MORE ON RC-LINDELÖF SETS AND ALMOST RC-LINDELÖF SETS

MOHAMMAD S. SARSAK

Received 25 November 2004; Revised 3 April 2006; Accepted 30 May 2006

We study new properties and characterizations of rc-Lindelöf sets and almost rc-Lindelöf sets; a special interest is given to the mapping properties of such sets. We also obtain some product theorems concerning rc-Lindelöf spaces.

Copyright © 2006 Hindawi Publishing Corporation. All rights reserved.

1. Introduction and preliminaries

A subset $A$ of a space $X$ is called regular open if $A = \text{Int} A$, and regular closed if $X \setminus A$ is regular open, or equivalently, if $A = \text{Int} A$. $A$ is called semiopen [16] (resp., preopen [17], semi-preopen [3], $b$-open [4]) if $A \subset \text{Int} A$ (resp., $A \subset \text{Int} A$, $A \subset \text{Int} A$, $A \subset \text{Int} A \cup \text{Int} A$). The concept of a preopen set was introduced in [6] where the term locally dense was used and the concept of a semi-preopen set was introduced in [1] under the name $\beta$-open. It was pointed out in [3] that $A$ is semi-preopen if and only if $P \subset A \subset P$ for some preopen set $P$. Clearly, every open set is both semiopen and preopen, semiopen sets as well as preopen sets are $b$-open, and $b$-open sets are semi-preopen. $A$ is called semiclosed (resp., preclosed, semi-preclosed, $b$-closed) if $X \setminus A$ is semiopen (resp., preopen, semi-preopen, $b$-open). $A$ is called semiregular [8] if it is both semiopen and semiclosed, or equivalently, if there exists a regular open set $U$ such that $U \subset A \subset \overline{U}$.

Clearly, every regular closed (regular open) set is semiregular. The semiclosure (resp., preclosure, semi-preclosure, $b$-closure) denoted by $\text{scl} A$ (resp., $\text{pcl} A$, $\text{spcl} A$, $\text{bcl} A$) is the intersection of all semiclosed (resp., preclosed, semi-preclosed, $b$-closed) subsets of $X$ containing $A$, or equivalently, is the smallest semiclosed (resp., preclosed, semi-preclosed, $b$-closed) set containing $A$. Dually, the semi-interior (resp., preinterior, semi-preinterior, $b$-interior) denoted by $\text{sint} A$ (resp., $\text{pint} A$, $\text{spint} A$, $\text{bint} A$) is the union of all semiopen (resp., preopen, semi-preopen, $b$-open) subsets of $X$ contained in $A$, or equivalently, is the largest semiopen (resp., preopen, semi-preopen, $b$-open) set contained in $A$.

A function $f$ from a space $X$ into a space $Y$ is called almost open [20] if $f^{-1}(U) \subset f^{-1}(\overline{U})$ whenever $U$ is open in $Y$, semicontinuous [16] if the inverse image of each...
open set is semiopen, $\beta$-continuous [1] if the inverse image of each open set is $\beta$-open, weakly $\theta$-irresolute [13] if the inverse image of each regular closed set is semiopen, rc-continuous [14] if the inverse image of each regular closed set is regular closed, and wrc-continuous [2] if the inverse image of each regular closed set is semi-preopen. We will use the term semiprecontinuous to indicate $\beta$-continuous. Clearly, every semicontinuous function is semiprecontinuous, every rc-continuous function is weakly $\theta$-irresolute, and every weakly $\theta$-irresolute function is wrc-continuous. It is also easy to see that a function that is both semicontinuous (resp., semiprecontinuous) and almost open is weakly $\theta$-irresolute (resp., wrc-continuous).

A function $f$ from a space $X$ into a space $Y$ is called somewhat continuous [12] if for each nonempty open set $V$ in $Y$, $\text{int} f^{-1}(V) \neq \phi$.

A space $X$ is called a weak $P$-space [18] if for each countable family $\{U_n : n \in \mathbb{N}\}$ of open subsets of $X$, $\bigcup U_n = \bigcup U_n$. Clearly, $X$ is a weak $P$-space if and only if the countable union of regular closed subsets of $X$ is regular closed (closed).

A space $X$ is called rc-Lindelöf [15] (resp., nearly Lindelöf [5]) if every regular closed (resp., regular open) cover of $X$ has a countable subcover, and called almost rc-Lindelöf [10] if every regular closed cover of $X$ has a countable subfamily whose union is dense in $X$.

A subset $A$ of a space $X$ is called an $S$-set in $X$ [7] if every cover of $A$ by regular closed subsets of $X$ has a finite subcover, and called an rc-Lindelöf set in $X$ (resp., an almost rc-Lindelöf set in $X$) [9] if every cover of $A$ by regular closed subsets of $X$ admits a countable subfamily that covers $A$ (resp., the closure of the union of whose members contains $A$). Obviously, every $S$-set is an rc-Lindelöf set and every rc-Lindelöf set is an almost rc-Lindelöf set; it is also clear that a subset $A$ of a weak $P$-space $X$ is rc-Lindelöf in $X$ if and only if it is almost rc-Lindelöf in $X$.

