ON AN EXTENSION OF SINGULAR INTEGRALS ALONG MANIFOLDS OF FINITE TYPE

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We extend the $L^p$-boundedness of a class of singular integral operators under the $H^1$ kernel condition on a compact manifold from the homogeneous Sobolev space $L^p_0(\mathbb{R}^n)$ to the Lebesgue space $L^p(\mathbb{R}^n)$. Copyright © 2006 Hindawi Publishing Corporation. All rights reserved.

1. Introduction

Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$, $n \geq 2$, with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega(x')$ be a homogeneous function of degree 0, with $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0,$$

(1.1)

where $x' = x/|x|$ for any $x \neq 0$.

Suppose that $h$ is an $L^\infty(\mathbb{R}^+)$ function; the singular integral operator $SI_{\Omega,h}$ is defined by

$$SI_{\Omega,h}(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} h(|y|) \frac{\Omega(y')}{|y|^n} f(x - y) dy$$

(1.2)

for all test functions $f$, where $y' = y/|y| \in S^{n-1}$.

We denote $SI_{\Omega,h}(f)$ by $SI_{\Omega}(f)$ if $h = 1$. The operator $SI_{\Omega}$ was first studied by Calderón and Zygmund in their well-known papers (see [1, 2]). They proved that $SI_{\Omega}$ is $L^p(\mathbb{R}^n)$ bounded, $1 < p < \infty$, provided that $\Omega \in L\log^+ L(S^{n-1})$ satisfying (1.1). They also showed that the space $L\log^+ L(S^{n-1})$ cannot be replaced by any Orlicz space $L^\phi(S^{n-1})$ with a monotonically increasing function $\phi$ satisfying $\phi(t) = o(t \log t)$, $t \to \infty$, that is, $L(\log^+ L)^{1-\epsilon}(S^{n-1}), 0 < \epsilon \leq 1$. The idea of their proof was as follows.

Suppose that $\Omega \in L^1(S^{n-1})$ is an odd function, then one can easily show that

$$SI_{\Omega}(f)(x) = \frac{1}{2} \int_{S^{n-1}} \Omega(y') \left\{ \int_{-\infty}^{\infty} f(x - ty') t^{-1} dt \right\} d\sigma(y').$$

(1.3)
Singular integrals along manifolds

By the method of rotation and the well-known $L^p$-boundedness of the Hilbert transform, one then obtains the $L^p$-boundedness of $SI_Ω$ under the weak condition $Ω \in L^1(S^{n-1})$.

For even kernels, the condition $Ω \in L^1(S^{n-1})$ is insufficient. It turns out that the right condition is $Ω \in L^{\log^+ L}(S^{n-1})$ (as far as the size of $Ω$ is concerned). The idea of Calderón and Zygmund is to compose the operator $SI_Ω$ with the Riesz transforms $R_j$, $1 \leq j \leq n$, and to show that $R_j(SI_Ω)$ is a singular integral operator with an appropriate odd kernel. Thus

$$\|R_j(SI_Ω)(f)\|_p \leq C_p \|f\|_p$$

(1.4)

for all test functions $f \in \mathcal{F}$. Furthermore, one can obtain

$$\|SI_Ω(f)\|_p = \left\| \left( \sum_{j=1}^n R_j^2 \right) SI_Ω(f) \right\|_p \leq \sum_{j=1}^n \|R_j(R_j SI_Ω(f))\|_p$$

$$\leq n\mathcal{C} \sum_{j=1}^n \|R_j SI_Ω(f)\|_p \leq n^2 C_p \|f\|_p$$

(1.5)

for all test functions $f \in \mathcal{F}$, since $-\sum_{j=1}^n R_j^2$ is the identity map. Using the above method, Connett [7] and Ricci and Weiss [15] independently obtained the same $L^p$-boundedness of $SI_Ω$ under the weak condition $Ω \in H^1(S^{n-1})$, where $H^1(S^{n-1})$ is the Hardy space which contains $L^{\log^+ L}(S^{n-1})$ as a proper subspace.

In [12], Fefferman generalized this Calderón-Zygmund singular integral by replacing the kernel $Ω(x')|x|^{-n}$ by $h(|x|)Ω(x')|x|^{-n}$, where $h$ is an arbitrary $L^\infty$ function. This allows the kernel to be rough not only on the sphere but also in the radial direction. For the singular integral operator $SI_{Ω,h}$ with the kernel $K(x) = h(|x|)(Ω(x')/(|x|^n))$, the formula (1.3) now is

$$SI_{Ω,h}(f)(x) = \int_{S^{n-1}} Ω(y') \left\{ \int_0^∞ f(x - ty')h(t)dt \right\} dσ(y').$$

(1.6)

Clearly, the method of Calderón and Zygmund can no longer be used to estimate the above integral in (1.6) even if $Ω$ is odd, since the integral in parentheses cannot be reduced to the Hilbert transform for an arbitrary $h(t)$. Thus, one needs to find a new approach.

Using a method which is different from Calderón and Zygmund, Fefferman showed in [12] that if $Ω$ satisfies a Lipschitz condition, then $SI_{Ω,h}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Later in [8], using Littlewood-Paley theory and Fourier transform methods, Duoandikoetxea and Rubio de Francia improved Fefferman’s results by assuming a roughness condition $Ω \in L^q(S^{n-1})$ (see also [3, 13, 14]). By modifying the method in [8], recently, Fan and Pan [11] have improved the above results on $SI_{Ω,h}$ by assuming a roughness condition $Ω \in H^1(S^{n-1})$. 
Noting that $S^{n-1}$ is an $(n-1)$-dimensional compact manifold in $\mathbb{R}^{n-1}$, Duoandikoetxea and Rubio de Francia [8] introduced the following extension of the operator $SI_{\Omega,h}$.

