INTUITIONISTIC FUZZY $H_v$-IDEALS

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The notion of intuitionistic fuzzy sets was introduced by Atanassov as a generalization of the notion of fuzzy sets. In this paper, we apply the concept of intuitionistic fuzzy sets to $H_v$-rings. We introduce the notion of an intuitionistic fuzzy $H_v$-ideal of an $H_v$-ring and then some related properties are investigated. We state some characterizations of intuitionistic fuzzy $H_v$-ideals. Also we investigate some natural equivalence relations on the set of all intuitionistic fuzzy $H_v$-ideals of an $H_v$-ring.

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1. Introduction and preliminaries

Hyperstructure theory was born in 1934 when Marty [11] defined hypergroups as a generalization of groups. This theory has been studied in the following decades and nowadays by many mathematicians. A recent book [3] contains a wealth of applications. There are applications to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence, and probabilities. Vougiouklis in the fourth Algebraic Hyperstructures and Applications Congress (1990) [15] introduced the notion of $H_v$-structures. The $H_v$-structures are hyperstructures where the equality is replaced by the nonempty intersection. The main tool in the study of $H_v$-structure is the fundamental structure which is the same as in the classical hyperstructures. In this paper, we deal with $H_v$-rings. $H_v$-rings are the largest class of algebraic systems that satisfy ring-like axioms. In [4], Darafsheh and Davvaz defined the $H_v$-ring of fractions of a commutative hyperring which is a generalization of the concept of ring of fractions. For the notion of an $H_v$-near-ring module, you can see [7]. In [13], Spartalis studied a wide class of $H_v$-rings resulting from an arbitrary ring by using the $P$-hyperoperations. In [18], Vougiouklis introduced the classes of $H_v$-rings useful in the theory of representations.

A hyperstructure is a nonempty set $H$ together with a map $*: H \times H \to \mathcal{P}(H)$ called hyperoperation, where $\mathcal{P}(H)$ denotes the set of all nonempty subsets of $H$. The image of the pair $(x, y)$ is denoted by $x \ast y$. If $x \in H$ and $A, B \subseteq H$, then by $A \ast B, A \ast x$, and $x \ast B$,
we mean

\[ A \ast B = \bigcup_{a \in A, b \in B} a \ast b, \quad A \ast x = A \ast \{x\}, \quad x \ast B = \{x\} \ast B. \quad (1.1) \]

A hyperstructure \((H, \ast)\) is called an \(H\)-semigroup if

\[ (x \ast (y \ast z)) \cap ((x \ast y) \ast z) \neq \emptyset \quad \forall x, y, z \in H. \quad (1.2) \]

Definition 1.1. An \(H\)-ring is a system \((R, +, \cdot)\) with two hyperoperations satisfying the following ring-like axioms:

(i) \((R, +, \cdot)\) is an \(H\)-group, that is,

\[ ((x + y) + z) \cap (x + (y + z)) \neq \emptyset \quad \forall x, y \in R, \]
\[ a + R = R + a = R \quad \forall a \in R; \quad (1.3) \]

(ii) \((R, \cdot)\) is an \(H\)-semigroup;

(iii) \((\cdot)\) is weak distributive with respect to (+), that is, for all \(x, y, z \in R,

\[ (x \cdot (y + z)) \cap (x \cdot y + x \cdot z) \neq \emptyset, \]
\[ ((x + y) \cdot z) \cap (x \cdot z + y \cdot z) \neq \emptyset. \quad (1.4) \]

An \(H\)-ring \((R, +, \cdot)\) is called dual \(H\)-ring if \((R, \cdot, +)\) is an \(H\)-ring. If both operations (+) and (\(\cdot\)) are weak commutative, then \(R\) is called a weak commutative dual \(H\)-ring.

We see that \(H\)-rings are a nice generalization of rings. For more definitions, results, and applications on \(H\)-rings, see [4, 5, 7, 8, 13–15, 17, 18].

Example 1.2 (cf. Vougiouklis [18]). Let \((H, \ast)\) be an \(H\)-group, then for every hyperoperation \((\circ)\) such that \(\{x, y\} \subseteq x \circ y\) for all \(x, y \in H\), the hyperstructure \((H, \ast, \circ)\) is a dual \(H\)-ring.

Example 1.3 (cf. Dramalidis [8]). On the set \(\mathbb{R}^n\), where \(\mathbb{R}\) is the set of real numbers, we define three hyperoperations:

\[ x \uplus y = \{r(x + y) \mid r \in [0, 1]\}, \]
\[ x \otimes y = \{x + r(y - x) \mid r \in [0, 1]\}, \]
\[ x \boxtimes y = \{x + ry \mid r \in [0, 1]\}. \quad (1.5) \]

Then the hyperstructure \((\mathbb{R}^n, \ast, \circ)\), where \(\ast, \circ \in \{\uplus, \otimes, \boxtimes\}\), is a weak commutative dual \(H\)-ring.

