We consider the semilinear elliptic problem 

\[-\Delta u + u = \lambda K(x)u^p + f(x)\text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u \in H_0^1(\Omega),\]

where \(\lambda \geq 0\), \(N \geq 3\), \(1 < p < (N+2)/(N-2)\), and \(\Omega\) is an exterior strip domain in \(\mathbb{R}^N\). Under some suitable conditions on \(K(x)\) and \(f(x)\), we show that there exists a positive constant \(\lambda^*\) such that the above semilinear elliptic problem has at least two solutions if \(\lambda \in (0, \lambda^*)\), a unique positive solution if \(\lambda = \lambda^*\), and no solution if \(\lambda > \lambda^*\).

We also obtain some bifurcation results of the solutions at \(\lambda = \lambda^*\).

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1. Introduction

In this paper, we consider the semilinear elliptic problem

\[-\Delta u + u = \lambda K(x)u^p + f(x)\text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u \in H_0^1(\Omega),\]

where \(\lambda \geq 0\), \(N = m + n \geq 3\), \(m \geq 2\), \(n \geq 1\), \(1 < p < (N+2)/(N-2)\), \(\omega \subseteq \mathbb{R}^m\) is a bounded smooth domain, \(S = \omega \times \mathbb{R}^n\) is a strip domain, \(D\) is a bounded smooth domain in \(\mathbb{R}^N\) such that \(D \subset \subset S\), \(\Omega = S \setminus \overline{D}\) is an exterior strip domain, \(0 \not= f(x) \geq 0\) in \(\Omega\), \(f(x) \in L^2(\Omega) \cap L^{q_0}(\Omega)\) for some \(q_0 > N/2\) if \(N \geq 4\), \(q_0 = 2\) if \(N = 3\), and \(K(x)\) is a positive, bounded, and continuous function on \(\overline{\Omega}\). Moreover \(K(x)\) satisfies the following conditions:

\[(k1) \lim_{|z| \to \infty} K(y,z) = K_{\infty} > 0 \text{ uniformly for } y \in \omega;\]

\[(k2) \text{there exist some constants } K_{\infty} > 0, \gamma > (n-1)/2, \text{ and } \theta > 0 \text{ such that} \quad K(y,z) \geq K_{\infty} - \theta \exp\left(-\frac{p+1}{p}\sqrt{1+\mu_1|z|}\right)|z|^{-\gamma} \quad \text{as } |z| \to \infty, \text{ uniformly for } y \in \omega,\]

where \(\mu_1\) is the first eigenvalue of the Dirichlet problem \(-\Delta\) in \(\omega\).

If \(\Omega\) is bounded (\(n = 0\) in our case), then (1.1)\(_\lambda\) has been studied by many authors: see for instance Bahri and Lions [4] and the references therein. We only consider that \(\Omega\)
is unbounded \((n \geq 1\) in our case). If \(\Omega\) is an exterior domain \((m = 0\) in our case), Zhu and Zhou [18] and Zhu [17] established the existence of multiple positive solutions of equations with structure unlike that here. If \(\Omega\) is an exterior strip domain, Hsu and Wang [12] have investigated the following problem:

\[-\Delta u + u = u^p + f(x) \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u \in H_0^1(\Omega), \tag{1.3}\]

where \(1 < p < (N + 2)/(N - 2)\) and \(N \geq 4\). Hsu and Wang [12] have proved that (1.3) has at least two positive solutions if \(f\) is nonzero positive \(L^2\) function with the \(L^2\) norm small enough and the decay fast enough.

Throughout this paper, let \(x = (y, z)\) be the generic point of \(\mathbb{R}^N\) with \(y \in \mathbb{R}^m, z \in \mathbb{R}^n, N = m + n \geq 3, m \geq 2, n \geq 1, 1 < p < (N + 2)/(N - 2)\), \(S\) is a smooth strip domain in \(\mathbb{R}^N\), \(\Omega\) is a smooth exterior strip domain in \(\mathbb{R}^N\), \(u_0\) is the unique positive solution of (1.1) \(0\), and we denote by \(c\) and \(c_i\) \((i = 1, 2, \ldots)\) the universal constants, unless otherwise specified. We set

\[
\|u\| = \left(\int_{\Omega} (|\nabla u|^2 + |u|^2) \, dx\right)^{1/2},
\]

\[
\|u\|_{L^q(\Omega)} = \left(\int_{\Omega} |u|^q \, dx\right)^{1/q}, \quad 2 \leq q < \infty,
\]

\[
\|u\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |u(x)|,
\]

\[
M = \inf \left\{ \int_S (|\nabla u|^2 + |u|^2) \, dx : \int_S |u|^{p+1} \, dx = 1 \right\}.
\]

Now, we state our main results in the following.

**Theorem 1.1.** Suppose \(f(x) \geq 0, f(x) \not\equiv 0\) in \(\Omega\), \(f(x) \in L^2(\Omega) \cap L^{q_0}(\Omega)\) for some \(q_0 > N/2\) if \(N \geq 4\), \(q_0 = 2\) if \(N = 3\), \(K(x)\) is a positive, bounded, and continuous function on \(\Omega\) and \(K(x)\) satisfies conditions (k1) and (k2). Then there is a \(\lambda^* > 0, \lambda^*\) depending on \(K\) and \(f\), such that

(i) equation (1.1) has at least two solutions \(u_1, U_\lambda\) and \(u_\lambda < U_\lambda\) if \(\lambda \in (0, \lambda^*)\);

(ii) equation (1.1) has a unique solution \(u_\lambda^*\);

(iii) equation (1.1) has no positive solutions if \(\lambda > \lambda^*\).

Furthermore,

\[
\lambda_1 = \frac{(p + 1)(p - 1)^{p-1} \|K\|_{L^\infty(\Omega)} \|f\|_{L^p(\Omega)}^{p-1}}{(2p)^p} \leq \lambda^* \leq \frac{\|w\|^2_{L^p(\Omega)}}{\inf_{w \in H_0^1(\Omega) \setminus \{0\}} \left(\frac{\|w\|^2_{L^p(\Omega)}}{\int_{\Omega} K u_0^{p-1} w^2 \, dx}\right)} \equiv \lambda_2 \tag{1.5}\]

\[
\leq \frac{\|f\|^2_{L^p(\Omega)}}{(p - 1)^2} \int_{\Omega} K u_0^{p+1} \, dx \equiv \lambda_3,
\]
where $u_{\lambda}$ is the minimal solution of $(1.1)_{\lambda}$ and $U_{\lambda}$ is the second solution of $(1.1)_{\lambda}$ constructed in Section 5.

**Theorem 1.2.** Under the assumptions of Theorem 1.1,

(i) $u_{\lambda}$ is strictly increasing with respect to $\lambda$, $u_{\lambda}$ is uniformly bounded in $L^\infty(\Omega) \cap H^1_0(\Omega)$ for all $\lambda \in [0, \lambda^*]$, and

$$u_{\lambda} \rightarrow u_0 \quad \text{in} \quad L^\infty(\Omega) \cap H^1_0(\Omega) \quad \text{as} \quad \lambda \rightarrow 0^+; \quad (1.6)$$

(ii) $U_{\lambda}$ is unbounded in $L^\infty(\Omega) \cap H^1_0(\Omega)$, that is,

$$\lim_{\lambda \rightarrow 0^+} \|U_{\lambda}\| = \lim_{\lambda \rightarrow 0^+} \|U_{\lambda}\|_{L^\infty(\Omega)} = \infty; \quad (1.7)$$

(iii) moreover, assume that $K(x)$ and $f(x)$ are in $C^\alpha(\Omega) \cap L^2(\Omega)$, then all solutions of $(1.1)_{\lambda}$ are in $C^{2,\alpha}(\Omega) \cap H^2(\Omega)$, and $(\lambda^*, u_{\lambda^*})$ is a bifurcation point for $(1.1)_{\lambda}$, and

$$u_{\lambda} \rightarrow u_0 \quad \text{in} \quad C^{2,\alpha}(\Omega) \cap H^2(\Omega) \quad \text{as} \quad \lambda \rightarrow 0^+. \quad (1.8)$$

This paper is organized as follows. In Section 2, we give some notations and preliminary results. In Section 3, we assert that there exists a positive constant $\lambda^*$, depending on $K$ and $f$, such that $(1.1)_{\lambda}$ has a minimal solution for $\lambda \in [0, \lambda^*]$. In Section 4, we establish several lemmas for the regularity and asymptotic behaviors of the solution of $(1.1)_{\lambda}$. In Section 5, we establish the existence of a second solution $U_{\lambda}$ for $\lambda \in (0, \lambda^*)$. In Section 6, we analyze the set of solutions.

**2. Preliminaries**

In this section, we give some notations and some known results. In order to get the existence of positive solutions of $(1.1)_{\lambda}$, we consider the energy functional $I_{\lambda}: H^1_0(\Omega) \rightarrow \mathbb{R}$ defined by

$$I_{\lambda}(u) = \int_{\Omega} \left[ \frac{1}{2} (|\nabla u|^2 + |u|^2) - \frac{\lambda}{p+1} K(x)(u^+)^{p+1} - f(x) u \right] dx, \quad (2.1)$$

where $u^+(x) = \max\{u(x), 0\}$.

Then the critical points of $I_{\lambda}$ are the positive solutions of $(1.1)_{\lambda}$.

Consider the equation

$$-\Delta u + u = \lambda K_{\infty} u^p \quad \text{in} \quad \mathbb{S}, \quad u > 0 \quad \text{in} \quad \mathbb{S}, \quad u \in H^1_0(\mathbb{S}), \quad (2.1)_{\lambda}$$

and its associated energy functional $I_{\lambda}^\infty$ defined by

$$I_{\lambda}^\infty(u) = \int_{\mathbb{S}} \left[ \frac{1}{2} (|\nabla u|^2 + |u|^2) - \frac{\lambda}{p+1} K_{\infty}(u^+)^{p+1} \right] dx, \quad u \in H^1_0(\mathbb{S}). \quad (2.2)$$
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By Esteban [8] and Lien et al. [14], (2.1) has a ground state solution $\bar{u}_\lambda$ such that

$$M^\infty = I^\infty(\bar{u}_\lambda) = \sup_{t \geq 0} I^\infty(t\bar{u}_\lambda),$$

and we also have

$$M = \inf \left\{ \int_\Omega (|\nabla u|^2 + |u|^2) : \int_\Omega |u|^{p+1} = 1 \right\} = \inf \left\{ \int_\Omega (|\nabla u|^2 + |u|^2) : \int_\Omega |u|^{p+1} = 1 \right\}.$$

Now, we quote here a precise asymptotic behavior result of Hsu [10] for positive solutions of (2.1) at infinity.

**Proposition 2.1.** Let $u$ be a positive solution of (2.1) in an unbounded strip $S = \omega \times \mathbb{R}^n \subseteq \mathbb{R}^{m+n}, m \geq 2, n \geq 1$, and let $\psi$ be the first positive eigenfunction of the Dirichlet problem

$$-\Delta \psi = \mu_1 \psi \text{ in } \omega,$$

then for any $\varepsilon > 0$, there exist constants $c_\varepsilon, \tilde{c}_\varepsilon > 0$ such that

$$u(x) \leq c_\varepsilon \psi(y) \exp \left( -\sqrt{1 + \mu_1} |z| \right) |z|^{-n-1/2+\varepsilon} \leq \tilde{c}_\varepsilon \psi(y) \exp \left( -\sqrt{1 + \mu_1} |z| \right) |z|^{-n-1/2-\varepsilon} \leq u(x)$$

as $|z| \to \infty, y \in \Omega$. (2.5)

3. Existence of minimal solution

In this section, by the standard barrier method, we will establish the existence of minimal positive solution $u_\lambda$ for all $\lambda$ in some finite interval $[0, \lambda^*]$ (i.e., for any positive solution $u$ of (1.1), then $u \geq u_\lambda$).