Let $m, n \in \mathbb{N}$, $m \leq n - 1$, and let $\mathcal{M}$ be a compact, smooth, $m$-dimensional manifold in $\mathbb{R}^n$. Suppose that $\mathcal{M} \cap \{rv : r > 0\}$ contains at most one point for any $v \in S^{n-1}$. Let $\mathcal{M}$ denote the cone \{r$\theta$ : $r > 0$, $\theta \in \mathcal{M}$\} equipped with the measure $ds(r\theta) = r^m dr d\sigma(\theta)$, where $d\sigma$ represents the induced Lebesgue measure on $\mathcal{M}$. For a locally integrable function in $\mathcal{M}$ of the form
\[
K(r\theta) = r^{-m-1} h(r)\Omega(\theta),
\] (1.7)

where $\Omega$ satisfies
\[
\int_{\mathcal{M}} \Omega(\theta) d\sigma(\theta) = 0,
\] (1.8)

they defined the corresponding singular integral operator $SI_{\mathcal{M},\Omega,h}$ on $\mathbb{R}^n$ by
\[
( SI_{\mathcal{M},\Omega,h} f ) (x) = \text{p.v.} \int_{\mathcal{M}} f(x - y) K(y) ds(y)
\] (1.9)

initially for $f \in \mathcal{S}(\mathbb{R}^n)$.

In [8], Duoandikoetxea and Rubio de Francia obtained the following results regarding $SI_{\mathcal{M},\Omega,h}$.

**Theorem 1.1.** Let $SI_{\mathcal{M},\Omega,h}$ be given as in (1.7)–(1.9). Suppose that

(i) $\Omega \in L^q(\mathcal{M})$,
(ii) $\sup_{R>0} \left[ (1/R) \int_0^R |h(r)|^2 dr \right] < \infty$,
(iii) $\mathcal{M}$ has a contact of finite order with every hyperplane.

Then $SI_{\mathcal{M},\Omega,h}$ extends to a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Inspired by the earlier result of Fan and Pan regarding $\Omega \in H^1(S^{n-1})$, Cheng and Pan [5] established the following.

**Theorem 1.2.** Let $SI_{\mathcal{M},\Omega,h}$ be given as in Theorem 1.1, and let $h$ and $\mathcal{M}$ satisfy (ii) and (iii), respectively. If $\Omega \in H^1(\mathcal{M})$, then $SI_{\mathcal{M},\Omega,h}$ extends to a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

The main purpose of this paper is to extend Theorem 1.2 to the case $\Omega \in H^r(\mathcal{M})$ with $0 < r < 1$. The space $H^r(\mathcal{M})$ is a distribution space when $0 < r < 1$. The definition of $H^r(\mathcal{M})$ can be found in Section 2, but here we must define the operator in the sense of distribution.
4 Singular integrals along manifolds

Let $(\Omega, \phi)$ be the pairing between $\Omega \in H^r(\mathcal{M})$ and a $C^\infty$ function $\phi$ on $\mathcal{M}$. For $0 \leq \alpha$, we define the singular integral operator $SI_{\mathcal{M}, \Omega, h, \alpha} f(x)$ by

$$SI_{\mathcal{M}, \Omega, h, \alpha} f(x) = \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \langle f(x - r \cdot), \Omega(\cdot) \rangle h(r) r^{-\alpha} dr,$$  \hfill (1.10)

where $f \in \mathcal{S}(\mathbb{R}^n)$, $h$, $\Omega$ satisfy (ii) and (iii) in Theorem 1.1, respectively, and $\Omega \in H^r(\mathcal{M})$ satisfies

$$\langle \Omega, P_m|_{\mathcal{M}} \rangle = 0$$  \hfill (1.11)

for all polynomials on $\mathbb{R}^n$ with degree $m \leq [\alpha]$ and $r = m/m + \alpha$.

When $\mathcal{M} = S^{n-1}$, the operator $SI_{S^{n-1}, \Omega, h, \alpha}$ was studied in [4]. It is not difficult to check that (1.10) is well defined and it is finite for all $x \in \mathbb{R}^n$.

When $\alpha = 0$, the operator $SI_{S^{n-1}, \Omega, h, 0}$ is exactly the operator $SI_{\mathcal{M}, \Omega, h}$.

The main result of this paper is as follows.

**Theorem 1.3.** Let $SI_{\mathcal{M}, \Omega, h, \alpha}$ be given as in (1.10), and let $h$, $\mathcal{M}$ satisfy (ii) and (iii) as in Theorem 1.1, respectively. If $\Omega \in H^r(\mathcal{M})$ satisfies (1.11), then $SI_{\mathcal{M}, \Omega, h, \alpha}$ extends to a bounded operator from the homogeneous Sobolev space $\dot{L}_p^\alpha(\mathbb{R}^n)$ to the Lebesgue space $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

2. Definitions and lemmas

Let $\mathcal{M}$ be a compact, smooth, $m$-dimensional manifold in $\mathbb{R}^n$, $m \leq n - 1$. The Hardy spaces $H^p(\mathcal{M})$ can be defined by using the maximal operator

$$\mathcal{A} : f \longrightarrow (\mathcal{A}f)(x) = \sup_{t > 0} \left| u(t, x) \right|,$$  \hfill (2.1)

where $u(t, x)$ is the solution of the boundary value problem

$$\left( \frac{\partial}{\partial t} - \Delta_x \right) u = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathcal{M},$$

$$u(0, x) = f(x), \quad x \in \mathcal{M}. \hfill (2.2)$$

Here $\Delta_x$ denotes the Laplace-Beltrami operator of $\mathcal{M}$.

**Definition 2.1.** Define

$$H^p(\mathcal{M}) = \{ f \in \mathcal{S}'(\mathcal{M}) : \| \mathcal{A} f \|_{L_p(\mathcal{M})} < \infty \}. \hfill (2.3)$$

For $f \in H^p(\mathcal{M})$, we set $\| f \|_{H^p(\mathcal{M})} = \| \mathcal{A} f \|_{L_p(\mathcal{M})}$.

It is well known that since $\mathcal{M}$ is compact,

$$H^p(\mathcal{M}) = L^p(\mathcal{M}) \subset L^{\log^+ L}(\mathcal{M}) \subset H^1(\mathcal{M}) \subset H^r(\mathcal{M}), \quad 0 < r < 1 < p, \hfill (2.4)$$

and all the inclusions are proper.
Let $B_n(x,r) = \{ y \in \mathbb{R}^n : |y-x| < r \}$. To give the atomic characterization of $H^r$, we need to define atoms on $\mathcal{M}$.