Definition 1.4. Let \(R\) be an \(H\)-ring. A nonempty subset \(I\) of \(R\) is called a left (resp., right) \(H\)-ideal if the following axioms hold:

(i) \((I, +)\) is an \(H\)-subgroup of \((R, +),\)

(ii) \(R \cdot I \subseteq I\) (resp., \(I \cdot R \subseteq I\).
2. Fuzzy sets and intuitionistic fuzzy sets

The concept of a fuzzy subset of a nonempty set was first introduced by Zadeh [19].

Let $X$ be a nonempty set, a mapping $\mu : X \to [0,1]$ is called a fuzzy subset of $X$. The complement of $\mu$, denoted by $\mu^c$, is the fuzzy set of $X$ given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$.

Note that using fuzzy subsets, we can introduce on any ring the structure of $H_v$-ring.

Example 2.1 (cf. Davvaz [5]). Let $(R,+,\cdot)$ be an ordinary ring and let $\mu$ be a fuzzy subset of $R$. We define hyperoperations $\cup, \otimes, \ast$ on $R$ as follows:

$$
\mu_x \cup \mu_y = \{ t \mid \mu(t) = \mu(x + y) \},
\mu_x \otimes \mu_y = \{ t \mid \mu(t) = \mu(x \cdot y) \},
\mu_x \ast \mu_y = \{ t \mid \mu(x) \leq \mu(t) \leq \mu(y) \}
$$

(2.1)

Then $(R,\ast,\ast), (R,\ast,\otimes), (R,\ast,\cup), (R,\cup,\ast), (R,\cup,\otimes)$, and $(R,\cup,\otimes)$ are $H_v$-rings.

Rosenfeld [12] applied the concept of fuzzy sets to the theory of groups and defined the concept of fuzzy subgroups of a group. Since then, many papers concerning various fuzzy algebraic structures have appeared in the literature. In [5–7], Davvaz applied the concept of fuzzy set theory in the algebraic hyperstructures, in particular in [5] he defined the concept of fuzzy $H_v$-ideal of an $H_v$-ring which is a generalization of the concept of fuzzy ideal.

Definition 2.2. Let $(R,+,\cdot)$ be an $H_v$-ring and $\mu$ a fuzzy subset of $R$. Then $\mu$ is said to be a left (resp., right) fuzzy $H_v$-ideal of $R$ if the following axioms hold:

(1) $\min \{\mu(x),\mu(y)\} \leq \inf \{\mu(z) \mid z \in x + y\}$ for all $x, y \in R$,

(2) for all $x, a \in R$, there exists $y \in R$ such that $x \in a + y$ and

$$
\min \{\mu(a),\mu(x)\} \leq \mu(y),
$$

(2.2)

(3) for all $x, a \in R$ there exists $z \in R$ such that $x \in z + a$ and

$$
\min \{\mu(a),\mu(x)\} \leq \mu(z),
$$

(2.3)

(4) $\mu(y) \leq \inf \{\mu(z) \mid z \in x \cdot y\}$ (resp., $\mu(x) \leq \inf \{\mu(z) \mid z \in x \cdot y\}$) for all $x, y \in R$.

Example 2.3 (cf. Davvaz [5]). Let $(R,+,\cdot)$ be an ordinary ring and let $\mu$ be a fuzzy ideal of $R$. We consider the $H_v$-ring $(R,\cup,\otimes)$ defined in Example 2.1. Then $\mu$ is a left fuzzy $H_v$-ideal of $(R,\cup,\otimes)$.

The concept of intuitionistic fuzzy set was introduced by Atanassov [1] as a generalization of the notion of fuzzy set. Some fundamental operations on intuitionistic fuzzy sets are defined by Atanassov in [2]. In [9], Kim et al. introduced the notion of an intuitionistic fuzzy subquasigroup of a quasigroup. Also in [10], Kim and Jun introduced the concept of intuitionistic fuzzy ideals of semirings.
4 Intuitionistic fuzzy $H_v$-ideals

**Defintion 2.4.** An intuitionistic fuzzy set $A$ of a nonempty set $X$ is an object having the form

$$A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\}, \quad (2.4)$$

where the functions $\mu_A : X \to [0, 1]$ and $\lambda_A : X \to [0, 1]$ denote the degree of membership (namely, $\mu_A(x)$) and the degree of nonmembership (namely, $\lambda_A(x)$) of each element $x \in X$ to the set $A$, respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$.

**Defintion 2.5.** For every two intuitionistic fuzzy sets $A$ and $B$, define the following operations:

1. $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$,
2. $A^c = \{(x, \lambda_A(x), \mu_A(x)) \mid x \in X\}$,
3. $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\}) \mid x \in X\}$,
4. $A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\}) \mid x \in X\}$,
5. $\bigcap A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\}$,
6. $\bigcup A = \{(x, \lambda_A(x), \mu_A(x)) \mid x \in X\}$.