**Lemma 3.1.** Let condition (k1) hold. Then (1.1) has a solution $u_\lambda$ if $0 \leq \lambda < \lambda_1$, where $\lambda_1$ is given by (1.5).

**Proof.** For $\lambda = 0$, the existence question is equivalent to the existence of $u_0 \in H^1_0(\Omega)$ such that

$$\int_\Omega \nabla u_0 \cdot \nabla \phi + u_0 \phi = \int_\Omega f \phi$$

for all $\phi \in H^1_0(\Omega)$ since

$$\left| \int_\Omega f \phi \right| \leq \|f\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)}$$

for all $\phi \in H^1_0(\Omega)$. According to the Lax-Milgram theorem, there exists a unique $u_0 \in H^1_0(\Omega)$ that satisfies (3.1). Since $0 \not\equiv f \geq 0$ in $\Omega$, by strong maximum principle (see Gilbarg and Trudinger [9]), we conclude that $u_0 > 0$ in $\Omega$.

We consider next the case $\lambda > 0$. We show first that for sufficiently small $\lambda$, say $\lambda = \lambda_0$, there exists $t = t_0(\lambda_0) > 0$ such that $I_{\lambda_0}(u) > 0$ for $\|u\| = t_0$. From the definitions of $I_\lambda$ and $M$ we have for any $u \in H^1_0(\Omega)$,

$$I_\lambda(u) \geq \frac{1}{2} \|u\|^2 - \frac{\lambda}{p+1} \|K\|_{L^\infty(\Omega)} M^{-(p+1)/2} \|u\|^{p+1} \|f\|_{L^2(\Omega)} \|u\|.$$
Set
\[ h(t) = \frac{1}{2} t - \lambda c_1 t^p - c_2, \] (3.4)
where \( c_1 = (1/(p+1)) \| K \|_{L^\infty(\Omega)} M^{-(p+1)/2} \) and \( c_2 = \| f \|_{L^2(\Omega)} \).

It then follows that \( h(t) \) achieves a maximum at \( t_\lambda = (2p\lambda c_1)^{-(p-1)} \). Set \( B_{t_\lambda} = \{ u \in H^1_0(\Omega) : \| u \| < t_\lambda \} \). Then for all \( u \in \partial B_{t_\lambda} = \{ u \in H^1_0(\Omega) : \| u \| = t_\lambda \} \),
\[ I_\lambda(u) \geq t_\lambda h(t_\lambda) \geq t_\lambda \left[ \frac{t_\lambda(p-1)}{2p - c_2} \right] > 0 \] (3.5)
provided that \( c_2 < t_\lambda(p-1)/2p \), which is satisfied for \( \lambda \in (0, \lambda_1) \). Fix a \( \lambda_0 \in (0, \lambda_1) \), and set \( t_0 = t_{\lambda_0} \). Let \( 0 \neq \phi \geq 0, \phi \in C_\infty^0(\Omega) \), such that \( \int_\Omega f \phi dx > 0 \). Then
\[ I_{\lambda_0}(t\phi) = \frac{t^2}{2} \| \phi \|_2^2 - \frac{\lambda_0}{p+1} t^{p+1} \int_\Omega K\phi^{p+1} - t \int_\Omega f \phi < 0 \] (3.6)
for sufficiently small \( t > 0 \), and it is easy to see that \( I_{\lambda_0} \) is bounded below on \( B_{t_0} \). Set \( \alpha_0 = \inf \{ I_{\lambda_0}(u) | u \in B_{t_0} \} \). Then \( \alpha_0 < 0 \), and since \( I_{\lambda_0}(u) > 0 \) on \( \partial B_{t_0} \), the continuity of \( I_{\lambda_0} \) on \( H^1_0(\Omega) \) implies that there exists \( 0 < t_1 < t_0 \) such that \( I_{\lambda_0}(u) > \alpha_0 \) for all \( u \in H^1_0(\Omega) \) and \( t_1 \leq \| u \| \leq t_0 \). By the Ekeland variational principle [7], there exists a sequence \( \{ u_k \}_{k=1}^\infty \subset B_{t_1} \) such that \( I_0(u_k) = \alpha_0 + o(1) \) and \( I_0'(u_k) = o(1) \) strongly in \( H^{-1}(\Omega) \), as \( k \rightarrow \infty \). It is easy to see that \( \{ u_k \} \) is bounded in \( H^1_0(\Omega) \). Hence, there exist a subsequence \( \{ u_k \} \) and \( u \) in \( B_{t_1} \) such that
\[ u_k \rightharpoonup u \quad \text{weakly in } H^1_0(\Omega), \]
\[ u_k \rightarrow u \quad \text{strongly in } L^q(\Omega) \text{ for } 2 \leq q < \frac{2N}{N-2} \text{ as } k \rightarrow \infty \] (3.7)
\[ u_k \rightarrow u \quad \text{a.e. in } \Omega. \]

For \( \phi \in C_\infty^0(\Omega) \), we get
\[ \int_\Omega \nabla u_k \cdot \nabla \phi \rightarrow \int_\Omega \nabla u \cdot \nabla \phi, \quad \int_\Omega u_k \phi \rightarrow \int_\Omega u \phi, \quad \int_\Omega K(u_k^+)\phi \rightarrow \int_\Omega K(u^+)\phi \] (3.8)
as \( k \rightarrow \infty \). Since \( I_0'(u_k), \phi = o(1) \) as \( k \rightarrow \infty \), \( I_0'(u) = 0 \) in \( H^{-1}(\Omega) \). Therefore \( u \) is a weak positive solution of (1.1)\( \lambda_0 \).

Denote
\[ \lambda^* = \sup \{ \lambda \geq 0 : (1.1)_\lambda \text{ has a positive solution} \}. \] (3.9)

Now, by the standard barrier method, we get the following lemma.
Lemma 3.2. Let condition (k1) hold, then there exists $\lambda^* > 0$ such that for each $\lambda \in [0, \lambda^*)$, problem (1.1)$_\lambda$ has a minimal positive solution $u_\lambda$ and $u_\lambda$ is strictly increasing in $\lambda$.

Proof. By Lemma 3.1 and the definition of $\lambda^*$, we deduce that $\lambda^* \geq \lambda_1 > 0$. Now, consider $\lambda \in [0, \lambda^*)$. By the definition of $\lambda^*$, we know that there exists $\lambda' > \lambda^*$ and (1.1)$_{\lambda'}$ has a positive solution $u_{\lambda'} > 0$, that is,

$$-\Delta u_{\lambda'} + u_{\lambda'} = \lambda' K(x)u_{\lambda'}^p + f(x)$$

$$> \lambda K(x)u_{\lambda}^p + f(x).$$

Then $u_{\lambda'}$ is a supersolution of (1.1)$_\lambda$. From $f(x) \geq 0$ and $f(x) \neq 0$, it is easily verified that 0 is a subsolution of (1.1)$_\lambda$. By the standard barrier method, there exists a solution $u_{\lambda}$ of (1.1)$_\lambda$ such that $0 \leq u_{\lambda} \leq u_{\lambda'}$. Since 0 is not a solution of (1.1)$_\lambda$ and $\lambda' > \lambda$, the maximum principle implies that $0 < u_{\lambda} < u_{\lambda'}$. Again using a result of Amann [2], we can choose a minimum positive solution $u_{\lambda}$ of (1.1)$_\lambda$. This completes the proof of Lemma 3.2. $\square$

Now, we consider a solution $u$ of (1.1)$_\lambda$. Let $\sigma_{\lambda}(u)$ be defined by

$$\sigma_{\lambda}(u) = \inf \left\{ \int_\Omega (|\nabla w|^2 + |w|^2)dx : w \in H^1_0(\Omega), \int_\Omega pKu^{p-1}w^2dx = 1 \right\}. \quad (3.11)$$

By the standard direct minimization procedure, we can show that $\sigma_{\lambda}(u)$ is attained by a function $\varphi_{\lambda} > 0$, $\varphi_{\lambda} \in H^1_0(\Omega)$, satisfying

$$-\Delta \varphi_{\lambda} + \varphi_{\lambda} = \sigma_{\lambda}(u)pKu^{p-1}\varphi_{\lambda} \quad \text{in } \Omega. \quad (3.12)$$

Lemma 3.3. Assume condition (k1) holds. For $\lambda \in [0, \lambda^*)$, let $u_{\lambda}$ be the minimal solution of (1.1)$_\lambda$ and let $\sigma_{\lambda}(u_{\lambda})$ be the corresponding number given by (3.11). Then

(i) $\sigma_{\lambda}(u_{\lambda}) > \lambda$ and is strictly decreasing in $\lambda$, $\lambda \in [0, \lambda^*)$;

(ii) $\lambda^* < \infty$, and (1.1)$_{\lambda^*}$ has a minimal solution $u_{\lambda^*}$.

Proof. Consider $u_{\lambda'}$, $u_{\lambda}$, where $\lambda^* > \lambda' > \lambda \geq 0$. Let $\varphi_{\lambda}$ be a minimizer of $\sigma_{\lambda}(u_{\lambda})$, then by Lemma 3.2, we obtain

$$\int_\Omega pKu_{\lambda'}^{p-1}\varphi_{\lambda}^2dx > \int_\Omega pKu_{\lambda}^{p-1}\varphi_{\lambda}^2dx = 1, \quad (3.13)$$

and there is $t$, $0 < t < 1$, such that

$$\int_\Omega pKu_{\lambda'}^{p-1}(t\varphi_{\lambda})^2 = 1. \quad (3.14)$$

Therefore,

$$\sigma_{\lambda'}(u_{\lambda'}) \leq t^2||\varphi_{\lambda}||^2 < ||\varphi_{\lambda}||^2 = \sigma_{\lambda}(u_{\lambda}), \quad (3.15)$$

showing the monotonicity of $\sigma_{\lambda}(u_{\lambda})$, $\lambda \in [0, \lambda^*)$. 

Consider now $\lambda \in (0, \lambda^*)$. Let $\lambda < \lambda' < \lambda^*$. From (3.12) and the monotonicity of $u_\lambda$, we get

$$
\sigma_\lambda (u_\lambda) p \int_\Omega (u_{\lambda'} - u_\lambda) K u_\lambda^{p-1} \varphi_\lambda dx = \int_\Omega \nabla (u_{\lambda'} - u_\lambda) \cdot \nabla \varphi_\lambda dx + \int_\Omega (u_{\lambda'} - u_\lambda) \varphi_\lambda dx
$$

$$
= (\lambda' - \lambda) \int_\Omega K u_\lambda^p \varphi_\lambda dx + \lambda \int_\Omega K (u_{\lambda'}^p - u_\lambda^p) \varphi_\lambda dx
$$

$$
> \lambda p \int_\Omega K \varphi_\lambda \int_0^{u_{\lambda'}} t^{p-1} dt dx
$$

$$
\ge \lambda p \int_\Omega K u_\lambda^{p-1} (u_{\lambda'} - u_\lambda) \varphi_\lambda dx,
$$

which implies that $\sigma_\lambda (u_\lambda) > \lambda$, $\lambda \in (0, \lambda^*)$. This completes the proof of (i).

We show next that $\lambda^* < \infty$. Let $\lambda_0 \in (0, \lambda^*)$ be fixed. For any $\lambda \ge \lambda_0$, (3.15) and (3.16) imply

$$
\sigma_{\lambda_0} (u_{\lambda_0}) \ge \sigma_\lambda (u_\lambda) > \lambda
$$

(3.17)

for all $\lambda \in [\lambda_0, \lambda^*)$. Thus, $\lambda^* < \infty$.