**Definition 2.2.** A function $a(\cdot)$ on $\mathcal{M}$ is called an $H^r$ atom if there are $\rho > 0$ and $\theta_0 \in \mathcal{M}$ such that

1. $\text{supp}(a) \subseteq B_n(\theta_0, \rho) \cap \mathcal{M}$,
2. $\|a\|_\infty \leq \rho^{-m/r}$,
3. $\int_{\mathcal{M}} a(\theta) P_k(\theta) d\sigma(\theta) = 0$,

for all polynomials $P_k$ on $\mathbb{R}^n$, with degrees $k \leq \lfloor m(1/r - 1) \rfloor$.

If $\Omega \in H^r(\mathcal{M})$, then there exist $H^r$ atoms $\{a_j\}$ and complex numbers $\{c_j\}$ such that

$$\Omega = \sum c_j a_j, \quad \sum |c_j|' \equiv \|\Omega\|_{H^r(\mathcal{M})} \quad (\text{see [6]}).$$

**Definition 2.3.** A smooth mapping $\phi$ from an open set $U$ in $\mathbb{R}^m$ into $\mathbb{R}^n$ is said to be of finite type at $u_0 \in U$ if, for every $\eta \in S^{n-1}$, there exists a nonzero multi-index $\omega = \omega(\eta)$ such that

$$\frac{\partial^\omega \left[ \eta \cdot \phi(u) \right]}{\partial u^\omega} \bigg|_{u = u_0} \neq 0.$$  

By the smoothness and compactness of $\mathcal{M}$, we may assume that there is a smooth mapping $\phi$ from a neighborhood of $\overline{B_m(0,1)}$ into $\mathbb{R}^n$ such that

(i) $\theta_0 \in \phi(B_m(0,1/2))$ and $\mathcal{M} \cap B_n(\theta_0, \rho) \subseteq \phi(B_m(0,1)) \subseteq \mathcal{M}$;

(ii) the vectors $\partial \phi/\partial u_1, \ldots, \partial \phi/\partial u_m$ are linearly independent for each $u \in \overline{B_m(0,1)}$;

(iii) $\phi$ is of finite type at every point in $\overline{B_m(0,1)}$ (see [16, page 350]).

Thus there is a smooth function $J(u)$ such that

$$\int_{\phi(B_m(0,1))} F d\sigma = \int_{B_m(0,1)} F(\phi(u)) J(u) du,$$

for any integrable function $F$ on $\mathcal{M}$. Since $\mathcal{M}$ is compact, we may assume that all $\phi$ raised from atoms $a$ satisfy $|\phi(u) - \phi(u_0)| \leq |u - u_0|$.

Now given $\Omega \in H^r(\mathcal{M})$, then for each $H^r$ atom, $a(\theta)$ supported in $\mathcal{M} \cap B_m(\theta_0, \rho)$, write $b(u) = a(\phi(u)) J(u) \chi_{B_m(0,1)}$. Let $u_0 = \phi^{-1}(\theta_0)$. It follows from (i)–(iii) that

$$\text{supp}(b) \subseteq B_m(u_0, \rho), \quad \|b\|_\infty \leq C \rho^{-m/r}, \quad \text{we may assume that } C = 1,$$

$$\int_{\mathbb{R}^m} b(u)(\phi(u) - \phi(u_0))^k du = 0,$$

for all $|k| \leq [\alpha]$, where $k = (k_1, k_2, \ldots, k_m)$ is a multi-index and $k = \sum_{i=1}^m k_i$.

We will need the following result (see [8]).
6 Singular integrals along manifolds

Lemma 2.4. Let \( \{a_k\} \) be a lacunary sequence of positive numbers such that \( a_k > 0 \) and \( \inf_{k \in \mathbb{Z}} |a_{k+1}/a_k| = \tau > 1 \). Let \( \tau_k \) be a sequence of Borel measures in \( \mathbb{R}^n \). Suppose that \( \|\tau_k\| \leq 1 \) and

1. \( |\hat{\tau}_k| \leq C |a_{k+1}|^{\gamma} \),
2. \( |\hat{\tau}_k| \leq C |a_k|^{-\gamma} \),

for all \( k \in \mathbb{Z} \), and suppose also that for some \( q > 1 \),

3. \( \|\tau^* (f)\| \leq C\|f\|_q \),

where \( \tau^* \) is the maximal operator: \( \tau^* (f) = \sup_k \|\tau_k * f\| \). Then

\[
Tf(x) = \sum_{k=-\infty}^{\infty} \tau_k * f(x) \tag{2.9}
\]

is a bounded operator on \( L^p(\mathbb{R}^n) \) for \( |1/p - 1/2| < 1/2q \).

We will also need the following result (see [8, 9, 11]).

Lemma 2.5. Let \( l,n \in \mathbb{N} \), and \( \{\tau_{s,k} : 0 \leq s \leq l, \text{ and } k \in \mathbb{Z} \} \) be a family of measures on \( \mathbb{R}^n \) with \( \tau_{0,k} = 0 \) for every \( k \in \mathbb{Z} \). Let \( \{\alpha_{ij} : 1 \leq s \leq l, \text{ and } j = 1,2 \} \subset \mathbb{R}^4 \), \( \{\eta_s : 1 \leq s \leq l \} \subset \mathbb{R}^{+}\setminus\{1\} \), \( \{M_s : 1 \leq s \leq l \} \subset \mathbb{N} \), and \( L_s : \mathbb{R}^n \to \mathbb{R}^m \) be linear transformations for \( 1 \leq s \leq l \). Suppose that