For the sake of simplicity, we will use the symbol $A = (\mu_A, \lambda_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\}$.

**Defintion 2.6.** Let $(R, +, \cdot)$ be an ordinary ring. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in $R$ is called a left (resp., right) intuitionistic fuzzy ideal of $R$ if

1. $\min\{\mu_A(x), \mu_A(y)\} \leq \mu_A(x - y)$ for all $x, y \in R$,
2. $\mu_A(y) \leq \mu_A(x \cdot y)$ (resp., $\mu_A(x) \leq \mu_A(x \cdot y)$) for all $x, y \in R$,
3. $\lambda_A(x - y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$ for all $x, y \in R$,
4. $\lambda_A(x \cdot y) \leq \lambda_A(x)$ (resp., $\lambda_A(x \cdot y) \leq \lambda_A(x)$) for all $x, y \in R$.

3. Intuitionistic fuzzy $H_v$-ideals

In what follows, let $R$ denote an $H_v$-ring, and we start by defining the notion of intuitionistic fuzzy $H_v$-ideals.

**Defintion 3.1.** An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in $R$ is called a left (resp., right) intuitionistic fuzzy $H_v$-ideal of $R$ if

1. $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) \mid z \in x + y\}$ for all $x, y \in R$,
2. for all $x, a \in R$, there exist $y, z \in R$ such that $x \in (a + y) \cap (z + a)$ and
   $$\min\{\mu_A(a), \mu_A(x)\} \leq \min\{\mu_A(y), \mu_A(z)\},$$
   (3.1)
3. $\mu_A(y) \leq \inf\{\mu_A(z) \mid z \in x \cdot y\}$ (resp., $\mu_A(x) \leq \inf\{\mu_A(z) \mid z \in x \cdot y\}$) for all $x, y \in R$,
4. $\sup\{\lambda_A(z) \mid z \in x + y\} \leq \max\{\lambda_A(x), \lambda_A(y)\}$ for all $x, y \in R$,
5. for all $x, a \in R$, there exist $y, z \in R$ such that $x \in (a + y) \cap (z + a)$ and
   $$\max\{\lambda_A(y), \lambda_A(z)\} \leq \max\{\lambda_A(a), \lambda_A(x)\},$$
   (3.2)
6. $\sup\{\lambda_A(z) \mid z \in x \cdot y\} \leq \lambda_A(y)$ (resp., $\sup\{\lambda_A(z) \mid z \in x \cdot y\} \leq \lambda_A(x)$) for all $x, y \in R$. 

Example 3.2. Let \( \mu \) be a left fuzzy \( H_v \)-ideal of \((R, \bigvee, \otimes)\) defined in Example 2.3. Then, as it is not difficult to see, \( A = (\mu_A, \mu'^*_A) \) is a left intuitionistic fuzzy \( H_v \)-ideal of \((R, \odot, \otimes)\).

Here we present all the proofs for left \( H_v \)-ideals. For right \( H_v \)-ideals, similar results hold as well.

Lemma 3.3. If \( A = (\mu_A, \lambda_A) \) is a left intuitionistic fuzzy \( H_v \)-ideal of \( R \), then so is \( \Box A = (\mu_A, \mu'^*_A) \).

Proof. It is sufficient to show that \( \mu'^*_A \) satisfies the conditions (4), (5), (6) of Definition 3.1.

For \( x, y \in R \), we have
\[
\min \{ \mu_A(x), \mu_A(y) \} \leq \inf \{ \mu_A(z) \mid z \in x + y \},
\] (3.3)
and so
\[
\min \{ 1 - \mu'^*_A(x), 1 - \mu'^*_A(y) \} \leq \inf \{ 1 - \mu'_A(z) \mid z \in x + y \}.
\] (3.4)
Hence
\[
\min \{ 1 - \mu'^*_A(x), 1 - \mu'^*_A(y) \} \leq 1 - \sup \{ \mu'_A(z) \mid z \in x + y \},
\] (3.5)
which implies that
\[
\sup \{ \mu'_A(z) \mid z \in x + y \} \leq 1 - \min \{ 1 - \mu'^*_A(x), 1 - \mu'^*_A(y) \}.
\] (3.6)
Therefore
\[
\sup \{ \mu'_A(z) \mid z \in x + y \} \leq \max \{ \mu'^*_A(x), \mu'^*_A(y) \},
\] (3.7)
in this way, Definition 3.1(4) is verified.

