We establish the weak convergence of a sequence of Mann iterates of an I-nonexpansive map in a Banach space which satisfies Opial’s condition.

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1. Introduction and preliminaries

Let $K$ be a closed convex bounded subset of uniformly convex Banach space $X = (X, \| \cdot \|)$ and $T$ self-mappings of $X$. Then $T$ is called nonexpansive on $K$ if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$. Let $F(T) = \{x \in K : Tx = x\}$ be denoted as the set of fixed points of a mapping $T$.

The first nonlinear ergodic theorem was proved by Baillon [1] for general nonexpansive mappings in Hilbert space $\mathcal{H}$: if $K$ is a closed and convex subset of $\mathcal{H}$ and $T$ has a fixed point, then for every $x \in K$, $\{T^n x\}$ is weakly almost convergent, as $n \to \infty$, to a fixed point of $T$. It was also shown by Pazy [7] that if $\mathcal{H}$ is a real Hilbert space and $(1/n) \sum_{i=0}^{n-1} T^i x$ converges weakly, as $n \to \infty$, to $y \in K$, then $y \in F(T)$.

The concept of a quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real functions. Diaz and Metcalf [2] and Dotson [3] studied quasi-nonexpansive mappings in Banach spaces. Recently, this concept was given by Kirk [5] in metric spaces which we adapt to a normed space as follows: $T$ is called a quasi-nonexpansive mapping provided

$$\|Tx - f\| \leq \|x - f\|$$

for all $x \in K$ and $f \in F(T)$.

Remark 1.1. From the above definitions it is easy to see that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and linear quasi-nonexpansive mappings are nonexpansive. But it is easily seen that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive.
2 Convergence theorems for $I$-nonexpansive mapping

There are many results on fixed points on nonexpansive and quasi-nonexpansive mappings in Banach spaces and metric spaces. For example, the strong and weak convergence of the sequence of certain iterates to a fixed point of quasi-nonexpansive maps was studied by Petryshyn and Williamson [8]. Their analysis was related to the convergence of Mann iterates studied by Dotson [3]. Subsequently, the convergence of Ishikawa iterates of quasi-nonexpansive mappings in Banach spaces was discussed by Ghosh and Debnath [4]. In [10], the weakly convergence theorem for $I$-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved. In [11], convergence theorems of iterative schemes for nonexpansive mappings have been presented and generalized.

In this paper, we consider $T$ and $I$ self-mappings of $K$, where $T$ is an $I$-nonexpansive mapping. We establish the weak convergence of the sequence of Mann iterates to a common fixed point of $T$ and $I$.

Let $X$ be a normed linear space, let $K$ be a nonempty convex subset of $X$, and let $T : K \to K$ be a given mapping. The Mann iterative scheme $\{x_n\}$ is defined by $x_0 = x \in K$ and

$$x_{n+1} = (1 - k_n)x_n + k_nTx_n$$

(1.3)

for every $n \in \mathbb{N}$, where $k_n$ is a sequence in $(0, 1)$.

Recall that a Banach space $X$ is said to satisfy Opial’s condition [6] if, for each sequence $\{x_n\}$ in $X$, the condition $x_n \rightharpoonup x$ implies that

$$\lim_{n \to \infty} \|x_n - x\| < \lim_{n \to \infty} \|x_n - y\|$$

(1.4)

for all $y \in X$ with $y \neq x$. It is well known from [6] that all $l_p$ spaces for $1 < p < \infty$ have this property. However, the $L_p$ spaces do not, unless $p = 2$.

The following definitions and statements will be needed for the proof of our theorem.

Let $K$ be a subset of a normed space $X = (X, \| \cdot \|)$ and $T$ and $I$ self-mappings of $K$. Then $T$ is called $I$-nonexpansive on $K$ if

$$\|Tx - Ty\| \leq \|Ix - Iy\|$$

(1.5)

for all $x, y \in K$ [9].

$T$ is called $I$-quasi-nonexpansive on $K$ if

$$\|Tx - f\| \leq \|Ix - f\|$$

(1.6)

for all $x \in K$ and $f \in F(T) \cap F(I)$.

2. The main result

Theorem 2.1. Let $K$ be a closed convex bounded subset of uniformly convex Banach space $X$, which satisfies Opial’s condition, and let $T, I$ self-mappings of $K$ with $T$ an $I$-nonexpansive mapping, $I$ a nonexpansive on $K$. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of Mann iterates converges weakly to common fixed point of $F(T) \cap F(I)$. 
Proof. If \( F(T) \cap F(I) \) is nonempty and a singleton, then the proof is complete. We will assume that \( F(T) \cap F(I) \) is nonempty and that \( F(T) \cap F(I) \) is not a singleton.

\[
\|x_{n+1} - f\| = \|(1-k_n)x_n + k_nTx_n - (1-k_n+k_n)f\| \\
= \|(1-k_n)(x_n - f) + k_n(Tx_n - f)\| \\
\leq (1-k_n)\|x_n - f\| + k_n\|Tx_n - f\| \\
\leq (1-k_n)\|x_n - f\| + k_n\|x_n - f\| \\
= \|x_n - f\|,
\]

where \( \{k_n\} \) is a sequence in \((0,1)\).

Thus, for \( k_n \neq 0 \), \( \{\|x_n - f\|\} \) is a nonincreasing sequence. Then, \( \lim_{n \to \infty} \|x_n - f\| \) exists.

Now we show that \( \{x_n\} \) converges weakly to a common fixed point of \( T \) and \( I \). The sequence \( \{x_n\} \) contains a subsequence which converges weakly to a point in \( K \). Let \( \{x_{n_k}\} \) and \( \{x_{m_j}\} \) be two subsequences of \( \{x_n\} \) which converge weakly to \( f \) and \( q \), respectively. We will show that \( f = q \). Suppose that \( X \) satisfies Opial’s condition and that \( f \neq q \) is in weak limit set of the sequence \( \{x_n\} \). Then \( \{x_{n_k}\} \to f \) and \( \{x_{m_j}\} \to q \), respectively. Since \( \lim_{n \to \infty} \|x_n - f\| \) exists for any \( f \in F(T) \cap F(I) \), by Opial’s condition, we conclude that

\[
\lim_{n \to \infty} \|x_n - f\| = \lim_{k \to \infty} \|x_{n_k} - f\| < \lim_{k \to \infty} \|x_{n_k} - q\| \\
= \lim_{n \to \infty} \|x_n - q\| = \lim_{j \to \infty} \|x_{m_j} - q\| \\
< \lim_{j \to \infty} \|x_{m_j} - f\| = \lim_{n \to \infty} \|x_n - f\|. \tag{2.2}
\]

This is a contradiction. Thus \( \{x_n\} \) converges weakly to an element of \( F(T) \cap F(I) \). \( \square \)

References


4 Convergence theorems for $I$-nonexpansive mapping


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