1. \( \|\tau_{s,k}\| \leq 1 \) for \( k \in \mathbb{Z} \) and \( 0 \leq s \leq l \);
2. \( \|\hat{\tau}_{s,k}(\xi)\| \leq C (\eta_s \langle L_s \xi \rangle_0)^{-\alpha_{s,j}} \) for \( \xi \in \mathbb{R}^m \), \( k \in \mathbb{Z} \), and \( 0 \leq s \leq l \);
3. \( \|\hat{\tau}_{s,k}(\xi) - \hat{\tau}_{s-1,k}(\xi)\| \leq C (\eta_s \langle L_s \xi \rangle_0)^{\alpha_{s,j}} \) for \( \xi \in \mathbb{R}^m \), \( k \in \mathbb{Z} \), and \( 0 \leq s \leq l \);
4. \( \) for some \( p_0 > 2 \), there exists a \( C > 0 \) such that

\[
\left\| \sum_{k \in \mathbb{Z}} \left( \|\tau_{s,k} * g_k\| \right)^2 \right\|_{L^n(\mathbb{R}^e)}^{1/2} \leq C \left\| \sum_{k \in \mathbb{Z}} \left( \|g_k\|_p \right)^2 \right\|_{L^n(\mathbb{R}^e)}^{1/2}, \tag{2.10}
\]

for all \( \{g_k\} \in L^{p_0}(\mathbb{R}^n, L^2) \) and \( 1 \leq s \leq l \).

Then for every \( p \in (p_0, p_0) \), there exists a positive constant \( C_p \) such that

\[
\left\| \sum_{k \in \mathbb{Z}} \tau_{l,k} * f \right\|_{L^p(\mathbb{R}^e)} \leq C_p \|f\|_{L^p(\mathbb{R}^e)}, \tag{2.11}
\]

\[
\left\| \left( \sum_{k \in \mathbb{Z}} \|\tau_{l,k} * f\|_p \right)^2 \right\|_{L^p(\mathbb{R}^e)} \leq C_p \|f\|_{L^p(\mathbb{R}^e)}
\]

hold for all \( f \in L^p(\mathbb{R}^n) \). The constant \( C_p \) is independent of the linear transformations \( \{L_s\}_{s=1}^l \).

3. Proof of theorem

We will prove the theorem in three different cases: \( 0 < \alpha < 1 \), \( \alpha = 1,2,3, \ldots \), and \( \alpha > 1 \), \( \alpha \notin \mathbb{Z} \). Without loss of generality, we may assume that \( \Omega(\theta) = a(\theta) \) is an \( H^\alpha \) atom as defined in Definition 2.2, the details can be found in [4].
Case 1 \((0 < \alpha < 1)\). Using the “lift” property of the Riesz potential and the definition of the space \(L^\alpha(\mathbb{R}^n)\), it is known that for any \(\alpha > 0\) and \(f \in L^\alpha(\mathbb{R}^n)\), one can write \(f = G_\alpha * f_a\) with \(|\hat{G}_\alpha(\xi)| \approx |\xi|^{-\alpha}\), \(|G_\alpha(y)| \approx |y|^{-n+\alpha}\), and \(\|f_a\|_p \approx \|f\|_{L^\alpha_p}\).

We write

\[
(\text{SI}_{\mu, \Omega, h, \alpha} f)(x) = \sum_{k} \mu_{k, \alpha} * f_a(x),
\]

where

\[
\mu_{k, \alpha}(x) = \int_{2^k}^{2^{k+1}} \int_{B_m(0,1)} G_\alpha(x - r\theta) \Omega(\theta) h(r) r^{-1-a} d\sigma(\theta) dr.
\]

In light of Lemma 2.4, in order to show that \(\|\text{SI}_{\mu, \Omega, h, \alpha} f\|_{L^p} \leq C\|f\|_{L^\alpha_p}\), it suffices to show that

(i) \(\|\mu_{k, \alpha}\|_{L^1(\mathbb{R}^n)} \leq C\),

(ii) \(|\hat{\mu}_{k, \alpha}(\xi)| \leq C|2^k \xi|^{|-\alpha|}\),

(iii) \(\|\mu_{k, \alpha}(\xi)\| \leq C|2^k \xi|^{|-\alpha|}\),

(iv) \(\|\text{supp}\{\mu_{k, \alpha} \ast f\}\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^\alpha_q}\), for all \(q \in (1, \infty)\).

Now, by the cancellation condition of \(b(u) = \Omega(\phi(u))f(u)\chi_{B_m(0,1)}(u)\), we have

\[
\|\mu_{k, \alpha}\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left| \int_{2^k}^{2^{k+1}} \left[ \int_{B_m(0,1)} (G_\alpha(x - r\phi(u)) - G_\alpha(x - r\phi(u_0))) b(u) du \right] \right| h(r) r^{-1-a} dr dx
\]

\[
\leq \int_{2^k}^{2^{k+1}} r^{-1-a} \int_{B_m(0,1)} |b(u)| dx \times \int_{\mathbb{R}^n} |G_\alpha(x - r\phi(u)) - G_\alpha(x - r\phi(u_0))| dx |h(r)| du dr.
\]

(3.3)

Letting \(y = x - r\phi(u_0)\), we have

\[
\int_{\mathbb{R}^n} |G_\alpha(x - r\phi(u)) - G_\alpha(x - r\phi(u_0))| dx = \int_{\mathbb{R}^n} |G_\alpha(y + r(\phi(u) - \phi(u_0))) - G_\alpha(y)| dy.
\]

(3.4)

As we mentioned before, \(|\phi(u) - \phi(u_0)| \leq |u - u_0| \leq \rho\), for \(u \in \text{supp}(b)\).