By (3.11) and $\sigma_\lambda (u_\lambda) > \lambda$, we have

$$
\int_\Omega \left( |\nabla u_\lambda|^2 + |u_\lambda|^2 \right) dx - \lambda p \int_\Omega K u_\lambda^{p+1} dx > 0,
$$

(3.18)

$$
\int_\Omega \left( |\nabla u_\lambda|^2 + |u_\lambda|^2 \right) dx - \int_\Omega \lambda K u_\lambda^{p+1} dx - \int f u_\lambda = 0.
$$

Thus

$$
\int_\Omega \left( |\nabla u_\lambda|^2 + |u_\lambda|^2 \right) dx = \int_\Omega \lambda K u_\lambda^{p+1} dx + \int f u_\lambda dx
$$

$$
< \frac{1}{p} \int_\Omega \left( |\nabla u_\lambda|^2 + |u_\lambda|^2 \right) dx + \|f\|_{L^p(\Omega)} \|u_\lambda\|
$$

(3.19)

$$
< \left( \frac{1}{p} + \frac{\delta}{2} \right) \|u_\lambda\|^2 + \frac{1}{2}\|f\|_{L^p(\Omega)}^2
$$

for any $\delta > 0$. Since $p > 1$, we can obtain $\|u_\lambda\| \le c < +\infty$ for all $\lambda \in (0, \lambda^*)$ by taking $\delta$ small enough. By Lemma 3.2, the solution $u_\lambda$ is strictly increasing with respect to $\lambda$; we may suppose that

$$
u_\lambda \rightharpoonup u_{\lambda^*} \quad \text{weakly in } H_0^1(\Omega),
$$

$$
 u_\lambda \rightarrow u_{\lambda^*} \quad \text{strongly in } L^q_{\text{loc}}(\Omega) \text{ for } 2 \le q < \frac{2N}{N-2} \text{ as } \lambda \rightarrow \lambda^*,
$$

(3.20)

$$
 u_\lambda \rightarrow u_{\lambda^*} \quad \text{a.e. in } \Omega,
$$

Thus
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For $\phi \in C_0^\infty(\Omega)$, we get

$$
\int_\Omega \nabla u_\lambda \nabla \phi \to \int_\Omega \nabla u_{\lambda^*} \nabla \phi, \quad \int_\Omega u_\lambda \phi \to \int_\Omega u_{\lambda^*} \phi, \quad \lambda \int_\Omega K u_\lambda^p \phi \to \lambda^* \int_\Omega K u_{\lambda^*}^p \phi,
$$

as $\lambda \to \lambda^*$. From $\langle I'_\lambda(u_\lambda), \phi \rangle = 0$ and letting $\lambda \to \lambda^*$, we deduce $I'_\lambda(u_{\lambda^*}) = 0$ in $H^{-1}(\Omega)$. Hence $u_{\lambda^*}$ is a positive solution of (1.1)$_{\lambda^*}$.

Let $u$ be any positive solution of (1.1)$_{\lambda^*}$. By adopting the argument as in Lemma 3.1, we have $u \geq u_\lambda$ in $\Omega$ for $\lambda \in (0, \lambda^*)$, where $u_\lambda$ is the minimal solution of (1.1)$_\lambda$. Therefore $u \geq u_{\lambda^*}$ in $\Omega$. This implies that $u_{\lambda^*}$ is a minimal solution of (1.1)$_{\lambda^*}$. \qed

In the following lemma, we give an estimate of $\lambda^*$.

**Lemma 3.4.** If condition (k1) holds, then $\lambda_1 \leq \lambda^* \leq \lambda_2 \leq \lambda_3$, where $\lambda_1$, $\lambda_2$, and $\lambda_3$ are given by (1.5).

**Proof.** By Lemma 3.1 and the definition of $\lambda^*$, we conclude that $\lambda^* \geq \lambda_1$.

As in Lemma 3.3, we have $\sigma_\lambda(u_\lambda) > \lambda$ for all $\lambda \in (0, \lambda^*)$, so for any $w \in H_0^1(\Omega) \setminus \{0\}$, we have

$$
\int_\Omega (|\nabla w + |w|^2) \, dx > \lambda p \int_\Omega K u_\lambda^{p-1} w^2 \, dx.
$$

(3.22)

Let $u_0$ be the unique solution of (1.1)$_0$, then by (3.22) and $u_\lambda > u_0$ for all $\lambda \in (0, \lambda^*)$, we obtain that

$$
\int_\Omega (|\nabla w + |w|^2) \, dx > \lambda p \int_\Omega K u_0^{p-1} w^2 \, dx,
$$

(3.23)

that is,

$$
\lambda \leq \inf_{w \in H_0^1(\Omega) \setminus \{0\}} \left( \frac{\|w\|^2}{p \int_\Omega K u_0^{p-1} w^2 \, dx} \right) = \lambda_2.
$$

(3.24)

This implies that $\lambda^* \leq \lambda_2$.

For all $\lambda \in [0, \lambda^*]$, let $u_\lambda$ be a minimal solution of (1.1)$_\lambda$ and take $w = u_\lambda$ in (3.22), then we have

$$
\|u_\lambda\|^2 = \lambda \int_\Omega K u_\lambda^{p+1} \, dx + \int_\Omega f u_\lambda \, dx \leq \frac{1}{p} \|u_\lambda\|^2 + \|f\|_{L^2(\Omega)} \|u_\lambda\|.
$$

(3.25)

This implies that

$$
\|u_\lambda\| \leq \frac{p}{p-1} \|f\|_{L^2(\Omega)}.
$$

(3.26)
Take \( w = u_\lambda \) in (3.24), and by (3.26) and the monotonicity of \( u_\lambda \), we get

\[
\lambda_2 \leq \frac{\|u_\lambda\|^2}{p \int_\Omega K u_0^{p-1} u_\lambda^2 dx} \leq \frac{p \|f\|^2_{L^p(\Omega)}}{(p - 1)^2 \int_\Omega K u_0^{p+1} dx} = \lambda_3.
\]

(3.27)

\[\square\]

4. Asymptotic behaviors of solutions

In this section, we will prove that a solution of (1.1)\( _\lambda \) belongs to \( C_b(\overline{\Omega}) \) and derive several precise estimates on its behavior at infinity. Now, let \( \mathbb{N} \) be all natural numbers, let \( X \) be a smooth domain in \( \mathbb{R}^N \), and hence we have the extension lemma, embedding lemma, interpolation lemma (see Adams [1] for the proof), and for regularity Lemmas 4.1–4.7.

**Lemma 4.1 (extension).** There is a positive constant \( c = c(\ell, q) \) such that for any \( u \in W^{\ell,q}(X) \), \( \ell \in \mathbb{N} \), \( 1 < q < \infty \), there exists some \( \tilde{u} \in W^{\ell,q}(\mathbb{R}^N) \) such that \( \tilde{u} = u \) a.e. in \( X \) and \( \|\tilde{u}\|_{W^{\ell,q}(\mathbb{R}^N)} \leq \|u\|_{W^{\ell,q}(X)} \).

**Lemma 4.2 (embedding).** There exists the following continuous embedding:

\[ W^{j+\ell,q}(X) \hookrightarrow C^{j,\alpha}(X), \quad 0 \leq \alpha \leq \ell - \frac{N}{q}, \]

provided \((\ell - 1)q < N < \ell q \) and \( j \in \mathbb{N} \cup \{0\} \).

**Lemma 4.3 (interpolation).** Given \( \ell \in \mathbb{N} \), \( 1 < q < \infty \), there exists a positive constant \( c = c(\ell, q, N) \) such that for any \( 0 < \varepsilon < 1 \), \( 0 \leq j \leq \ell - 1 \), and any \( u \in W^{\ell,q}(X) \),

\[
\|u\|_{W^{j,q}(X)} \leq c \varepsilon \|u\|_{W^{\ell,q}(X)} + \frac{c}{\varepsilon^{j/(\ell - j)}} \|u\|_{W^{0,q}(X)}.
\]

(4.2)

**Lemma 4.4 (regularity Lemma 1).** Let \( g : X \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function such that for almost every \( x \in X \), there holds

\[
|g(x, u)| \leq c(|u| + |u|^p) \quad \text{uniformly in } x \in X,
\]

where \( c, p \) are some positive constants, \( N \geq 3 \), and \( 1 < p < (N + 2)/(N - 2) \). Also, let \( u \in H^1_0(X) \) be a weak solution of equation \(-\Delta u = g(x, u) + f(x) \) in \( X \), where \( f \in L^{N/2}(X) \cap L^2(X) \). Then \( u \in L^q(X) \) for \( q \in [2, \infty) \).


Now, we quote regularity Lemmas 4.5–4.7 (see Gilbarg and Trudinger [9, Theorems 8.8, 9.11, 9.16] for the proof).
By Lemma 4.2, \( \psi \) is the first positive eigenfunction of the Dirichlet problem
\[-\Delta u + u = g \quad \text{in} \quad \Omega, \]
and by Lemma 4.4, we have \( u \) satisfies
\[-\Delta u + u = g \quad \text{in} \quad H^{-1}(\Omega), \quad (4.7) \]
since \( K \) is bounded in \( \Omega \) and \( f \in L^2(\Omega) \cap L^q(\Omega) \) for some \( q > N/2 \). Hence \( f \in L^{N/2}(\Omega) \) and by Lemma 4.4, we have \( u \in L^q(\Omega) \) for \( q \in [2, \infty) \). Hence \( \lambda K(x)u^p + f(x) \in L^2(\Omega) \cap L^{q_0}(\Omega) \) for some \( q_0 > N/2 \). Then by Lemma 4.7, we have \( u \in W^{2,q}(\Omega) \) for some \( q_0 > N/2 \). By Lemma 4.2, \( u \in C_b(\overline{\Omega}) \) and there exists a constant \( c > 0 \), such that for any \( r > 1 \),
\[ ||u||_{L^\infty(\overline{\Omega})} \leq c ||u||_{W^{2,q}(\overline{\Omega})}, \quad (4.8) \]
where \( B_r = \{ x = (y,z) \in \Omega : |z| > r \} \). Hence \( \lim_{|z| \to \infty} u(y,z) = 0 \) uniformly for \( y \in \omega \). \( \square \)

Lemma 4.8 (asymptotic Lemma 1). Let condition \((k1)\) hold. If \( u \) is a weak solution of (1.1), then \( u(y,z) \to 0 \) as \( |z| \to \infty \) uniformly for \( y \in \omega \).

