \{ f_{m,n} \} \) defined by
\[
f_{m,n}(x) = \psi_{m,n}(Az) \tag{2.16}
\]
is an element of \( \text{ces}^*[(p), (q)] \). Since \( \text{ces}[(p), (q)] \) is complete and further \( \sup_{m,n} |\psi_{m,n}(Az)| < \infty \) on \( \text{ces}[(p), (q)] \), so by arguing with uniform boundedness principle, we have the condition.
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**Sufficiency.** Suppose that condition (2.15) holds. Fix \( n \in \mathbb{N} \). For every integer \( s \geq 1 \) we have

\[
\sum_{r=0}^{s} \left( Q_{2r} \max_{k} (q_{k}^{-1} | a(n,k,m) |) \right)^{t_{r}} E^{-t_{r}} \leq \sup_{m,n} \sum_{r=0}^{\infty} \left( Q_{2r} \max_{k} (q_{k}^{-1} | a(n,k,m) |) \right)^{t_{r}} E^{-t_{r}}.
\] (2.17)

So

\[
\lim_{s \to \infty} \sum_{r=0}^{s} \left( Q_{2r} \max_{k} (q_{k}^{-1} | a(n,k,m) |) \right)^{t_{r}} E^{-t_{r}} \leq \sup_{m,n} \sum_{r=0}^{\infty} \left( Q_{2r} \max_{k} (q_{k}^{-1} | a(n,k,m) |) \right)^{t_{r}} E^{-t_{r}} < \infty.
\] (2.18)

Hence \( \{ a(n,k,m) \} \in \text{ces}^{*}[(p,q)] \). Therefore, the series \( \sum_{k=1}^{\infty} a(n,k,m)z_{k} \) converges for each \( m \) and \( n \) and \( z \in \text{ces}[(p,q)] \).

This completes the proof. \( \Box \)

**Remark 2.2.** The matrix class \( (\text{ces}(p),\hat{l}_{\infty}) \), was characterized by F. M. Khan and M. A. Khan [3] which we can obtain directly from the above theorem by letting \( q_{n} = 1 \) for all \( n \) and \( \sigma(n) = n + 1 \). Besides, we can further deduce many corollaries for different choices of \( p \), \( q \), and \( \sigma \).

3. Sequence-to-series transformations

For all integers \( m,n \geq 1 \), we write

\[
t^{*}_{mn}(Ax) = \sum_{i=1}^{m} t_{in}(Ax) = \sum_{k=1}^{m} \frac{1}{i+1} \sum_{j=0}^{i} a(\sigma^{j}(n),k)x_{k} = \sum_{k} t^{*}(m,n,k)x_{k},
\] (3.1)

where

\[
t^{*}(m,n,k) = \sum_{i=1}^{m} \frac{1}{i+1} \sum_{j=0}^{i} a(\sigma^{j}(n),k).
\] (3.2)

**Theorem 3.1.** Let \( 1 < p_{r} \leq \sup_{r} p_{r} < \infty \). Then \( A \in (\text{ces}[(p),(q)],c^{\sigma}) \) if and only if

(i) there exists an integer \( E > 1 \) such that for all \( n \),

\[
U(E) = \sup_{m} \sum_{r=0}^{\infty} \left( Q_{2r} \max_{k} \left( \frac{|t^{*}(n,k,m)|}{q_{k}} \right) \right)^{t_{r}} E^{-t_{r}} < \infty,
\] (3.3)

where \( 1/p_{r} + 1/t_{r} = 1 \), \( r = 0,1,2,\ldots \), and \( \max_{r} \) means maximum over \( 2^{r} \leq k \leq 2^{r+1} \);
(ii) \( a_{(k)} = \{a_{nk}\}_{n=1}^{\infty} \subseteq c_0^s \) for each \( k \), that is, \( \lim_{m} t^*(n,k,m) = u_k \) uniformly in \( n \), for each \( k \).

In this case, the \( \sigma \)-limit of \( Ax \) is \( \sum_{k=1}^{\infty} u_k x_k \).

**Theorem 3.2.** Let \( 1 < p_r \leq \sup_r p_r < \infty \). Then \( A \in (\text{ces}[ (p),(q)] , b_{\sigma}^s) \) if and only if

\[
\sup_{m,n} \sum_{r=0}^{\infty} \left( Q_{2r} \max_{r} (q_k^{-1} | t^*(n,k,m) |) \right)^{t_r} E^{-t_r} < \infty, \tag{3.4}
\]

where \( E \) is an integer greater than 1 and \( 1/p_r + 1/t_r = 1, r = 0,1,2, \ldots \).

Proofs of Theorems 3.1 and 3.2 are similar to those of Theorems 2.1 and 2.2, respectively.

**Remark 3.1.** If \( \sigma \) is translation, then Theorems 3.1 and 3.2 give the characterization for the classes \( (\text{ces}[ (p),(q)] , \hat{c}_s) \) and \( (\text{ces}[ (p),(q)] , \hat{b}_s) \). As Remarks 2.1 and 2.2, for different choices of \( p, q \), and \( \sigma \), we can deduce many corollaries.

**References**


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