We write

\[
\int_{\mathbb{R}^n} |G_\alpha(y + r(\phi(u) - \phi(u_0))) - G_\alpha(y)| dy = \int_{|y| > 3\rho} |G_\alpha(y + r(\phi(u) - \phi(u_0))) - G_\alpha(y)| dy
\]

\[
+ \int_{|y| < 3\rho} |G_\alpha(y + r(\phi(u) - \phi(u_0))) - G_\alpha(y)| dy
\]

(3.5)

\[
= I_1 + I_2,
\]

where \(u\) is in the support of \(b(u)\).
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By the definition of $G_{\alpha}(x)$, we have, if $y \geq 3r\rho \geq 3r|\phi(u) - \phi(u_0)|$,

$$|G_{\alpha}(y + r(\phi(u) - \phi(u_0))) - G_{\alpha}(y)| \leq C \frac{r\rho}{|y|^{n-\alpha+1}}.$$  (3.6)

Thus,

$$I_1 \leq C \int_{|y| \geq 3r\rho} \frac{r\rho}{|y|^{n-\alpha+1}} dy \approx (r\rho)\alpha. \ (3.7)$$

It is easy to see that

$$I_2 \leq 2 \int_{|y| \leq 5r\rho} |G_{\alpha}(y)| \ dy \leq C \int_{|y| \leq 5r\rho} \frac{dy}{|y|^{n-\alpha}} \leq C(r\rho)^\alpha. \ (3.8)$$

Thus,

$$\|\mu_{k,\alpha}\|_{L^1(\mathbb{R}^n)} \leq \int_{2^{k}r}^{2^{k+1}r} r^{-1-\alpha} \int_{B_m(0,1)} |b(u)|
\times \int_{\mathbb{R}^n} |G_{\alpha}(x - r\phi(u)) - G_{\alpha}(x - r\phi(u_0))| \ dx \ h(r) \ du \ dr \leq C \int_{2^{k}r}^{2^{k+1}r} r^{-1-\alpha} \int_{B_m(0,1)} |b(u)| (r\rho)^\alpha \ h(r) \ du \ dr \leq C. \ (3.9)$$

To prove (ii), we write

$$\hat{\mu}_{k,\alpha}(\xi) = (\sigma_{k,\alpha} * G_{\alpha})(\xi) = \hat{\sigma}_{k,\alpha}(\xi) \ |\hat{G}_{\alpha}(\xi)| \leq C|\xi|^{-\alpha} |\hat{\sigma}_{k,\alpha}(\xi)|. \ (3.10)$$

Thus,

$$\begin{align*}
|\hat{\mu}_{k,\alpha}(\xi)| &\leq C|\xi|^{-\alpha} \int_{2^{k}r}^{2^{k+1}r} \left( \int_{B_m(0,1)} e^{-ir\xi \cdot \phi(u)} b(u) du \right) r^{-1-\alpha} h(r) dr \\
&\leq C|\xi|^{-\alpha} 2^{-k\alpha} \int_{2^{k}r}^{2^{k+1}r} \left( e^{-ir\xi \cdot \phi(u)} - e^{ir\xi \cdot \phi(u_0)} \right) b(u) du \ r^{-1} \ h(r) \ dr \\
&\leq C|\xi|^{-\alpha} 2^{-k\alpha} |2^k \xi| \int_{B_m(0,1)} |\phi(u) - \phi(u_0)| \ |b(u)| \ du \leq C |2^k \xi r|^{-\alpha},
\end{align*} \ (3.11)$$

which proves (ii).

On the other hand,

$$|\hat{\mu}_{k,\alpha}(\xi)| \leq C|\xi|^{-\alpha} 2^{-k\alpha} \int_{2^{k}r}^{2^{k+1}r} |b(u)| \ |du| \ h(r) \ dr = C |2^k \xi r|^{-\alpha}, \ (3.12)$$

which proves (iii).
It remains to show that

\[
\left\| \sup_{k \in \mathbb{Z}} |\mu_k, \alpha| \ast f \right\|_p \leq C \|f\|_p. \tag{3.13}
\]

Without loss of generality, assume that \( h(r) \geq 0 \). Then

\[
\left\| \sup_{k \in \mathbb{Z}} |\mu_k, \alpha| \ast f \right\|_{L^q(\mathbb{R}^n)} \leq C \sup_{k \in \mathbb{Z}} 2^{-k-\alpha} \int_{2^k}^{2^{k+1}} h(r) \int_{B_m(0,1)} |b(u)| \int_{\mathbb{R}^n} |f(x-z)| |G_\alpha(z-r\phi(u)) - G_\alpha(z-r\phi(u_0))| \, dz \, du \, dr. \tag{3.14}
\]

In the above integral, we write

\[
\int_{\mathbb{R}^n} |f(x-z)| |G_\alpha(z-r\phi(u)) - G_\alpha(z-r\phi(u_0))| \, dz = \int_{|z-r\phi(u_0)|>3r} |f(x-z)| |G_\alpha(z-r\phi(u)) - G_\alpha(z-r\phi(u_0))| \, dz + \int_{|z-r\phi(u_0)|\leq3r} |f(x-z)| |G_\alpha(z-r\phi(u)) - G_\alpha(z-r\phi(u_0))| \, dz \tag{3.15}
\]

where \( u \in B_n(u_0, \rho) \cap \mathcal{M} \).

In the integral \( I_1(f) \), we change variables \( z - r\phi(u_0) \rightarrow y \) and again write \( y \) as \( z \), then

\[
I_1(f)(x) = C \int_{|z|>3r} |f(x-z+r\phi(u_0))| |G_\alpha(z+r\phi(u_0)) - G_\alpha(z)| \, dz. \tag{3.16}
\]

Note that \(|r\phi(u_0) - r\phi(u)| \leq r\rho < |z|/2\). By the mean value theorem,

\[
I_1(f)(x) \leq C \int_{|z|>3r} r\rho |f(x-z+r\phi(u_0))| |z|^\alpha-1-n \, dz \tag{3.17}
\]

\[
\cong \int_{S^{n-1}} \int_{3r\rho}^{\infty} r\rho s^{\alpha-2} |f(x-sz'+r\phi(u_0))| \, ds \, d\sigma(z').
\]

Using integration by parts, it is easy to see that

\[
I_1(f)(x) \leq C \int_{S^{n-1}} (r\rho)^{\alpha-1} \int_0^{3r\rho} |f(x-tz'+r\phi(u_0))| \, dt \, d\sigma(z') + C \int_{S^{n-1}} \int_{3r\rho}^{\infty} r\rho s^{\alpha-3} \int_0^s |f(x-tz'+r\phi(u_0))| \, ds \, dt \, d\sigma(z'). \tag{3.18}
\]
Let $M_z f(x)$ be the maximal function

$$M_z f(x) = \sup_{t>0} t^{-1} \int_0^t |f(x-rz)| dr.$$  \hfill (3.19)