Proof. Let \( u \) satisfy
\[-\Delta u + u = \lambda K(x)u^p + f(x) \quad \text{in} \quad H^{-1}(\Omega), \quad (4.7) \]
since \( K \) is bounded in \( \Omega \) and \( f \in L^2(\Omega) \cap L^q(\Omega) \) for some \( q > N/2 \). Hence \( f \in L^{N/2}(\Omega) \) and by Lemma 4.4, we have \( u \in L^q(\Omega) \) for \( q \in [2, \infty) \). Hence \( \lambda K(x)u^p + f(x) \in L^2(\Omega) \cap L^{q_0}(\Omega) \) for some \( q_0 > N/2 \). Then by Lemma 4.7, we have \( u \in W^{2,q}(\Omega) \) for some \( q_0 > N/2 \). By Lemma 4.2, \( u \in C_b(\overline{\Omega}) \) and there exists a constant \( c > 0 \), such that for any \( r > 1 \),
\[ ||u||_{L^\infty(\overline{\Omega})} \leq c ||u||_{W^{2,q}(\overline{\Omega})}, \quad (4.8) \]
where \( B_r = \{ x = (y,z) \in \Omega : |z| > r \} \). Hence \( \lim_{|z| \to \infty} u(y,z) = 0 \) uniformly for \( y \in \omega \). \( \square \)

Lemma 4.9 (asymptotic Lemma 2). Let \( u \) be a positive solution of (1.1) for \( \lambda \in [0, \lambda^*] \) and let \( \psi \) be the first positive eigenfunction of the Dirichlet problem –\( \Delta \psi = \mu_1 \psi \) in \( \omega \), then there exists a positive constant \( c \) such that
\[ u(x) \geq c \psi(y) \exp \left( -\sqrt{1 + \mu_1} |z| \right) |z|^{-(n-1)/2} \quad \text{as} \quad |z| \to \infty, \quad y \in \omega. \quad (4.9) \]

Proof. Let \( \Phi(x) = (1 + 1/|z|) \psi(y) \exp( -\sqrt{1 + \mu_1} |z|) |z|^{-(n-1)/2} \) for \( x = (y,z) \in \overline{\Omega} \) and \( |z| > 0 \). It is very easy to show that there is a \( R_0 > 0 \) such that
\[-\Delta \Phi + \Phi \leq 0, \quad \forall |z| \leq R_0. \quad (4.10) \]
Let $u_\lambda$ be the minimal solution of (1.1)$_\lambda$, let $q = (q_y, q_z)$, $q_y \in \partial \omega$, $|q_z| = R_0$, and $B$ a small ball in $\Omega$ such that $q \in \partial B$. Since $\psi(y) > 0$ for $x = (y, z) \in B, \psi(q_y) = 0, u(x) > 0$ for $x \in B, u_\lambda(q) = 0$, by the strongly maximum principle $(\partial \psi/\partial y)(q_y) < 0, (\partial u_\lambda/\partial x)(q) < 0$. Thus

$$\lim_{x \to y, |z| = R_0} \frac{u_\lambda(x)}{\psi(y)} = \frac{(\partial u_\lambda/\partial x)(q)}{(\partial \psi/\partial y)(q_y)} > 0. \quad (4.11)$$

Note that $u_\lambda(x)\psi^{-1}(y) > 0$ for $x = (y, z), y \in \omega, |z| = R_0$. Thus $u_\lambda(x)\psi^{-1}(y) > 0$ for $x = (y, z), y \in \omega, |z| = R_0$.

Since $\Phi(x)$ and $u_\lambda(x)$ are $C^1(\overline{\omega \times \partial B_{R_0}(0)})$, if we set

$$\alpha = \inf_{y \in \omega, |z| = R_0} (u_\lambda(x)\Phi^{-1}(x)), \quad (4.12)$$

then $\alpha > 0$ and

$$\alpha \Phi(x) \leq u_\lambda(x) \quad \text{for } y \in \omega, \quad |z| = R_0. \quad (4.13)$$

For $|z| \geq R_0$, we have

$$-\Delta(u_\lambda - \alpha \Phi)(x) + (u_\lambda - \alpha \Phi)(x) = \lambda K(x)u_\lambda^p(x) + f(x) + \alpha(\Delta \Phi + \Phi)(x) \geq 0. \quad (4.14)$$

By the maximum principle, we obtain

$$u_\lambda(x) \geq \alpha \Phi(x) \quad \text{for } y \in \omega, \quad |z| \geq R_0. \quad (4.15)$$

Let $\epsilon = \alpha > 0$, we get

$$u_\lambda(x) \geq c \psi(y) \exp \left(-\sqrt{1 + \mu_1 |z|} \right) |z|^{-(n-1)/2} \quad \text{for } y \in \omega, \quad |z| \geq R_0. \quad (4.16)$$

This implies that (4.9) holds for $u_\lambda$ and hence for arbitrary positive solution $u$. \hfill $\square$

5. Existence of second solution

The existence of a second solution of (1.1)$_\lambda$, $\lambda \in (0, \lambda^*)$, will be established via the mountain pass theorem. When $0 < \lambda < \lambda^*$, we have known that (1.1)$_\lambda$ has a minimal positive solution $u_\lambda$ by Lemma 3.2, then we need only to prove that (1.1)$_\lambda$ has another positive solution in the form of $U_\lambda = u_\lambda + v_\lambda$, where $v_\lambda$ is a solution of the following problem:

$$-\Delta v + v = \lambda K[(v + u_\lambda)^p - u_\lambda^p] \quad \text{in } \Omega, \quad v \in H_0^1(\Omega), \quad v > 0 \quad \text{in } \Omega. \quad (5.1)_\lambda$$

The corresponding variational functional of (5.1)$_\lambda$ is

$$J_\lambda(v) = \frac{1}{2} \int_\Omega (|\nabla v|^2 + v^2) - \lambda \int_\Omega \int_0^v \nabla K[(s + u_\lambda)^p - u_\lambda^p] ds \, dx, \quad v \in H_0^1(\Omega). \quad (5.1)$$

To verify the conditions of the mountain pass theorem, we need the following lemmas.
Lemma 5.1. For any $\epsilon > 0$, there is a positive constant $c_\epsilon$ such that

\[(\xi + s)^p - \xi^p - p\xi^{p-1}s \leq \epsilon\xi^{p-1}s + c_\epsilon s^p, \quad \forall s \geq 0, \xi > 0. \tag{5.2}\]

Proof. From the fact

\[
\lim_{t \to 0^+} \frac{(1 + t)^p - 1 - pt}{t} = 0, \quad \lim_{s \to \infty} \frac{(1 + t)^p - 1 - pt}{tp} = 1, \tag{5.3}
\]

we obtain that for any $\epsilon > 0$, there is a positive constant $c_\epsilon$ such that

\[(1 + t)^p - 1 - pt \leq \epsilon t + c_\epsilon t^p, \quad \forall t \geq 0. \tag{5.4}\]

Let $\xi > 0$, $s \geq 0$, and take $t = s/\xi$ in (5.4), we can deduce that

\[(\xi + s)^p - \xi^p - p\xi^{p-1}s \leq \epsilon\xi^{p-1}s + c_\epsilon s^p. \tag{5.5}\]

Lemma 5.2. There exist positive constants $\rho$ and $\alpha$, such that

\[J_\lambda(v) \geq \alpha > 0, \quad v \in H^1_0(\Omega), \quad \|v\| = \rho. \tag{5.6}\]

Proof. For any $\epsilon > 0$ there is by Lemma 5.1 (with $\xi = u_\lambda$) a positive constant $c_\epsilon$ such that

\[J_\lambda(v) = \frac{1}{2} \int_\Omega (|\nabla v|^2 + v^2) dx - \frac{1}{2} \lambda p \int_\Omega K u_\lambda^{p-1}(v^+)^2 dx

- \lambda \int_\Omega \int_0^{v^+} K (u_\lambda + s)^p - u_\lambda^p - pu_\lambda^{p-1}s ds dx

\geq \frac{1}{2} \left[ \int_\Omega (|\nabla v|^2 + v^2) dx - \lambda p \int_\Omega K u_\lambda^{p-1}(v^+)^2 dx \right]

- \lambda \int_\Omega K \left[ \frac{\epsilon}{2} u_\lambda^{p-1}(v^+)^2 + c_\epsilon \left( \frac{v^+}{p+1} \right)^{p+1} \right] dx. \tag{5.7}\]

Furthermore, from the definition $\sigma_\lambda(u_\lambda)$ in (3.11), we have

\[\int_\Omega (|\nabla v|^2 + v^2) dx \geq \sigma_\lambda(u_\lambda) p \int_\Omega K u_\lambda^{p-1}(v^+)^2 dx, \tag{5.8}\]

and, therefore, by (5.7) we obtain

\[J_\lambda(v) \geq \frac{1}{2} \sigma_\lambda(u_\lambda)^{-1} \left( \sigma_\lambda(u_\lambda) - \lambda - \frac{\xi}{2\lambda} \right) \|v\|^2 - \lambda c_\epsilon (p+1)^{-1} \int_\Omega K (v^+)^{p+1} dx. \tag{5.9}\]
Since \( \sigma_1(u_\lambda) > \lambda \), by property (ii) in Lemma 3.3, the boundedness of \( K \) and the Sobolev inequality imply that for small \( \epsilon > 0 \),

\[
J_\lambda(v) \geq \frac{1}{4} \sigma_1(u_\lambda)^{-1}(\sigma_1(u_\lambda) - \lambda)\|v\|^2 - \lambda c\|v\|^{p+1},
\]
and the conclusion in Lemma 5.2 follows.

Now, we give the following decomposition lemma for later use.

**Lemma 5.3.** Assume condition (k1) holds. Let \( \{v_k\} \) be a (PS)\(_c\) sequence of \( J_\lambda \) in \( H^1_0(\Omega) \):

\[
\begin{align*}
J_\lambda(v_k) &= c + o(1) \quad \text{as } k \to \infty, \\
J'_\lambda(v_k) &= o(1) \quad \text{strongly in } H^{-1}(\Omega).
\end{align*}
\]

Then there exists a subsequence (still denoted by) \( \{v_k\} \) for which the following holds: there exist an integer \( l \geq 0 \), sequence \( \{x_i^k\} \subset \mathbb{R}^N \) of the form \( (0, z_i^k) \in S \) for \( 1 \leq i \leq l \), a solution \( v_\lambda \) of (5.1), and solutions \( v_i^\lambda \) of (2.1) for \( 1 \leq i \leq l \), such that as \( k \to \infty \)

\[
v_k \rightharpoonup v_\lambda \quad \text{weakly in } H^1_0(\Omega); \\
v_k - \left[ v_\lambda + \sum_{i=1}^{l} \overline{v}_i^\lambda \left( \cdot - x_i^k \right) \right] \to 0 \quad \text{strongly in } H^1_0(\Omega); \\
J_\lambda(v_k) = J_\lambda(v_\lambda) + \sum_{i=1}^{l} I_\infty^\lambda(v_i^\lambda) + o(1),
\]

where it is agreed upon that in the case \( l = 0 \), the above holds without \( \overline{v}_i^\lambda \), \( \{x_i^k\} \).

**Proof.** The proof can be obtained by using the arguments in Bahri and Lions [5] (also see [15, 16]). We omit it.

Now, let \( \delta \) be small enough, \( D^\delta \) a \( \delta \)-tubular neighborhood of \( D \) such that \( D^\delta \subset \subset S \). Let \( \eta(x) : S \to [0, 1] \) be a \( C^\infty \) cutoff function such that \( 0 \leq \eta \leq 1 \) and

\[
\eta(x) = \begin{cases} 
0 & \text{if } x \in D; \\
1 & \text{if } x \in \Omega \setminus D^\delta.
\end{cases}
\]

Let \( e_N = (0,0,\ldots,0,1) \in \mathbb{R}^N \), denote

\[
\tau_0 = 2 \sup_{x \in D^\delta} |x| + 1, \\
\tilde{u}_\lambda(x) = \eta(x)\overline{u}_\lambda(x - \tau e_N), \quad \tau \in [0, \infty),
\]

where \( \overline{u}_\lambda \) is a ground state solution of (2.1).**

**Lemma 5.4.** Assume condition (k1) holds, then there exist some constants \( t_0 > 0 \), \( \tau_* \geq \tau_0 \) such that \( J_\lambda(t\tilde{u}_\lambda) < 0 \) for all \( \tau \geq \tau_* \), \( t \geq t_0 \).
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Proof. By the inequality \((a + b)^p \geq a^p + b^p\) for all \(a \geq 0, b \geq 0, p > 1\) and \(\bar{u}_\lambda\) is a ground state solution of (2.1)\(_\lambda\), denote \(\eta_t(x) = \eta(x + \tau e_N)\), then we have

\[
J_\lambda(t\tilde{u}_t) = \frac{1}{2} t^2 \int_\Omega \left( |\nabla \tilde{u}_t|^2 + |\tilde{u}_t|^2 \right) dx - \int_\Omega \int_0^{t\tilde{u}_t} \lambda K(x)\tilde{u}_t^{p+1} dx
- \frac{1}{p+1} \int_\Omega (s + u_\lambda)^p - u_\lambda^p - s^p ds dx
\leq \frac{1}{2} t^2 \int_S (-\Delta \bar{u}_\lambda + \bar{u}_\lambda)(\eta_t^2 \bar{u}_\lambda) dx + \frac{1}{2} t^2 \int_S |\nabla \eta_t|^2 |\bar{u}_\lambda|^2 dx
- \frac{1}{p+1} t^{p+1} \int_S \lambda K(x)\eta_t^{p+1}(x)\bar{u}_\lambda^{p+1}(x - \tau e_N) dx
\leq \frac{1}{2} t^2 \int_S \lambda K_\infty \bar{u}_\lambda^{p+1} dx + \frac{1}{2} t^2 (\max_{x \in S} |\nabla \eta|^2) \int_S |\bar{u}_\lambda|^2 dx
- \frac{t^{p+1}}{p+1} \int_S \lambda K(x)\eta_t^{p+1}(x)\bar{u}_\lambda^{p+1}(x - \tau e_N) dx.
\] (5.15)