Now, let \( a, x \in R \), then there exist \( y, z \in R \) such that \( x \in (a + y) \cap (z + a) \) and
\[
\min \{ \mu_A(a), \mu_A(x) \} \leq \min \{ \mu_A(y), \mu_A(z) \}.
\] (3.8)
So
\[
\min \{ 1 - \mu'^*_A(a), 1 - \mu'^*_A(x) \} \leq \min \{ 1 - \mu'_A(y), 1 - \mu'_A(z) \}.
\] (3.9)
Hence
\[
\max \{ \mu'_A(y), \mu'_A(z) \} \leq \max \{ \mu'^*_A(a), \mu'^*_A(x) \},
\] (3.10)
and Definition 3.1(5) is satisfied.

For the condition (6), let \( x, y \in R \), then since \( \mu_A \) is a left fuzzy \( H_v \)-ideal of \( R \), we have
\[
\mu_A(y) \leq \inf \{ \mu_A(z) \mid z \in x \cdot y \},
\] (3.11)
and so
\[
1 - \mu'^*_A(y) \leq \inf \{ 1 - \mu'^*_A(z) \mid z \in x \cdot y \},
\] (3.12)
which implies that
\[
\sup \{ \mu'_A(z) \mid z \in x \cdot y \} \leq \mu'_A(y). \tag{3.13}
\]

Therefore Definition 3.1(6) is satisfied.

**Lemma 3.4.** If \( A = (\mu_A, \lambda_A) \) is a left intuitionistic fuzzy \( H_v \)-ideal of \( R \), then so is \( \Diamond A = (\lambda'_A, \lambda_A) \).

The proof is similar to the proof of Lemma 3.3.

Combining the above two lemmas, it is not difficult to see that the following theorem is valid.

**Theorem 3.5.** \( A = (\mu_A, \lambda_A) \) is a left intuitionistic fuzzy \( H_v \)-ideal of \( R \) if and only if \( \Box A \) and \( \Diamond A \) are left intuitionistic fuzzy \( H_v \)-ideals of \( R \).

**Corollary 3.6.** \( A = (\mu_A, \lambda_A) \) is a left intuitionistic fuzzy \( H_v \)-ideal of \( R \) if and only if \( \mu_A \) and \( \lambda'_A \) are left fuzzy \( H_v \)-ideals of \( R \).

**Definition 3.7.** For any \( t \in [0,1] \) and fuzzy set \( \mu \) of \( R \), the set
\[
U(\mu; t) = \{ x \in R \mid \mu(x) \geq t \} \quad \text{(resp., } L(\mu; t) = \{ x \in R \mid \mu(x) \leq t \}) \tag{3.14}
\]

is called an upper (resp., lower) \( t \)-level cut of \( \mu \).

**Theorem 3.8.** If \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy \( H_v \)-ideal of \( R \), then for every \( t \in \text{Im}(\mu_A) \cap \text{Im}(\lambda_A) \), the sets \( U(\mu_A; t) \) and \( L(\mu_A; t) \) are \( H_v \)-ideals of \( R \).

**Proof.** Let \( t \in \text{Im}(\mu_A) \cap \text{Im}(\lambda_A) \subseteq [0,1] \) and let \( x, y \in U(\mu_A; t) \). Then \( \mu_A(x) \geq t \) and \( \mu_A(y) \geq t \), and so \( \min\{\mu_A(x), \mu_A(y)\} \geq t \). It follows from Definition 3.1(1) that \( \inf\{\mu_A(z) \mid z \in x \cdot y\} \geq t \). Therefore for all \( z \in x \cdot y \), we have \( z \in U(\mu_A; t) \), so \( x \cdot y \subseteq U(\mu_A; t) \). Hence for all \( a \in U(\mu_A; t) \), we have \( a + U(\mu_A; t) \subseteq U(\mu_A; t) \) and \( U(\mu_A; t) \). Now, let \( x \in U(\mu_A; t) \), then there exist \( y, z \in R \) such that \( x \in (a + y) \cap (z + a) \) and \( \inf\{\mu_A(x), \mu_A(a)\} \leq \inf\{\mu_A(y), \mu_A(z)\} \). Since \( x, a \in U(\mu_A; t) \), we have \( t \leq \inf\{\mu_A(x), \mu_A(a)\} \), and so \( t \leq \inf\{\mu_A(y), \mu_A(z)\} \), which implies that \( y \in U(\mu_A; t) \) and \( z \in U(\mu_A; t) \). This proves that \( U(\mu_A; t) \leq a + U(\mu_A; t) \) and \( U(\mu_A; t) \leq U(\mu_A; t) + a \).

Now, for every \( x \in R \) and \( y \in U(\mu_A; t) \), we show that \( x \cdot y \subseteq U(\mu_A; t) \). Since \( A \) is a left intuitionistic fuzzy \( H_v \)-ideal of \( R \), we have
\[
t \leq \mu_A(y) \leq \inf\{\mu_A(z) \mid z \in x \cdot y\}. \tag{3.15}
\]

Therefore, for every \( z \in x \cdot y \), we get \( \mu_A(z) \geq t \), which implies that \( z \in U(\mu_A; t) \), so \( x \cdot y \subseteq U(\mu_A; t) \).