It is known in [16, page 477] that there is a constant $C$ independent of $z$ such that

$$\|M_z(f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}. \hfill (3.20)$$

Thus we have

$$I_1(f)(x) \leq C(r\rho)^\alpha \int_{S_n-1} M'_z f(x + r\phi(u_0)) d\sigma(z'). \hfill (3.21)$$

For the second integral $I_2(f)(x)$, we have

$$J_1(f)(x) = \int_{|z - r\phi(u_0)| < 3r\rho} |f(x-z) G_\alpha(z - r\phi(u))| dz,$$

$$J_2(f)(x) = \int_{|z| < 3r\rho} |f(x-z + r\phi(u_0)) G_\alpha(z)| dz. \hfill (3.22)$$

Let $w = z - r\phi(u)$. Then, in $J_1(f)(x)$, we have

$$|w| \leq |z - r\phi(u_0)| + |r\phi(u) - r\phi(u_0)| \leq 4r\rho. \hfill (3.23)$$

This gives (again write $z$ instead of $w$)

$$J_1(f)(x) \leq C \int_{|z| < 4r\rho} |f(x-z - r\phi(u))| |z|^{n-\alpha-n} dz$$

$$= C \int_{S^{n-1}} \int_{4r\rho} |f(x-tz' - r\phi(u))| dt d\sigma(z'). \hfill (3.24)$$

Using integration by parts, we obtain

$$J_1(f)(x) \leq C \int_{S^{n-1}} (r\rho)^\alpha M'_z (f(x - r\phi(u))) d\sigma(z'). \hfill (3.25)$$

Similarly, we can have the same estimate on $J_2(f)(x)$ so that

$$J_2(f)(x) \leq C \int_{S^{n-1}} (r\rho)^\alpha \{M'_z f(x + r\phi(u_0)) + M'_z (f(x - r\phi(u)))\} d\sigma(z'). \hfill (3.26)$$

Thus

$$\int_{\mathbb{R}^n} |f(x-z)| |G_\alpha(z - r\phi(u)) - G_\alpha(z - r\phi(u_0))| dz$$

$$\leq C(r\rho)^\alpha \int_{S^{n-1}} \{M'_z f(x + r\phi(u_0)) + M'_z (f(x - r\phi(u)))\} d\sigma(z'). \hfill (3.27)$$
Therefore, we have
\[
\left\| \sup_{k \in \mathbb{Z}} |\mu_{k,\alpha}| \ast f \right\|_{L^q(\mathbb{R}^n)} \leq C \int_{B_m(0,1) \times S^{n-1}} |b(u)| \rho^\alpha \left\{ \|M_{\phi(u_0)}M_{\epsilon^*}(f)\|_{L^q(\mathbb{R}^n)} + \|M_{\phi(u)}M_{\epsilon^*}f\|_{L^q(\mathbb{R}^n)} \right\} d\sigma(z') du.
\]
(3.28)

Since \(b\) is an \((r, \infty)\) atom supported in \(B_m(u_0, \rho) \cap M\) with \(r = m/(m+\alpha)\), it is easy to see that
\[
\int_{B_m(0,1)} |b(u)| \rho^\alpha du \leq C
\]
uniformly for \(b\) and \(\rho\). Thus
\[
\left\| \sup_{k \in \mathbb{Z}} |\mu_{k,\alpha}| \ast f \right\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^q(\mathbb{R}^n)}.
\]
(3.30)

By Lemma 2.4, Case 1 is established.

Case 2 \((\alpha = 1, 2, 3, \ldots)\). Using Taylor’s expansion about \(\theta_0\), we have, for \(j = (j_1, \ldots, j_m)\),
\[
(SM_{\phi,h,a})f(x) = \sum_{|j| = \alpha} C_j \int_0^1 (1-t)^{a-1} \int_0^\infty \mathcal{B}(u)r^{-1}h(r)
\times D^j f(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) du dr dt,
\]
(3.31)
where \(C_j\)'s are constants and \(\mathcal{B}(u) = b(u)(\phi(u) - \phi(u_0))^j\). Clearly, \(\mathcal{B}(u)\) is an \(H^1\) atom with the same support as \(b\).

For each \(j\), \(|j| = \alpha\), define the measures \(\{\sigma_{\phi,h,a}\} \in \mathbb{Z}\) by
\[
\int_{\mathbb{R}^n} F(x) d\sigma_{\phi,h,a} = \int_0^1 (1-t)^{a-1} \int_{2^k}^{2^{k+1}}F(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) \mathcal{B}(u)r^{-1}h(r) du dr dt.
\]
(3.32)

**Lemma 3.1.** Suppose that \(h\) satisfies (ii) in Theorem 1.1. Then for \(1 < p < \infty\), there exists a constant \(C_p > 0\) such that
\[
\left\| \left( \sum_{k \in \mathbb{Z}} \left| \sigma_{\phi,h,a} \ast g_k \right|^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left( \sum_{k \in \mathbb{Z}} \left| g_k \right|^2 \right)^{1/2} \right\|_p
\]
holds for all continuous mappings \(\phi\) and measurable functions \(\{g_k\}\) on \(\mathbb{R}^n\).
12 Singular integrals along manifolds

Proof. For $\xi \in \mathbb{R}^n$, we define the maximal operator $M_\xi$ on $\mathbb{R}^n$ by

$$ (M_\xi f)(x) = \sup_{k \in \mathbb{Z}} \left[ 2^{-k} \int_{2^k}^{2^{k+1}} |f(x + r\xi)| \, dr \right]. \tag{3.34} $$

It follows from the $L^p$-boundedness of the one-dimensional Hardy-Littlewood maximal operator that

$$ \|M_\xi f\|_p \leq A_p \|f\|_p, \tag{3.35} $$

for $1 < p < \infty$, where $A_p$ is independent of $\xi$.