Set \(B_1(\tau e_N) = \{x = (y, z) \in S : y \in \omega, |z - \tau e_N| < 1\}\). By condition (k1), there exists \(\tau_* \geq \tau_0\) such that \(K(x) \geq K_\infty/2\) for \(x \in B_1(\tau e_N)\) for all \(\tau \geq \tau_*\) and note that \(\eta(x) \equiv 1\) on \(B_1(\tau e_N)\) for \(\tau \geq \tau_*\), then we obtain that

\[
\int_S \lambda K(x)\eta_t^{p+1}(x)\bar{u}_\lambda^{p+1}(x - \tau e_N) dx
\geq \int_{B_1(\tau e_N)} \frac{\lambda}{2} K_\infty \bar{u}_\lambda^{p+1}(x - \tau e_N) dx
= \frac{\lambda}{2} K_\infty \bar{u}_\lambda^{p+1}(x) dx = c > 0,
\] (5.16)

where \(c\) is independent of \(\tau\). Combining (5.15) and (5.16), there exist some positive constants \(c_1, c_2\), independent of \(\tau\), such that

\[
J_\lambda(t\tilde{u}_t) \leq c_1 t^2 - c_2 t^{p+1} \quad \forall \tau \geq \tau_*.
\] (5.17)

From (5.17), we conclude the result. \(\square\)

**Lemma 5.5.** Assume conditions (k1) and (k2) hold, then there exists a constant \(\tau^* > 0\), such that the following inequality holds for \(\tau \geq \tau^*\):

\[
0 < \sup_{t \geq 0} J_\lambda(t\tilde{u}_t) < I^\infty_\lambda(\bar{u}_\lambda) = M^\infty_\lambda.
\] (5.18)
Proof. From (5.6), we easily see that the left-hand of (5.18) holds and we need only to show that the right-hand side of (5.18) holds. By Lemma 5.4, we have that there exists a constant \( t_2 > 0 \) such that

\[
\sup_{t \geq 0} J_\lambda(t\tilde{u}_\tau) = \sup_{0 \leq s \leq t_1} J_\lambda(t\tilde{u}_\tau) \quad \text{for any } \tau \geq \tau_*. \quad (5.19)
\]

Since \( J \) is continuous in \( H_0^1(\Omega) \) and \( J(0) = 0 \), there exists a constant \( t_1 > 0 \) such that

\[
J_\lambda(t\tilde{u}_\tau) < M_\lambda^\infty \quad \text{for any } \tau \in (0, \infty), \ 0 \leq t < t_1. \quad (5.20)
\]

Then, to prove (5.18) we now need only to prove the following inequality:

\[
\sup_{t_1 \leq t \leq t_2} J_\lambda(t\tilde{u}_\tau) < M_\lambda^\infty \quad \text{for } \tau \text{ large enough.} \quad (5.21)
\]

By the definition of \( J_\lambda \), we get

\[
J_\lambda(t\tilde{u}_\tau) = \frac{t^2}{2} \int_\Omega \left( |\nabla \tilde{u}_\tau|^2 + \tilde{u}_\tau^2 \right) dx - \frac{tp^{p+1}}{p+1} \int_S \lambda K_\infty \tilde{u}^{p+1}_\tau dx
\]

\[
+ \frac{tp^{p+1}}{p+1} \int_S \lambda (K_\infty - K(x)) \tilde{u}^{p+1}_\tau dx
\]

\[
- \int_\Omega \int_0^{t\tilde{u}_\tau} \lambda K(x) [ (s + u_\lambda)^p - u_\lambda^p - sp] ds dx. \quad (5.22)
\]

Since \( \bar{u}_\lambda \) is a ground state solution of (2.1)\(_\lambda\), denote \( \eta_\tau(x) = \eta(x + \tau e_N) \), then we have

\[
J_\lambda(t\tilde{u}_\tau) \leq \frac{t^2}{2} \int_\Omega \left( -\Delta \bar{u}_\lambda + \bar{u}_\lambda \right) (\eta_\tau^2 \bar{u}_\lambda) dx - \frac{tp^{p+1}}{p+1} \int_S \lambda K_\infty \bar{u}^{p+1}_\lambda dx
\]

\[
+ \frac{t^2}{2} \int_S |\nabla \eta_\tau|^2 |\bar{u}_\lambda|^2 dx
\]

\[
+ \frac{tp^{p+1}}{p+1} \int_S \lambda K_\infty (\bar{u}^{p+1}_\lambda - \tilde{u}^{p+1}_\tau) dx
\]

\[
+ \frac{tp^{p+1}}{p+1} \int_S \lambda (K_\infty - K(x)) \tilde{u}^{p+1}_\tau dx
\]

\[
- \int_\Omega \int_0^{t\tilde{u}_\tau} \lambda K(x) [ (s + u_\lambda)^p - u_\lambda^p - sp] ds dx. \quad (5.23)
\]
It follows from (2.5) that for any $\epsilon > 0$, there exists a constant $c_1 > 0$, independent of $\tau$, such that, for all $\tau \geq \tau_*$,

$$
\frac{t_2^2}{2} \int_{S} \left| \nabla \eta_{\tau} \right|^2 \left| \tilde{u}_{\lambda} \right|^2 \, dx \leq c_1 \exp \left( -2 \sqrt{1 + \mu_1 \tau} \right) \tau^{-n+1+2\epsilon},
$$

(5.24)

$$
\frac{t_2^p}{p+1} \int_{S} \lambda K_{\infty} \left( \tilde{u}_{\lambda}^{p+1} - u_{\tau}^{p+1} \right) \, dx \leq c_1 \exp \left( -2 \sqrt{1 + \mu_1 \tau} \right) \tau^{-n+1+2\epsilon}.
$$

From condition (k2) and (2.5), there exists a constant $c_2 > 0$ such that, for all $\tau \geq \tau_1$,

$$
\frac{t_2^p}{p+1} \int_{S} \lambda (K_{\infty} - K(x))^{+} \tilde{u}_{\tau}^{p+1} \, dx
$$

$$
\leq c \left( \int_{S \cap \{|z| \geq \tau/(p+1)\}} + \int_{S \cap \{|z| = \tau/(p+1)\}} \right) (K_{\infty} - K(x))^{+} \tilde{u}_{\lambda}^{p+1} (x + \tau e_N) \, dx
$$

$$
\leq c \exp \left( -\sqrt{1 + \mu_1 \tau} \right) \left( \frac{p}{p+1} \right)^{p-1} \tau^{-n+1+2\epsilon} + c_1 \exp \left( -\sqrt{1 + \mu_1 \tau} \right) \left( \frac{1}{p+1} \right)^{(p+1)(-(n-1)/2+\epsilon)}
$$

(5.25)

$$
\leq c_2 \exp \left( -\sqrt{1 + \mu_1 \tau} \right) \tau^{-\gamma_0},
$$

where $c_2 > 0$ is a constant independent of $\tau$ and $\gamma_0 = \min\{|\gamma,(p+1)(-(n-1)/2+\epsilon)\}$.

Let $\omega_0 \subset \subset \omega$ be a smooth bounded domain in $\mathbb{R}^n$. Set $D_1(\tau e_N) = \{x = (y,z) \in S : y \in \omega_0, |z - \tau e_N| < 1\}$. Noting that $(a+b)^p \geq a^p + b^p$ for all $a \geq 0, b \geq 0, p > 1$, then for $\tau \geq \tau_0$, we have $\eta(x) = 1$ on $D_1(\tau e_N)$ and

$$
\int_{\Omega} \int_{0}^{\tau e_N} \lambda K(x) \left[(s + \eta_{\tau})^{p} - u_{\lambda}^{p} - s^{p} \right] ds \, dx
$$

$$
\geq \int_{D_1(\tau e_N)} \int_{0}^{\tau e_N} \lambda K(x) \left[(s + \eta_{\tau})^{p} - u_{\lambda}^{p} - s^{p} \right] ds \, dx
$$

$$
= \int_{D_1(\tau e_N)} \int_{0}^{\tau e_N} \lambda K(x) \left[(s + \eta_{\tau})^{p-1} - u_{\lambda}^{p-1} \right] ds \, dx
$$

$$
\geq \int_{D_1(\tau e_N)} \int_{0}^{\tau e_N} \lambda K(x) \left[(s + \eta_{\tau})^{p-1} - u_{\lambda}^{p-1} \right] u_{\lambda} ds \, dx
$$

$$
= \int_{D_1(\tau e_N)} \lambda K(x) \left[ \left( \frac{\tilde{u}_{\tau} + \eta_{\tau}}{\tilde{u}_{\lambda}} \right)^{p} - u_{\lambda}^{p-1} \right] \tilde{u}_{\tau} \, u_{\lambda} \, ds \, dx.
$$

(5.26)

By Lemma 4.8, there exist some constants $\tau_2 \geq \tau_0 + \tau_1$ and $\alpha > 0$, such that

$$
\left( \frac{\tilde{u}_{\tau} + \eta_{\tau}}{\tilde{u}_{\lambda}} \right)^{p} - u_{\lambda}^{p-1} \geq \alpha \quad \text{for } \tau \geq \tau_2, x \in D_1(\tau e_N), t \in \left[t_1, t_2\right],
$$

(5.27)
then by (k1), (4.9), and (5.26), there exists a constant $\tau_3 \geq \tau_2$ such that $K(x) \geq K_\infty/2$ for $x \in D_1(\tau e_N)$ and

$$
\int_\Omega \int_0^{\tau e_N} \lambda K(x)[(s + u_\lambda)^p - u^p_\lambda - s^p] ds dx
\geq \frac{1}{2} \lambda a K_\infty \int_{D_1(\tau e_N)} u_\lambda(x - \tau e_N) u_\lambda(x) dx
\geq c \int_{D_1(\tau e_N)} u_\lambda(x - \tau e_N) \exp \left[ - (\tau + 1) \sqrt{1 + \mu_1} \right] (\tau + 1)^{- (n - 1)/2}
\geq c_3 \exp \left( - \sqrt{1 + \mu_1} \right) \tau^{-(n - 1)/2},
$$

(5.28)

where $c_3 > 0$ is a constant independent of $\tau$ for all $\tau \geq \tau_3$ and $t \in [t_1, t_2]$.

From (5.23)–(5.28), we get, for $\tau \geq \tau_3 + \tau^*$ and $t \in [t_1, t_2]$,

$$
J_\lambda(t \tilde{u}_\tau) \leq M_\lambda^\infty + 2c_1 \exp \left( -2\sqrt{1 + \mu_1} \right) \tau^{-n + 1 + 2\epsilon} + c_2 \exp \left( -\sqrt{1 + \mu_1} \right) \tau^{-\gamma_0} - c_3 \exp \left( -\sqrt{1 + \mu_1} \right) \tau^{-(n - 1)/2},
$$

(5.29)

where $c_i, 1 \leq i \leq 3$, are independent of $\tau$.