If \( x, y \in L(\lambda_A; t) \), then \( \max\{\lambda_A(x), \lambda_A(y)\} \leq t \). It follows from Definition 3.1(4) that \( \sup\{\lambda_A(z) \mid z \in x \cdot y\} \leq t \). Therefore for all \( z \in x \cdot y \), we have \( z \in L(\lambda_A; t) \), so \( x \cdot y \subseteq L(\lambda_A; t) \). Hence for all \( a \in L(\lambda_A; t) \), we have \( a + L(\lambda_A; t) \subseteq L(\lambda_A; t) \) and \( L(\lambda_A; t) + a \subseteq L(\lambda_A; t) \). Now, let \( x \in L(\lambda_A; t) \), then there exist \( y, z \in R \) such that \( x \in (a + y) \cap (z + a) \) and \( \max\{\lambda_A(y), \lambda_A(z)\} \leq \max\{\lambda_A(x), \lambda_A(x)\} \). Since \( x, a \in L(\lambda_A; t) \), we have \( \max\{\lambda_A(a), \lambda_A(x)\} \leq t \), and so \( \max\{\lambda_A(y), \lambda_A(z)\} \leq t \). Thus \( y \in L(\lambda_A; t) \) and \( z \in L(\lambda_A; t) \). Hence \( L(\lambda_A; t) \leq a + L(\lambda_A; t) \) and \( L(\lambda_A; t) \leq L(\lambda_A; t) + a \).
Proof. Therefore, Definition 3.1 (2) is verified. If we put $x = L(\lambda_A; t)$, then $t = \max\{\lambda_A(x), \lambda_A(y)\}$ for $x, y \in R$, then $x, y \in U(\mu_A; t_0)$ and $x, y \in L(\lambda_A; t_1)$. Therefore for all $z \in x + y$, we have $\mu_A(z) \geq t_0$ and $\lambda_A(z) \leq t_1$, that is,

$$\inf \{\mu_A(z) \mid z \in x + y\} \geq \min \{\mu_A(x), \mu_A(y)\},$$

$$\sup \{\lambda_A(z) \mid z \in x + y\} \leq \max \{\lambda_A(x), \lambda_A(y)\},$$

which verifies the conditions (1) and (4) of Definition 3.1.

Now, if $t_2 = \min\{\mu_A(a), \mu_A(x)\}$ for $x, a \in R$, then $a, x \in U(\mu_A; t_2)$. So there exist $y_1, z_1 \in U(\mu_A; t_2)$ such that $x = a + y_1$ and $x \in z_1 + a$. Also we have $t_2 \leq \min\{\mu_A(y_1), \mu_A(z_1)\}$. Therefore, Definition 3.1(2) is verified. If we put $t_3 = \max\{\lambda_A(a), \lambda_A(x)\}$, then $a, x \in L(\lambda_A; t_3)$. So there exist $y_2, z_2 \in L(\lambda_A; t_3)$ such that $x = a + y_2$ and $x \in z_2 + a$, and we have $\max\{\lambda_A(y_2), \lambda_A(z_2)\} \leq t_3$, and so Definition 3.1(5) is verified.

Now, we verify the conditions (3) and (6). Let $t_4 = \mu_A(y)$ and $t_5 = \lambda_A(y)$ for some $x, y \in R$. Then $y \in U(\mu_A; t_4)$, $y \in L(\lambda_A; t_5)$. Since $U(\mu_A; t_4)$ and $L(\lambda_A; t_5)$ are $H_v$-ideals of $R$, then $x \cdot y \subseteq U(\mu_A; t_4)$ and $x \cdot y \subseteq L(\lambda_A; t_5)$. Therefore for every $z \in x \cdot y$, we have $z \in U(\mu_A; t_4)$ and $z \in L(\lambda_A; t_5)$ which imply that $\mu_A(z) \geq t_4$ and $\lambda_A(z) \leq t_5$. Hence

$$\inf \{\mu_A(z) \mid z \in x \cdot y\} \geq t_4 = \mu_A(y),$$

$$\sup \{\lambda_A(z) \mid z \in x \cdot y\} \leq t_5 = \lambda_A(y).$$

This completes the proof. \qed

Corollary 3.10. Let $I$ be a left $H_v$-ideal of an $H_v$-ring $R$. If fuzzy sets $\mu$ and $\lambda$ are defined on $R$ by

$$\mu(x) = \begin{cases} \alpha_0 & \text{if } x \in I, \\ \alpha_1 & \text{if } x \in R \setminus I, \end{cases}$$

$$\lambda(x) = \begin{cases} \beta_0 & \text{if } x \in I, \\ \beta_1 & \text{if } x \in R \setminus I, \end{cases}$$

where $0 \leq \alpha_i < \alpha_0$, $0 \leq \beta_0 < \beta_1$, and $\alpha_i + \beta_i \leq 1$ for $i = 0, 1$, then $A = (\mu, \lambda)$ is an intuitionistic fuzzy $H_v$-ideal of $R$ and $U(\mu; \alpha_0) = I = L(\lambda; \beta_0)$.