By duality, we may assume that $p > 2$, then for $\{g_k\} \in L^p(\mathbb{R}^n, L^2)$, there exists a function $w \in L^{(p/2)'}(\mathbb{R}^n)$ such that $\|w\|_{(p/2)'} = 1$ and

$$ \left\| \left( \sum_{k \in \mathbb{Z}} |\sigma_{\phi, \beta, h, k, \alpha} \ast g_k|^2 \right)^{1/2} \right\|_p^2 = \int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{Z}} |\sigma_{\phi, \beta, h, k, \alpha} \ast g_k|^2 \right) w(x) dx. \tag{3.36} $$

By Hölder’s inequality and (3.35),

$$ \left\| \left( \sum_{k \in \mathbb{Z}} |\sigma_{\phi, \beta, h, k, \alpha} \ast g_k|^2 \right)^{1/2} \right\|_p^2 \leq \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} \left( \int_0^1 (1 - t)^{n-1} \int_{2^k}^{2^{k+1}} g_k(\nu_x \phi(u_0) + rt(\phi(u_0) - \phi(u))) \right. $$

$$ \times B(u)r^{-1}h(r)dudt \left| \int w(x) dx \right|^2 $$

$$ \leq C\|B\|_1 \sum_{k \in \mathbb{Z}} 2^{-k} \int_{\mathbb{R}^n} \int_0^1 \int_{2^k}^{2^{k+1}} B_{m(0,1)} \left| g_k(\nu_x \phi(u_0) + rt(\phi(u_0) - \phi(u))) \right|^2 $$

$$ \times B(u)w(x) \left| du \right| dr \left| dt \right| dx $$

$$ = C\|B\|_1 \sum_{k \in \mathbb{Z}} 2^{-k} \int_{\mathbb{R}^n} \int_0^1 B_{m(0,1)} \left| g_k(x) \right|^2 \left| w(x + rt(\phi(u_0) - \phi(u))) \right| dx dr dt $$

$$ \leq C\|B\|_1 \left[ \int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{Z}} |g_k(x)|^2 \right) (M_{\phi(u_0) + rt(\phi(u_0) - \phi(u))}w)(x) dx \right] B(u) \left| du \right| $$

$$ \leq C\|B\|_1 \left( \sum_{k \in \mathbb{Z}} |g_k|^2 \right)^{1/2} \left| B(u) \right|^2. \tag{3.37} $$

We also have the following estimates for $\sigma_{\phi, \beta, h, k, \alpha}$.

□
Lemma 3.2. Suppose that \( \phi \) is smooth and of finite type at every point in \( B_m(0,1) \) and \( h \) satisfies (ii) in Theorem 1.1. Then there exists a \( \delta > 0 \) such that

\[
|\hat{\sigma}_{\phi, h, k, \alpha, a}(\xi)| \leq C \| B \|_2 (2^k |\xi|)^{-\delta}. \tag{3.38}
\]

Proof.

\[
|\hat{\sigma}_{\phi, h, k, \alpha, a}(\xi)| = \left| \int_0^1 (1-t)^{\alpha-1} \int_{2^k}^{2^{k+1}} h(r)r^{-1} e^{i \xi r \phi(u_0)} e^{-i \xi r \phi(u_0)} B(u) e^{i \xi r \phi(u)} du dr dt \right|.
\]

Changing variables \((s = rt)\), we have

\[
|\hat{\sigma}_{\phi, h, k, \alpha, a}(\xi)| = \left| \int_0^1 (1-t)^{\alpha-1} \int_{2^k}^{2^{k+1}} h\left(\frac{s}{t}\right) s^{-1} e^{i \xi (s/t) \phi(u_0)} e^{-i \xi s \phi(u_0)} B(u) e^{i \xi \phi(u)} du ds dt \right|
\]

\[
\leq \int_0^1 |1-t|^{\alpha-1} \int_{2^k}^{2^{k+1}} |h(s/t) s^{-1}| \left( \int_{B_m(0,1)} B(u) e^{i \xi \phi(u)} du \right) ds dt. \tag{3.40}
\]

The remainder of the proof is similar to the proof of Lemma 3.3 in [5].

The following result is similar to those in [10], see also [5].

Lemma 3.3. Let \( B(\cdot) \) be a function satisfying \( \text{supp}(B) \subset B_m(0, \rho) \) and \( \| B \|_{\infty} \leq \rho^{-m} \) for some \( \rho < 1 \). Suppose that \( h \) satisfies (ii) in Theorem 1.1. Then there exists a constant \( C > 0 \) such that

\[
\left| \int_0^1 (1-t)^{\alpha-1} \int_{2^k}^{2^{k+1}} h(r)r^{-1} B(u) e^{-i rt [Q(u) + \sum_{|\beta|=s} d_{\beta}]} du dr dt \right|
\]

\[
\leq C \left( 2^k \rho^s \sum_{|\beta|=s} |d_{\beta}| \right)^{-1/(4s)} \tag{3.41}
\]

holds for all polynomials \( Q : \mathbb{R}^m \to \mathbb{R} \) with \( \text{deg}(Q) < s \) and \( \{d_{\beta}\} \subset \mathbb{R} \). The constant \( C \) is independent of \( \rho \).

Now, by Lemma 3.2, there exists a \( \delta > 0 \) such that

\[
|\hat{\sigma}_{\phi, h, k, \alpha, a}(\xi)| \leq C (2^k |\xi|)^{-\delta} \rho^{-m/2}. \tag{3.42}
\]
Let $l = \lfloor m/(2\delta) \rfloor + 1$. Following the proof of Theorem 3.7 in [5], we define a sequence of mappings $\{\Phi^s\}_{s=0}^{l-1}$ by

$$
\Phi^0 = \phi = (\phi_1, \ldots, \phi_n),
$$

$$
\Phi^s(u) = \left( \sum_{|\beta|\leq s} \frac{1}{\beta!} \partial_\beta \phi_1(u_0)(u-u_0)^\beta, \ldots, \sum_{|\beta|\leq s} \frac{1}{\beta!} \partial_\beta \phi_n(u_0)(u-u_0)^\beta \right)
$$

(3.43)

for $s = 0, 1, \ldots, l-1$.