Let $\epsilon = p(n - 1)/4(p + 1)$ and by $\gamma > (n - 1)/2$, we have $\gamma_0 > (n - 1)/2$. Hence, we can find some constant $\tau^* > \tau_3 + \tau^*$ large enough such that

$$
2c_1 \exp \left( -2\sqrt{1 + \mu_1} \right) \tau^{-n + 1 + 2\epsilon} + c_2 \exp \left( -\sqrt{1 + \mu_1} \right) \tau^{-\gamma_0} - c_3 \exp \left( -\sqrt{1 + \mu_1} \right) \tau^{-(n - 1)/2} < 0
$$

(5.30)

and (5.18) is proved. \hfill \Box

**Proposition 5.6.** Let conditions (k1) and (k2) hold, then (5.1) has at least one solution for $\lambda \in (0, \lambda^*)$.

**Proof.** For the constant $\tau^*$ in Lemma 5.5, by Lemma 5.4, we know that there is a constant $t_0 > 0$ such that $J_\lambda(t_0 \tilde{u}_{\tau^*}) < 0$. We set

$$
\Gamma = \{ \gamma \in C([0, 1], H^1_0(\Omega)) : \gamma(0) = 0, \gamma(1) = t_0 \tilde{u}_{\tau^*} \},
$$

(5.31)

then, from (5.6) and (5.18) we get

$$
0 < c = \inf_{\gamma \in \Gamma_0} \max_{s \leq 1} J_\lambda(\gamma(s)) < M_\lambda^\infty.
$$

(5.32)

Applying the mountain pass lemma of Ambrosetti and Rabinowitz [3], there exists a \((PS)_c\)-sequence $\{v_k\}$ such that

$$
J_\lambda(v_k) \longrightarrow c, \quad J'_\lambda(v_k) \longrightarrow 0 \quad \text{in} \ H^{-1}(\Omega).
$$

(5.33)
By Lemma 5.3, there exist a subsequence, still denoted by \( \{v_k\} \), an integer \( l \geq 0 \), a solution \( v_\lambda \) of (5.1), and solutions \( v_i^\lambda \) of (2.1), for \( 1 \leq i \leq l \), such that

\[
c = J_\lambda(v_\lambda) + \sum_{i=1}^{l} I_\lambda^\infty(v_i^\lambda).
\]

(5.34)

By the strongly maximum principle, to complete the proof, we only need to prove \( v_\lambda \not\equiv 0 \) in \( \Omega \). We proceed by contradiction. Assume that \( v_\lambda \equiv 0 \) in \( \Omega \). From (5.32) and (5.34), we have \( l \geq 1 \) and

\[
0 < M_\lambda^\infty \leq lM_\lambda^\infty \leq \sum_{i=1}^{l} I_\lambda^\infty(v_i^\lambda) = c < M_\lambda^\infty.
\]

(5.35)

This implies \( v_\lambda \not\equiv 0 \) in \( \Omega \).

\qed

6. Properties and bifurcation of solutions

Denote \( A = \{ (\lambda, u) : u \) satisfies (1.1) \( \lambda \in [0, \lambda^*] \} \). By Lemma 4.8, we have \( A \subset \mathbb{R} \times L^\infty(\Omega) \cap H_0^1(\Omega) \). Moreover, we assume that \( f(x), K(x) \in C^\alpha(\Omega) \cap L^2(\Omega) \). By elliptic regular theory \([9]\), we can deduce that \( A \subset \mathbb{R} \times C^2, \alpha(\Omega) \cap H^2(\Omega) \).

For each \( (\lambda, u) \in A \), let \( \sigma_\lambda(u) \) denote the number defined by (3.11), which is the first eigenvalue of the problem (3.12).

Lemma 6.1. Let \( u \) be a solution and let \( u_\lambda \) be the minimal solution of (1.1) for \( \lambda \in (0, \lambda^*) \). Then

(i) \( \sigma_\lambda(u) > \lambda \) if and only if \( u = u_\lambda \);

(ii) \( \sigma_\lambda(U_\lambda) < \lambda \), where \( U_\lambda \) is the second solution of (1.1) constructed in Section 5.

Proof. Now, let \( \phi \geq 0 \) and \( \phi \in H_0^1(\Omega) \). Since \( u \) and \( u_\lambda \) are the solution of (1.1), then

\[
\int_\Omega \nabla \phi \cdot \nabla (u_\lambda - u) \, dx + \int_\Omega \phi (u_\lambda - u) \, dx = \lambda \int_\Omega K(u_\lambda^p - u^p) \phi \, dx = \lambda \int_\Omega \left( \int_u^{u_\lambda} t^{p-1} \, dt \right) pK \phi \, dx \geq \lambda \int_\Omega pKu_\lambda^{p-1} (u_\lambda - u) \phi \, dx.
\]

(6.1)

Let \( \phi = (u - u_\lambda)^+ \geq 0 \) and \( \phi \in H_0^1(\Omega) \). If \( \phi \neq 0 \), then (6.1) implies

\[
-\int_\Omega (|\nabla \phi|^2 + \phi^2) \, dx \geq -\lambda \int_\Omega pKu_\lambda^{p-1} \phi^2 \, dx
\]

(6.2)

and, therefore, the definition of \( \sigma_\lambda(u) \) implies

\[
\int_\Omega (|\nabla \phi|^2 + \phi^2) \, dx \leq \lambda \int_\Omega pKu_\lambda^{p-1} \phi^2 \, dx < \sigma_\lambda(u) \int_\Omega pKu_\lambda^{p-1} \phi^2 \, dx \leq \int_\Omega (|\nabla \phi|^2 + \phi^2) \, dx,
\]

(6.3)

which is impossible. Hence \( \phi \equiv 0 \), and \( u = u_\lambda \) in \( \Omega \). On the other hand, by Lemma 3.3, we also have that \( \sigma_\lambda(u_\lambda) > \lambda \). This completes the proof of (i).
By (i), we get that $\sigma_\lambda(U_\lambda) \leq \lambda$ for $\lambda \in (0, \lambda^\ast)$. We claim that $\sigma_\lambda(U_\lambda) = \lambda$ cannot occur. We proceed by contradiction. Set $w_\lambda = U_\lambda - u_\lambda$; we have
\[
-\Delta w_\lambda + w_\lambda = \lambda K[U_\lambda^p - (U_\lambda - w_\lambda)^p], \quad w_\lambda > 0 \text{ in } \Omega. \tag{6.4}
\]
By $\sigma_\lambda(U_\lambda) = \lambda$, we have that the problem
\[
-\Delta \varphi + \varphi = \lambda p K U_\lambda^{p-1} \varphi, \quad \varphi \in H^1_0(\Omega), \tag{6.5}
\]
possesses a positive solution $\varphi_\lambda$.

Multiplying (6.4) by $\varphi_\lambda$ and (6.5) by $w_\lambda$, integrating, and subtracting we deduce that
\[
0 = \int_\Omega \lambda K[U_\lambda^p - (U_\lambda - w_\lambda)^p] \varphi_\lambda dx - \frac{1}{2} p(p-1) \int_\Omega \lambda \xi_\lambda^{p-2} w_\lambda^2 \varphi_\lambda dx,
\]
where $\xi_\lambda \in (u_\lambda, U_\lambda)$. Thus $w_\lambda \equiv 0$, that is, $U_\lambda = u_\lambda$ for $\lambda \in (0, \lambda^\ast)$. This is a contradiction. Hence, we have $\sigma_\lambda(U_\lambda) < \lambda$ for $\lambda \in (0, \lambda^\ast)$.  

**Lemma 6.2.** Let $u_\lambda$ be the minimal solution of (1.1) for $\lambda \in [0, \lambda^\ast]$ and $\sigma_\lambda(u_\lambda) > \lambda$. Then for any $g(x) \in H^{-1}(\Omega)$, problem
\[
-\Delta w + w = \lambda p K u_\lambda^{p-1} w + g(x), \quad w \in H^1_0(\Omega), \tag{6.4}_\lambda
\]
has a solution.

**Proof.** Consider the functional
\[
\Phi(w) = \frac{1}{2} \int_\Omega (|\nabla w|^2 + w^2) dx - \frac{1}{2} \lambda p \int_\Omega K u_\lambda^{p-1} w^2 dx - \int_\Omega g(x) w dx,
\]
where $w \in H^1_0(\Omega)$. From Hölder inequality and Young’s inequality, we have, for any $\epsilon > 0$,
\[
\Phi(w) \geq \frac{1}{2} \left(1 - \lambda \sigma_\lambda(u_\lambda)^{-1}\right) \|w\|^2 - \frac{1}{2} \epsilon \|w\|^2 - \frac{C\epsilon}{2} \|g\|^2_{H^{-1}(\Omega)} \tag{6.8}
\]
if we choose $\epsilon$ small.

Now, let $\{w_k\} \subset H^1_0(\Omega)$ be the minimizing sequence of variational problem
\[
d = \inf \{\Phi(w) \mid w \in H^1_0(\Omega)\}. \tag{6.9}
\]
From (6.8) and $\sigma_\lambda(u_\lambda) > \lambda$, we can also deduce that $\{w_k\}$ is bounded in $H^1_0(\Omega)$ if we choose $\epsilon$ small. So we may suppose that
\[
w_k \rightharpoonup w \quad \text{weakly in } H^1_0(\Omega), \tag{6.10}_{\text{a.e. in } \Omega}
w_k \to w \quad \text{strongly in } L^q_{\text{loc}}(\Omega) \quad \text{for } 2 \leq q < \frac{2N}{N-2} \quad \text{as } k \to \infty.
\]
By Fatou’s lemma,

\[ \|w\| \leq \liminf_{k \to \infty} \|w_k\|, \quad (6.11) \]

and by the weak convergence we have

\[ \int_{\Omega} g w_k dx \to \int_{\Omega} g w dx \quad \text{as} \quad k \to \infty. \quad (6.12) \]

By Lemma 4.8, we have \( u_\lambda(y, z) \to 0 \) as \( |z| \to \infty \) uniformly for \( y \in \omega \). It follows that there exists a constant \( c_1 > 0 \) such that

\[ |u_\lambda(x)| \leq c_1 \quad \forall x \in \Omega. \quad (6.13) \]

Furthermore, for any \( \varepsilon > 0 \), there exists \( R > 0 \) such that \( |u_\lambda^{p-1}(x)| < \varepsilon \) for all \( x = (y, z) \in \Omega \) and \( |y| \geq R \). Let \( \Omega_R = \{ x = (y, z) \in \Omega : |z| < R \} \), then we have

\[
\left| \int_{\Omega} \left( Ku_\lambda^{p-1}(w_k^2 - w^2) \right) dx \right| \\
\leq \|K\|_{L^\infty(\Omega)} \left( \int_{\Omega_R} u_\lambda^{p-1} |w_k - w|^2 dx + \int_{\Omega \setminus \Omega_R} u_\lambda^{p-1} |w_k - w|^2 dx \right) \\
\leq c_2 \int_{\Omega_R} |w_k - w|^2 dx + \varepsilon \int_{\Omega \setminus \Omega_R} |w_k - w|^2 dx .
\]

From \( w_k \to w \) strongly in \( L^q_{\text{loc}}(\Omega) \) for \( 2 \leq q < 2N/(N-2) \) as \( k \to \infty \), it follows that

\[ \lim_{k \to \infty} \int_{\Omega_R} |w_k - w|^2 dx = 0. \quad (6.15) \]

Since \( \{w_k\} \) is bounded in \( H^1_0(\Omega) \), this implies that there exists a constant \( c_3 > 0 \) such that

\[ \int_{\Omega \setminus \Omega_R} |w_k - w|^2 dx \leq c_3. \quad (6.16) \]

Therefore, we conclude that

\[ \lim_{k \to \infty} \left| \int_{\Omega} Ku_\lambda^{p-1}(w_k^2 - w^2) dx \right| \leq c_3 \varepsilon. \quad (6.17) \]

Take \( \varepsilon \to 0 \), we obtain

\[ \int_{\Omega} Ku_\lambda^{p-1} w_k^2 dx \to \int_{\Omega} Ku_\lambda^{p-1} w^2 dx \quad \text{as} \quad k \to \infty. \quad (6.18) \]

Therefore

\[ \Phi(w) \leq \lim_{n \to \infty} \Phi(w_k) = d \quad (6.19) \]

and \( \Phi(w) = d \) which gives that \( w \) is a solution of (6.4). \( \square \)
Remark 6.3. From Lemma 6.2, we know that (6.4)_λ has a solution \( w \in H^1_0(\Omega) \). Now, we also assume that \( K(x), f(x), \) and \( g(x) \) are in \( C^\infty(\Omega) \cap L^2(\Omega) \), then by Lemmas 4.4 and 4.6, we have that \( w \in H^1_0(\Omega) \). The standard elliptic regular theory yields \( w \in C^{2,\alpha}(\Omega) \).