Corollary 3.11. Let $\chi_i$ be the characteristic function of a left $H_v$-ideal $I$ of $R$. Then $A = (\chi_i, \chi_i^c)$ is a left intuitionistic fuzzy $H_v$-ideal of $R$. 
Theorem 3.12. If \( A = (\mu_A, \lambda_A) \) is a left intuitionistic fuzzy \( H_v \)-ideal of \( R \), then for all \( x \in R \),

\[
\begin{align*}
\mu_A(x) &= \sup \{ \alpha \in [0,1] \mid x \in U(\mu_A; \alpha) \}, \\
\lambda_A(x) &= \inf \{ \alpha \in [0,1] \mid x \in L(\lambda_A; \alpha) \}.
\end{align*}
\] (3.20)

Proof. Let \( \delta = \sup \{ \alpha \in [0,1] \mid x \in U(\mu_A; \alpha) \} \) and let \( \varepsilon > 0 \) be given. Then \( \delta - \varepsilon < \alpha \) for some \( \alpha \in [0,1] \) such that \( x \in U(\mu_A; \alpha) \). This means that \( \delta - \varepsilon < \mu_A(x) \) so that \( \delta \leq \mu_A(x) \) since \( \varepsilon \) is arbitrary.

We now show that \( \mu_A(x) \leq \delta \). If \( \mu_A(x) = \beta \), then \( x \in U(\mu_A; \beta) \), and so

\[
\beta \in \{ \alpha \in [0,1] \mid x \in U(\mu_A; \alpha) \}.
\] (3.21)

Hence

\[
\mu_A(x) = \beta \leq \sup \{ \alpha \in [0,1] \mid x \in U(\mu_A; \alpha) \} = \delta.
\] (3.22)

Therefore

\[
\mu_A(x) = \delta = \sup \{ \alpha \in [0,1] \mid x \in U(\mu_A; \alpha) \}.
\] (3.23)

Now let \( \eta = \inf \{ \alpha \in [0,1] \mid x \in L(\lambda_A; \alpha) \} \). Then

\[
\inf \{ \alpha \in [0,1] \mid x \in L(\lambda_A; \alpha) \} < \eta + \varepsilon
\] (3.24)

for any \( \varepsilon > 0 \), and so \( \alpha < \eta + \varepsilon \) for some \( \alpha \in [0,1] \) with \( x \in L(\lambda_A; \alpha) \). Since \( \lambda_A(x) \leq \alpha \) and \( \varepsilon \) is arbitrary, it follows that \( \lambda_A(x) \leq \eta \).

To prove that \( \lambda_A(x) \geq \eta \), let \( \lambda_A(x) = \zeta \). Then \( x \in L(\lambda_A; \zeta) \), and thus \( \zeta \in \{ \alpha \in [0,1] \mid x \in L(\lambda_A; \alpha) \} \). Hence

\[
\inf \{ \alpha \in [0,1] \mid x \in L(\lambda_A; \alpha) \} \leq \zeta,
\] (3.25)

that is, \( \eta \leq \zeta = \lambda_A(x) \). Consequently

\[
\lambda_A(x) = \eta = \inf \{ \alpha \in [0,1] \mid x \in L(\lambda_A; \alpha) \},
\] (3.26)

which completes the proof. \( \square \)

4. Relations

Let \( \alpha \in [0,1] \) be fixed and let \( \text{IF}(R) \) be the family of all intuitionistic fuzzy left \( H_v \)-ideals of an \( H_v \)-ring \( R \). For any \( A = (\mu_A, \lambda_A) \) and \( B = (\mu_B, \lambda_B) \) from \( \text{IF}(R) \), we define two binary relations \( \Upsilon^\alpha \) and \( \Sigma^\alpha \) on \( \text{IF}(R) \) as follows:

\[
(A,B) \in \Upsilon^\alpha \iff U(\mu_A; \alpha) = U(\mu_B; \alpha),
\]

\[
(A,B) \in \Sigma^\alpha \iff L(\lambda_A; \alpha) = L(\lambda_B; \alpha).
\] (4.1)

These two relations \( \Upsilon^\alpha \) and \( \Sigma^\alpha \) are equivalence relations. Hence \( \text{IF}(R) \) can be divided into
the equivalence classes of $\mathfrak{L}^a$ and $\mathfrak{L}^a$, denoted by $[A]_{\mathfrak{L}^a}$ and $[A]_{\mathfrak{L}^a}$ for any $A = (\mu_A, \lambda_A) \in \text{IF}(R)$, respectively. The corresponding quotient sets will be denoted as $\text{IF}(R)/\mathfrak{L}^a$ and $\text{IF}(R)/\mathfrak{L}^a$, respectively.