Let

$$
\sigma_{s,k,\alpha} = \sigma_{\Phi^s, \mathcal{B}, h, \alpha}
$$

(3.44)

for $0 \leq s \leq l$ and $k \in \mathbb{Z}$.

In order to show that $\| SI_{l,\Omega, h, \alpha} f \|_{L^p} \leq C \| f \|_{L^p}$, it suffices to show that the family of measures $\{\sigma_{s,k,\alpha}\}$ satisfies the conditions of Lemma 2.5.

By its definition and Lemma 3.2, the family of measures $\{\sigma_{s,k,\alpha}\}$ satisfies conditions (i) and (iv) in Lemma 2.5, for any $p_0 > 2$.

It is easy to see that

$$
\| \sigma_{s,k,\alpha} \| \leq \| \mathcal{B} \|_1 \int_0^1 (1-t)^{a-1} \int_{2^k}^{2^{k+1}} r^{-1} |h(r)| dr dt \leq C.
$$

(3.45)

Also we have

$$
\sigma_{0,k,\alpha}(x) = 0, \quad \text{by the cancellation condition of } \mathcal{B}(u).
$$

(3.46)

For $j = 1, \ldots, n$, let

$$
d_{j,\beta} = \frac{1}{\beta!} \partial_\beta \phi_j(u_0).
$$

(3.47)

By (3.42) and Lemma 3.3, we have

$$
|\hat{\sigma}_{l,k,\alpha}(\xi)| \leq C (2^k \rho |\xi|)^{-\delta},
$$

$$
|\hat{\sigma}_{s,k,\alpha}(\xi)| \leq C \left( 2^k \rho^s \sum_{|\beta|=s} \left| \sum_{j=1}^n d_{j,\beta} \xi_j \right| \right)^{-1/(4s)}
$$

(3.48)

for $1 \leq s \leq l-1, k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$. We also have,

$$
|\hat{\sigma}_{l,k,\alpha}(\xi) - \hat{\sigma}_{l-1,k,\alpha}(\xi)|
\leq \left| \int_0^1 (1-t)^{a-1} \int_{2^k}^{2^{k+1}} |h(r)| r^{-1} \int_{B_m(0,1)} |\mathcal{B}(u)| |e^{i\xi r \phi(u)} - e^{i\xi r \phi^{l-1}(u)}| du dr dt \right|
\leq C |\xi| 2^k \int_{B_m(0,1)} |\mathcal{B}(u)| |(\phi(u) - \phi^{l-1}(u))| du \leq C (2^k |\xi| \rho^l).
$$

(3.49)
Similarly,
\[
|\hat{u}_{s,k,a}(\xi) - \hat{u}_{s-1,k,a}(\xi)| \leq C2^k \int_{B_m(0,1)} |B(u)| |\xi \cdot (\phi^s(u) - \phi^{s-1}(u))| \, du
\]

\[
\leq C2^k \rho^z \sum_{|\beta| = s} \left| \sum_{j=1}^n d_{j\beta} \xi_j \right|
\]

for \(1 \leq s \leq l - 1, k \in \mathbb{Z} \) and \(\xi \in \mathbb{R}^n\).

Invoking Lemma 2.5, Case 2 is established.

Case 3 \((\alpha > 1, \alpha \notin \mathbb{Z})\). Write \(\alpha = [\alpha] + \gamma, \gamma \in (0,1)\).

Similar to the case \(\alpha = 1, 2, 3, \ldots\), by Taylor’s expansion, we have
\[
(S_{\Omega,\Omega,\Omega,\alpha} f)(x) = \sum_{|j| = \alpha} C_j \int_0^1 (1 - t)^{a-1} \int_0^\infty r^{-1-\gamma} h(r) \int_{B_m(0,1)} B(u) \times D^j f(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) \, du \, dt \, dr,
\]

where \(B(u) = b(u)(\phi(u) - \phi(u_0))^j\). Clearly, \(B(u)\) is an \(H^r\) atom, where \(r = m/(m + \gamma)\).

Similar to Case 1, again using the “lift” property of the Riesz potential and the definition of the space \(L^p_m(\mathbb{R}^n)\), it is known that for any \(\gamma > 0\) and \(f \in L^p_m(\mathbb{R}^n)\), one can write \(f = G_\gamma \ast f_\gamma\) with \(|\hat{G}_\gamma(\xi)| \approx |\xi|^{-\gamma}, |G_\gamma(y)| \approx |y|^{-\gamma},\) and \(\|f\|_p \approx \|f\|_{L^p_m}\).

We write
\[
(S_{\Omega,\Omega,\Omega,\alpha,k} f)(x) = \sum_k \sigma_{k,y} \ast f_\gamma,
\]

where
\[
\sigma_{k,y} = \int_0^1 (1 - t)^{a-1} \int_{2^k}^{2^{k+1}} r^{-1-\gamma} h(r) \int_{B_m(0,1)} B(u) G_\gamma(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) \, du \, dr \, dt
\]

\[
= \int_0^1 (1 - t)^{a-1} \int_{2^k}^{2^{k+1}} r^{-1-\gamma} h(r) \int_{B_m(0,1)} B(u) \times [G_\gamma(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) - G_\gamma(x - r\phi(u_0))] \, du \, dr \, dt.
\]

Again, by Lemma 2.4, in order to show that \(\|S_{\Omega,\Omega,\Omega,\alpha,k} f\|_p \leq C\|f\|_{L^p_m}\), it suffices to show that
\begin{enumerate}
  \item \(\|\sigma_{k,y}\|_{L^1(\mathbb{R}^n)} \leq C\),
  \item \(|\hat{\sigma}_{k,y}(\xi)| \leq C|2^k \xi \rho|^{1-\gamma},
  \item \(|\hat{\sigma}_{k,y}(\xi)| \leq C|2^k \xi \rho|^{-\gamma},
  \item \(\|\sup_{k \in \mathbb{Z}} |\sigma_{k,y}| \ast f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}\).
\end{enumerate}

The proof is similar to the proof for Case 1. We leave the details to the reader.

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References


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