**Lemma 6.4.** Suppose \( u_{\lambda^*} \) is a solution of (1.1)_{\lambda^*}, then \( \sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^* \) and the solution \( u_{\lambda^*} \) is unique.

**Proof.** Define \( F: \mathbb{R} \times H^1_0(\Omega) \to H^{-1}(\Omega) \) by

\[
F(\lambda, u) = \Delta u - u + \lambda K(u^+)^p + f(x).
\]

Since \( \sigma_\lambda(u_\lambda) \geq \lambda \) for \( \lambda \in (0, \lambda^*) \), so \( \sigma_\lambda(u_\lambda) \geq \lambda^* \). If \( \sigma_\lambda(u_\lambda) > \lambda^* \), the equation \( F_u(\lambda^*, u_{\lambda^*}) \phi = 0 \) has no nontrivial solution. From Lemma 6.2, \( F_u \) maps \( \mathbb{R} \times H^1_0(\Omega) \) onto \( H^{-1}(\Omega) \).

Applying the implicit function theorem to \( F \), we can find a neighborhood \((\lambda^* - \delta, \lambda^* + \delta)\) of \( \lambda^* \) such that (1.1) possesses a solution \( u_\lambda \) if \( \lambda \in (\lambda^* - \delta, \lambda^* + \delta) \). This is contradictory to the definition of \( \lambda^* \). Hence, we obtain \( \sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^* \).

Next, we are going to prove that \( u_{\lambda^*} \) is unique. In fact, suppose (1.1)_{\lambda^*} has another solution \( U_{\lambda^*} \geq u_{\lambda^*} \). Set \( w = U_{\lambda^*} - u_{\lambda^*} \); we have

\[
-\Delta w + w = \lambda^* K[(w + u_{\lambda^*})^p - u_{\lambda^*}^p], \quad w > 0 \text{ in } \Omega.
\]

By \( \sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^* \), we have that the problem

\[
-\Delta \phi + \phi = \lambda^* p K u_{\lambda^*}^{p-1} \phi, \quad \phi \in H^1_0(\Omega)
\]

possesses a positive solution \( \phi_1 \).

Multiplying (6.21) by \( \phi_1 \) and (6.22) by \( w \), integrating, and subtracting we deduce that

\[
0 = \int_\Omega \lambda^* K[(w + u_{\lambda^*})^p - u_{\lambda^*}^p - p u_{\lambda^*}^{p-1} w] \phi_1 dx
\]

\[
= \frac{1}{2} p(p-1) \int_\Omega \lambda^* K \xi_{\lambda^*}^{p-2} w^2 \phi_1 dx,
\]

where \( \xi_{\lambda^*} \in (u_{\lambda^*}, u_{\lambda^*} + w) \). Thus \( w \equiv 0 \). \( \square \)

**Proposition 6.5.** Let \( u_\lambda \) be the minimal solution of (1.1)\_\lambda. Then \( u_\lambda \) is uniformly bounded in \( L^\infty(\Omega) \cap H^1_0(\Omega) \) for all \( \lambda \in [0, \lambda^*] \) and

\[
\lambda^* \to u_0 \quad \text{in } L^\infty(\Omega) \cap H^1_0(\Omega) \text{ as } \lambda \to 0^+,
\]

where \( u_0 \) is the unique positive solution of (1.1) \_0.

**Proof.** By Lemmas 4.8, 3.3, and 6.4, we can deduce \( \|u_\lambda\|_{L^\infty(\Omega)} \leq \|u_{\lambda^*}\|_{L^\infty(\Omega)} \leq c \) for \( \lambda \in [0, \lambda^*] \). By (3.26), we have \( \|u_\lambda\| \leq (p/(p+1)) \|f\|_{H^{-1}} \). Hence, \( u_\lambda \) is uniformly bounded in \( L^\infty(\Omega) \cap H^1_0(\Omega) \) for \( \lambda \in [0, \lambda^*] \).

Now, let \( w_\lambda = u_\lambda - u_0 \), then \( w_\lambda \) satisfies the following equation:

\[
-\Delta w_\lambda + w_\lambda = \lambda K u_\lambda^p \quad \text{in } \Omega,
\]

(8.6)
and by $u_\lambda$ being uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$, we have

$$
\|w_\lambda\|^2 = \int_\Omega \lambda K u_\lambda^p w_\lambda \, dx \\
\leq \lambda \|K\|_{L^\infty(\Omega)} \|u_\lambda\|_{L^p(\Omega)}^{-1} \|u_\lambda\|_{L^2(\Omega)} \|w_\lambda\|_{L^2(\Omega)} \leq c\lambda, 
$$

(6.25)

where $c$ is independent of $\lambda$. Hence, we obtain $u_\lambda \to u_0$ in $H_0^1(\Omega)$ as $\lambda \to 0^+$.

By Lemma 4.4, $u_\lambda \in L^q(\Omega)$ for all $q \in [2, \infty)$ and $u_\lambda$ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$, then for any $q \in [2, \infty)$, there exists a positive constant $c_q$, independent of $u_\lambda$, $\lambda \in [0, \lambda^*)$, such that

$$
\|K u_\lambda^p\|_{L^q(\Omega)} \leq c_q. 
$$

(6.26)

Now, let $q_0 = N/2 + 1 > N/2$ and by Lemma 4.4, we have $\lambda K u_\lambda^p \in L^{q_0}(\Omega)$. Apply Lemmas 4.2, 4.4, 4.6, to (6.8) and by (6.25) and (6.26), we obtain

$$
\|w_\lambda\|_{L^\infty(\Omega)} \leq c_1 \|w_\lambda\|_{W^{2,q}(\Omega)} \\
\leq c_2 \left(\|\lambda K u_\lambda^p\|_{L^{q_0}(\Omega)} + \|w_\lambda\|_{L^{q_0}(\Omega)}\right) \\
\leq c_3 \lambda + c_2 \|w_\lambda\|_{L^{\infty}(\Omega)}^{1-2/q_0} \|w_\lambda\|_{L^2(\Omega)}^{2/q_0} \leq c(\lambda + \lambda^{1/q_0}),
$$

(6.27)

where $c$ is independent of $\lambda$. Hence, we obtain $u_\lambda \to u_0$ in $L^\infty(\Omega)$ as $\lambda \to 0^+$. \hfill \Box

**Proposition 6.6.** For $\lambda \in (0, \lambda^*)$, let $U_\lambda$ be the positive solution of (1.1) with $U_\lambda > u_\lambda$, then $U_\lambda$ is unbounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$, that is,

$$
\lim_{\lambda \to 0^+} \|U_\lambda\| = \lim_{\lambda \to 0^+} \|U_\lambda\|_{L^\infty(\Omega)} = \infty.
$$

(6.28)

**Proof.** Let $\varphi_\lambda$ be a minimizer of $\sigma_\lambda(U_\lambda)$ for $\lambda \in (0, \lambda^*)$, that is,

$$
\int_\Omega p K U_\lambda^{p-1} \varphi_\lambda^2 \, dx = 1, \quad \|\varphi_\lambda\|^2 = \sigma_\lambda(U_\lambda).
$$

(6.29)

(i) First, we show that $\{U_\lambda : \lambda \in (0, \lambda_0)\}$ is unbounded in $L^\infty(\Omega)$ for any $\lambda_0 \in (0, \lambda^*)$. We proceed by contradiction. Assume to the contrary that there exists $c_0 > 0$ such that

$$
\|U_\lambda\|_{\infty} \leq c_0 < \infty \quad \forall \lambda \in (0, \lambda_0),
$$

(6.30)

by (6.29) and $\sigma_\lambda(U_\lambda) < \lambda$ for all $\lambda \in (0, \lambda_0)$, we obtain

$$
1 = \int_\Omega p K U_\lambda^{p-1} \varphi_\lambda^2 \, dx \leq c \|\varphi_\lambda\|^2 = c \sigma_\lambda(U_\lambda) < c\lambda,
$$

(6.31)

where $c = p \|K\|_{L^\infty(\Omega)} \lambda_0^{p-1}$. This is a contradiction for all $\lambda < 1/c$. Hence, for any $\lambda_0 \in (0, \lambda^*)$, $\{U_\lambda : \lambda \in (0, \lambda^*)\}$ is unbounded in $L^\infty(\Omega)$. From this result, it is easy to see that $\lim_{\lambda \to 0^+} \|U_\lambda\|_{\infty} = \infty$. 

Proof of Theorems 1.1 and 1.2. and Proposition 5.6. The conclusions (i) and (ii) of Theorem 1.2 follow immediately from

\[ \| U_\lambda \| \leq c_0 \quad \forall \lambda \in (0, \lambda_0). \]  

(6.32)

By (6.29), (6.32), Hölder inequality, Sobolev embedding theorem, and \( \sigma_1(U_\lambda) < \lambda \) for all \( \lambda \in (0, \lambda^*) \), we have

\[
1 = \int_\Omega pK U_\lambda^{p-1} \phi_\lambda^2 \, dx \leq p K \| U_\lambda \|_{L^p(\Omega)}^{p-1} \| \phi_\lambda \|_{L^{p^*}(\Omega)}^2
\]

\[
\leq c_1 \| U_\lambda \|^{p-1} \| \phi_\lambda \|^2 \leq c_1 c_0^{p-1} \| \phi_\lambda \|^2 = c_1 c_0^{p-1} \sigma_1(U_\lambda) < c_1 c_0^{p-1} \lambda,
\]

where \( c_1 \) is a constant independent of \( \lambda \). Now, let \( \lambda \to 0^+ \), then we obtain a contradiction. Hence, \( \{ U_\lambda : \lambda \in (0, \lambda^*) \} \) is unbounded in \( H_0^1(\Omega) \) and \( \lim_{\lambda \to 0^+} \| U_\lambda \| = +\infty. \)

In order to get bifurcation results we need the following bifurcation theorem which can be found in Crandall and Rabinowitz [6].

Theorem 6.7. Let \( X, Y \) be Banach spaces. Let \( (\bar{\lambda}, \bar{x}) \in \mathbb{R} \times X \) and let \( F \) be a continuously differentiable mapping of an open neighborhood of \((\bar{\lambda}, \bar{x})\) into \( Y \). Let the null space \( N(F_x(\bar{\lambda}, \bar{x})) = \text{span}\{x_0\} \) be one-dimensional and \( \text{codim } R(F_x(\bar{\lambda}, \bar{x})) = 1 \). Let \( F_1(\bar{\lambda}, \bar{x}) \notin R(F_x(\bar{\lambda}, \bar{x})) \).