For the family $\mathfrak{L}(R)$ of all left $H^e$-ideals of $R$, we define two maps $U_\alpha$ and $L_\alpha$ from $\text{IF}(R)$ to $\mathfrak{L}(R) \cup \{\emptyset\}$ putting

$$U_\alpha(A) = U(\mu_A; \alpha), \quad L_\alpha(A) = L(\lambda_A; \alpha)$$

for each $A = (\mu_A, \lambda_A) \in \text{IF}(R)$.

It is not difficult to see that these maps are well defined.

**Lemma 4.1.** For any $\alpha \in (0, 1)$, the maps $U_\alpha$ and $L_\alpha$ are surjective.

**Proof.** Let $0$ and $1$ be fuzzy sets on $R$ defined by $0(x) = 0$ and $1(x) = 1$ for all $x \in R$. Then $0_\alpha = (0, 1) \in \text{IF}(R)$ and $U_{\alpha}(0_\alpha) = L_{\alpha}(0_\alpha) = \emptyset$ for any $\alpha \in (0, 1)$. Moreover, for any $K \in \mathfrak{L}(R)$, we have $L_\alpha = (\chi_K, \chi_K^e) \in \text{IF}(R)$, $U_\alpha(L_\alpha) = U(\chi_K; \alpha) = K$, and $L_\alpha(L_\alpha) = L(\chi_K^e; \alpha) = K$. Hence $U_\alpha$ and $L_\alpha$ are surjective.

**Theorem 4.2.** For any $\alpha \in (0, 1)$, the sets $\text{IF}(R)/\mathfrak{L}^a$ and $\text{IF}(R)/\mathfrak{L}^a$ are equipotent to $\mathfrak{L}(R) \cup \{\emptyset\}$.

**Proof.** Let $\alpha \in (0, 1)$. Putting $U_{\alpha}^*([A]_{\mathfrak{L}^a}) = U_\alpha(A)$ and $L_{\alpha}^*([A]_{\mathfrak{L}^a}) = L_\alpha(A)$ for any $A = (\mu_A, \lambda_A) \in \text{IF}(R)$, we obtain two maps:

$$U_{\alpha}^* : \text{IF}(R)/\mathfrak{L}^a \to \mathfrak{L}(R) \cup \{\emptyset\}, \quad L_{\alpha}^* : \text{IF}(R)/\mathfrak{L}^a \to \mathfrak{L}(R) \cup \{\emptyset\}.$$  

If $U(\mu_A; \alpha) = U(\mu_B; \alpha)$ and $L(\lambda_A; \alpha) = L(\lambda_B; \alpha)$ for some $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ from $\text{IF}(R)$, then $(A, B) \in \mathfrak{L}^a$ and $(A, B) \in \mathfrak{L}^a$, whence $[A]_{\mathfrak{L}^a} = [B]_{\mathfrak{L}^a}$ and $[A]_{\mathfrak{L}^a} = [B]_{\mathfrak{L}^a}$, which means that $U_{\alpha}^*$ and $L_{\alpha}^*$ are injective.

To show that the maps $U_{\alpha}^*$ and $L_{\alpha}^*$ are surjective, let $K \in \mathfrak{L}(R)$. Then for $L_\alpha = (\chi_K, \chi_K^e) \in \text{IF}(R)$, we have $U_{\alpha}^*([L_\alpha]_{\mathfrak{L}^a}) = U(\chi_K; \alpha) = K$ and $L_{\alpha}^*([L_\alpha]_{\mathfrak{L}^a}) = L(\chi_K^e; \alpha) = K$. Also $0_\alpha = (0, 1) \in \text{IF}(R)$. Moreover, $U_{\alpha}^*([0_\alpha]_{\mathfrak{L}^a}) = U(0; \alpha) = \emptyset$ and $L_{\alpha}^*([0_\alpha]_{\mathfrak{L}^a}) = L(1; \alpha) = \emptyset$. Hence $U_{\alpha}^*$ and $L_{\alpha}^*$ are surjective.

Now for any $\alpha \in [0, 1]$, we have the new relation $\mathfrak{R}^a$ on $\text{IF}(R)$ putting

$$(A, B) \in \mathfrak{R}^a \iff U(\mu_A; \alpha) \cap L(\lambda_A; \alpha) = U(\mu_B; \alpha) \cap L(\lambda_B; \alpha),$$

where $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$. Obviously, $\mathfrak{R}^a$ is an equivalence relation.

**Lemma 4.3.** The map $I_\alpha : \text{IF}(R) \to \mathfrak{L}(R) \cup \{\emptyset\}$ defined by

$$I_\alpha(A) = U(\mu_A; \alpha) \cap L(\lambda_A; \alpha),$$

where $A = (\mu_A, \lambda_A)$, is surjective for any $\alpha \in (0, 1)$.

**Proof.** Indeed, if $\alpha \in (0, 1)$ is fixed, then for $0_\alpha = (0, 1) \in \text{IF}(R)$, we have

$$I_\alpha(0_\alpha) = U(0; \alpha) \cap L(1; \alpha) = \emptyset,$$
and for any \( K \in LI(R) \), there exists \( I_\sim = (\chi_K, \chi_K^c) \in IF(R) \) such that \( I_a(I_\sim) = U(\chi_K; \alpha) \cap L(\chi_K^c; \alpha) = K \).

**Theorem 4.4.** For any \( \alpha \in (0,1) \), the quotient set \( IF(R)/\gamma^* \) is equipotent to \( LI(R) \cup \{ \emptyset \} \).

**Proof.** Let \( I_a^*: IF(R)/\gamma^* \to LI(R) \cup \{ \emptyset \} \), where \( \alpha \in (0,1) \), be defined by the formula

\[
I_a^*([A]\_\gamma^*) = I_a(A) \quad \text{for each} \quad [A]\_\gamma^* \in IF(R)/\gamma^*.
\]

(4.7)

If \( I_a^*([A]\_\gamma^*) = I_a^*([B]\_\gamma^*) \) for some \( [A]\_\gamma^*, [B]\_\gamma^* \in IF(R)/\gamma^* \), then

\[
U(\mu_A; \alpha) \cap L(\lambda_B; \alpha) = U(\mu_B; \alpha) \cap L(\lambda_B; \alpha),
\]

which implies that \((A,B) \in \gamma^* \) and, in the consequence, \([A]\_\gamma^* = [B]\_\gamma^* \). Thus \( I_a^* \) is injective.

It is also onto because \( I_a^*(0_\sim) = I_a(0_\sim) = \emptyset \) for \( 0_\sim = (0,1) \in IF(R) \), and \( I_a^*(I_\sim) = I_a(K) = K \) for \( K \in LI(R) \) and \( I_\sim = (\chi_K, \chi_K^c) \in IF(R) \).

The relation \( \gamma^* \) is the smallest equivalence relation on \( R \) such that the quotient \( R/\gamma^* \) is a ring. \( \gamma^* \) is called the fundamental equivalence relation on \( R \) and \( R/\gamma^* \) is called the fundamental ring, see [16].

According to [16], if \( \mathcal{U} \) denotes the set of all finite polynomials of elements of \( R \) over \( \mathbb{N} \), then a relation \( \gamma \) can be defined on \( R \) as follows:

\[
xyy \quad \text{iff} \quad \{x,y\} \subseteq u \quad \text{for some} \quad u \in \mathcal{U}.
\]

(4.9)

According to [16], the transitive closure of \( \gamma \) is the fundamental relation \( \gamma^* \), that is, \( ay^*b \) if and only if there exist \( x_1, \ldots, x_{m+1} \in R; u_1, \ldots, u_m \in \mathcal{U} \) with \( x_1 = a, x_{m+1} = b \) such that

\[
\{x_i, x_{i+1}\} \subseteq u_i, \quad i = 1, \ldots, m.
\]

(4.10)

Suppose that \( \gamma^*(a) \) is the equivalence class containing \( a \in R \). Then both the sum \( \oplus \) and the product \( \odot \) on \( R/\gamma^* \) are defined as follows:

\[
\gamma^*(a) \oplus \gamma^*(b) = \gamma^*(c) \quad \forall c \in \gamma^*(a) + \gamma^*(b),
\]

\[
\gamma^*(a) \odot \gamma^*(b) = \gamma^*(d) \quad \forall c \in \gamma^*(a) \cdot \gamma^*(b).
\]

(4.11)

We denote the unit of the group \( (R/\gamma^*, \oplus) \) by \( \omega_R \).

**Definition 4.5.** Let \( R \) be an \( H_\nu \)-ring and \( \mu \) a fuzzy subset of \( R \). The fuzzy subset \( \mu_{\gamma^*} \) on \( R/\gamma^* \) is defined as follows:

\[
\mu_{\gamma^*} : R/\gamma^* \to [0,1],
\]

\[
\mu_{\gamma^*}(\gamma^*(x)) = \sup \{\mu(a) \mid a \in \gamma^*(x)\}.
\]

(4.12)

**Theorem 4.6.** Let \( R \) be an \( H_\nu \)-ring and \( A = (\mu_A, \lambda_A) \) a left intuitionistic fuzzy \( H_\nu \)-ideal of \( R \). Then \( A/\gamma^* = (\mu_{\gamma^*}, \lambda_{\gamma^*}) \) is a left intuitionistic fuzzy ideal of the fundamental ring \( R/\gamma^* \).
References


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