If \( Z \) is the complement of \( \text{span}\{x_0\} \) in \( X \), then the solutions of \( F(\lambda, x) = F_1(\bar{\lambda}, \bar{x}) \) near \( (\bar{\lambda}, \bar{x}) \) form a curve \( (\lambda(s), x(s)) = (\bar{\lambda} + \tau(s), \bar{x} + sx_0 + z(s)), \) where \( s \to (\tau(s), z(s)) \in \mathbb{R} \times Z \) is continuously differentiable function near \( s = 0 \) and \( \tau(0) = \tau'(0) = 0, z(0) = z'(0) = 0. \)

Proof of Theorems 1.1 and 1.2. Theorem 1.1 now follows from Lemmas 3.2, 3.3, 6.1, 6.4, and Proposition 5.6. The conclusions (i) and (ii) of Theorem 1.2 follow immediately from Lemma 3.3, Remark 6.2 and Propositions 6.5, 6.6. Now we are going to prove that \( (\lambda^*, u_{\lambda^*}) \) is a bifurcation point in \( C^{2,\alpha}(\Omega) \cap H^2(\Omega) \) by using an idea in [13]. We also assume that \( K(x) \) and \( f(x) \) are in \( C^\alpha(\Omega) \cap L^2(\Omega) \) and define

\[
F : \mathbb{R}^1 \times C^{2,\alpha}(\Omega) \cap H^2(\Omega) \to C^\alpha(\Omega) \cap L^2(\Omega)
\]

(6.34)

by

\[
F(\lambda, u) = \Delta u - u + \lambda K(u^+)^p + f(x),
\]

(6.35)

where \( C^{2,\alpha}(\Omega) \cap H^2(\Omega) \) and \( C^\alpha(\Omega) \cap L^2(\Omega) \) are endowed with the natural norm; then they become Banach spaces. It can be verified easily that \( F(\lambda, u) \) is differentiable. From Lemma 6.2 and Remark 6.3, we know that

\[
F_u(\lambda, u)w = \Delta w - w + \lambda pK u_\lambda^{p-1} w
\]

(6.36)

is an isomorphism of \( \mathbb{R}^1 \times C^{2,\alpha}(\Omega) \cap H^2(\Omega) \) onto \( C^\alpha(\Omega) \cap L^2(\Omega) \). It follows from implicit function theorem that the solutions of \( F(\lambda, u) = 0 \) near \( (\lambda, u_\lambda) \) are given by a continuous curve.

Now we are going to prove that \( (\lambda^*, u_{\lambda^*}) \) is a bifurcation point of \( F \). We show first that at the critical point \( (\lambda^*, u_{\lambda^*}) \), Theorem 6.7 applies. Indeed, from Lemma 6.4, problem
(6.22) has a solution \( \phi_1 > 0 \) in \( \Omega \). By the standard elliptic regular theory, we have \( \phi_1 \in C^{2,\alpha}(\Omega) \cap H^2(\Omega) \) if \( f \in C^4(\Omega) \cap L^2(\Omega) \). Thus \( F_u(\lambda^*, u_{1^*}) \phi = 0, \phi \in C^{2,\alpha}(\Omega) \cap H^2(\Omega) \) has a solution \( \phi_1 > 0 \). This implies that \( N(F_u(\lambda^*, u_{1^*})) = \text{span}\{ \phi_1 \} = 1 \) is one dimensional and codim \( R(F_u(\lambda^*, u_{1^*})) = 1 \) by the Fredholm alternative. It remains to check that \( F_1(\lambda^*, u_{1^*}) \notin R(F_u(\lambda^*, u_{1^*})) \).

Assuming the contrary would imply the existence of \( v \neq 0 \) such that

\[
\Delta v - v + \lambda^* p K u_{1^*}^{p-1} v = K u_{1^*}^p, \quad v \in H_0^1(\Omega). \tag{6.37}
\]

From \( F_u(\lambda^*, u_{1^*}) \phi_1 = 0 \), we conclude that \( \int_{\Omega} K u_{1^*}^p \phi_1 dx = 0 \). This is impossible because \( K(x) \geq 0 \), \( K(x) \neq 0 \), \( u_{1^*}(x) > 0 \), and \( \phi_1(x) > 0 \) in \( \Omega \).

Applying Theorem 6.7, we conclude that \((\lambda^*, u_{1^*})\) is a bifurcation point near which the solution of (1.1) forms a curve \( (\lambda^* + \tau(s), u_{1^*} + s\phi_1 + z(s)) \) with \( s \) near \( s = 0 \) and \( \tau(0) = \tau'(0) = 0, z(0) = z'(0) = 0 \). We claim that \( \tau''(0) < 0 \) which implies that the bifurcation curve turns strictly to the left in \((\lambda, u)\) plane. In order to obtain that \( \tau''(0) < 0 \), we need the following lemma.

**Lemma 6.8.** Suppose condition (k1) holds, then

\[
\int_{\Omega} K u_{1^*}^{p-2} \phi_1^3 dx < +\infty. \tag{6.38}
\]

**Proof.** Since \( u_{1^*}(x) \to 0 \) as \( |x| \to \infty \), there is \( R_1 > 0 \) such that

\[
0 = \Delta \phi_1 - \phi_1 + \lambda^* p K u_{1^*}^{p-1} \phi_1 \leq \Delta \phi_1 - \frac{1}{4} \phi_1, \quad \text{for } \omega \in, |z| \geq R_1. \tag{6.39}
\]

It is well-known that the Dirichlet equation \( \Delta w - (1/4) w = -w^p \) in \( \Sigma \) has a positive ground-state solution, denoted by \( \overline{w} \) (see [14] and the references there). We can modify the proof in Hsu [10] and obtain that for any \( \epsilon > 0 \) with \( 0 < \epsilon < 1/4 + \mu_1 \), there exist constants \( c_\epsilon > 0 \) and \( R_2 > 0 \) such that

\[
\overline{w}(y, z) \leq c_\epsilon \psi(y) \exp\left(-\sqrt{\frac{1}{4} + \mu_1 - \epsilon |z|}\right) \quad \text{for } y \in \omega, |z| \geq R_2, \tag{6.40}
\]

where \( \psi \) is the first positive eigenfunction of the Dirichlet problem \( -\Delta \psi = \mu_1 \psi \) in \( \omega \). Now, let \( \epsilon = (1/2) \mu_1 \). Since \( \Delta \overline{w} - (1/4) \overline{w} = -\overline{w}^p \leq 0 \) in \( \Omega \), hence by the maximum principle we obtain that there exist constants \( c_1 > 0 \) and \( R_3 > 0 \) such that

\[
\phi_1(y, z) \leq c_1 \psi(y) \exp\left(-\frac{1}{2} \sqrt{1 + 2 \mu_1 |z|}\right) \quad \text{for } y \in \omega, |z| \geq R_3. \tag{6.41}
\]

Let \( q \in \partial \Omega \), and \( B \) a small ball in \( \Omega \) such that \( q \in \partial B \). Since \( \phi_1(x) > 0 \) for \( x \in B, \phi_1(q) = 0, u_{1^*}(x) > 0 \) for \( x \in B, u(q) = 0 \), by the strongly maximum principle \( (\partial \phi_1/\partial x)(q) < 0, (\partial u_{1^*}/\partial x)(q) < 0 \). Thus

\[
\lim_{x \to q} \frac{u_{1^*}(x)}{\phi_1(x)} = \frac{(\partial u_{1^*}/\partial x)(q)}{(\partial \phi_1/\partial x)(q)} > 0, \tag{6.42}
\]
and we have \( u_{\lambda^*}^{-1} \phi_1 \in C^1(\Omega) \) and \( u_{\lambda^*}^{-1} \phi_1 > 0 \) on \( \overline{\Omega} \). Therefore, there exists \( c_2 > 0 \) such that
\[
u_{\lambda^*}^{-1}(x) \phi_1(x) \leq c_2 \quad \text{for } x \in \Omega_{R_0},
\]
where \( \Omega_{R_0} = \{ x = (y, z) \in \Omega : |z| < R_0 \} \).

Now, by (4.9), (6.41), and (6.43), there exists \( c_3 > 0 \) such that
\[
u_{\lambda^*}^{-1}(x) \phi_1(x) \leq c_3 \quad \text{for } x \in \Omega.
\]

From (6.41), (6.44) and Hölder’s inequality, we derive
\[
\int_{\Omega} K u_{\lambda^*}^{p-2} \phi_1^3 \, dx 
\leq c_3 \int_{\Omega} K u_{\lambda^*}^{p-1} \phi_1 \, dx 
\leq c \left( \int_{\Omega} u_{\lambda^*}^{p+1} \, dx \right)^{(p-1)/(p+1)} \left( \int_{\omega} [\psi(y)]^{p+1/2} \, dy \cdot \int_{\mathbb{R}^n} e^{-(p+1)/4 \sqrt{1+2\mu} |z|} \, dz \right)^{2/(p+1)} < \infty.
\]
(6.45)

Since \( \lambda = \lambda^* + \tau(s) \), \( u = u_{\lambda^*} + s \phi_1 + z(s) \) in
\[
-\Delta u + u - \lambda K u^p - f = 0, \quad u > 0, \ u \in C^{2,\alpha}(\Omega) \cap H^2(\Omega).
\]
(6.46)

Differentiating (6.46) in \( s \) twice, we have
\[
-\Delta u_{ss} + u_{ss} - \lambda p K u^{p-1} u_{ss} - 2\lambda_s p K u^{p-1} u_s - \lambda p(p-1) K u^{p-2} u_s^2 - \lambda_{ss} K u^p = 0.
\]
(6.47)

Setting here \( s = 0 \) and using the facts that \( \tau'(0) = 0 \), \( u_s = \phi_1(x) \), and \( u = u_{\lambda^*} \) as \( s = 0 \), we obtain
\[
-\Delta u_{ss} + u_{ss} - \lambda^* p K u_{\lambda^*}^{p-1} u_{ss} - \lambda^* p(p-1) K u_{\lambda^*}^{p-2} \phi_1^2 - \tau''(0) K u_{\lambda^*}^p = 0.
\]
(6.48)

Multiplying \( F_s(\lambda^*, u_{\lambda^*}) \phi_1 = 0 \) by \( u_{ss} \) and (6.48) by \( \phi_1 \), integrating, and subtracting the result, and by (6.38) we obtain
\[
\int_{\Omega} \lambda^* p(p-1) K u_{\lambda^*}^{p-2} \phi_1^3 \, dx + \tau''(0) \int_{\Omega} K u_{\lambda^*}^p \phi_1^3 \, dx = 0,
\]
which immediately gives \( \tau''(0) < 0 \). Thus
\[
u_{\lambda^*} \longrightarrow u_{\lambda^*} \quad \text{in } C^{2,\alpha}(\Omega) \cap H^2(\Omega) \quad \text{as } \lambda \longrightarrow \lambda^*,
\]
\[
u_{\lambda^*} \longrightarrow u_{\lambda^*} \quad \text{in } C^{2,\alpha}(\Omega) \cap H^2(\Omega) \quad \text{as } \lambda \longrightarrow \lambda^*.
\]
(6.50)

Using Lemma 6.2, Remark 6.3, the implicit function theorem, and the uniqueness of the positive ground-state solution of (1.1) \( 0 '\), we can easily prove that
\[
u_{\lambda^*} \longrightarrow u_0 \quad \text{in } C^{2,\alpha}(\Omega) \cap H^2(\Omega) \quad \text{as } \lambda \longrightarrow 0^+,
\]
(6.51)

which proves Theorem 1.2.
Remark 6.9. If \( \Omega = S, \Omega = \mathbb{R}^N, \) or \( \Omega = \mathbb{R}^N \setminus D, \) the proof still holds after simple modification.

References


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