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers. For the concepts not defined here, we refer the reader to Engelking [11].

In concluding this section, we recall the following facts for their importance in the material of our paper.

**Theorem 1.1** [9]. If $A$ is an rc-Lindelöf (resp., almost rc-Lindelöf) set in a space $X$ and $B$ is a regular open subset of $X$, then $A \cap B$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$. In particular, a regular open subset $A$ of an rc-Lindelöf (resp., almost rc-Lindelöf) space $X$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$.

**Theorem 1.2** [9]. Let $A$ be a preopen subset of a space $X$ and $B \subset A$. Then $B$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$ if and only if $B$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $A$. In particular, a preopen subset $A$ of a space $X$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$ if and only if $A$ is an rc-Lindelöf (resp., almost rc-Lindelöf) subspace.

**Proposition 1.3** [19]. If $A$ is an almost rc-Lindelöf set in a space $X$ and $A \subset B \subset \overline{A}$, then $B$ is almost rc-Lindelöf in $X$.

**Proposition 1.4** [9]. The countable union of rc-Lindelöf (resp., almost rc-Lindelöf) sets in a space $X$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$. 


Proposition 1.5 [9]. A subset $A$ of a space $X$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$ if and only if every cover of $A$ by semiopen subsets of $X$ admits a countable subfamily the union of the closures of whose members (resp., the closure of the union of whose members) contains $A$.

Proposition 1.6 [19]. Let $A$ be a preopen, almost rc-Lindelöf set in a space $X$ and $B$ a regular closed subset of $X$, then $A \cap B$ is almost rc-Lindelöf in $X$. In particular, a regular closed subset $A$ of an almost rc-Lindelöf space $X$ is almost rc-Lindelöf in $X$.

Lemma 1.7. If $A$ is a preopen subset of a space $X$ and $U$ is open in $X$, then $A \cap U \cap A = U \cap A$.

2. Further properties

This section is devoted to study new properties concerning rc-Lindelöf sets and almost rc-Lindelöf sets. We obtain several characterizations of rc-Lindelöf sets and almost rc-Lindelöf sets.

The following proposition is an improvement of Proposition 1.6 and the fact of Theorem 1.1 that a regular open subset of an almost rc-Lindelöf space $X$ is almost rc-Lindelöf in $X$.

Proposition 2.1. Let $A$ be a preopen, almost rc-Lindelöf set in a space $X$ and $B$ a semiregular subset of $X$, then $A \cap B$ is almost rc-Lindelöf in $X$. In particular, a semiregular subset $A$ of an almost rc-Lindelöf space $X$ is almost rc-Lindelöf in $X$.

Proof. Since $B$ is a semiregular subset of $X$, there exists a regular open subset $U$ of $X$ such that $U \subset B \subset \overline{U}$, thus by Lemma 1.7, it follows that $A \cap U \subset A \cap B \subset \overline{U} \cap A \subset \overline{A \cap U}$. Since $A$ is almost rc-Lindelöf set in $X$, it follows from Theorem 1.1 that $A \cap U$ is almost rc-Lindelöf set in $X$. The result yields from Proposition 1.3. □

Proposition 2.2 [19]. If $A$ is a regular closed subset of a space $X$ such that $A$ is almost rc-Lindelöf in $X$, then $A$ is an almost rc-Lindelöf.

The following proposition includes an improvement of Proposition 2.2.

Proposition 2.3. Let $A$ be a semiopen subset of a space $X$ and $B \subset A$. If $B$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$, then $B$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $A$. In particular, if $A$ is a semiopen subset of a space $X$ such that $A$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$, then $A$ is an rc-Lindelöf (resp., almost rc-Lindelöf) subspace.

Proof. Follows from Proposition 1.5 and the fact that if $A$ is semiopen in $X$ and $B$ is semiopen in $A$, then $B$ is semiopen in $X$. □

Corollary 2.4 [2]. Let $X$ be an rc-Lindelöf weak $P$-space. If $U \subset A \subset \overline{U}$, where $U$ is a regular open subset of $X$, then $A$ is an rc-Lindelöf subspace.

Proof. By Theorem 1.1, $U$ is an rc-Lindelöf set in $X$ and thus almost rc-Lindelöf in $X$. By Proposition 1.3, $A$ is almost rc-Lindelöf in $X$, but $X$ is a weak $P$-space, so $A$ is rc-Lindelöf in $X$. Finally, since $A$ is semiopen (it is moreover semiregular), it follows from Proposition 2.3 that $A$ is an rc-Lindelöf subspace. □
Theorem 2.5. Let $A$ be a subset of a space $X$. Then the following are equivalent.

(i) $A$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$.

(ii) Every cover of $A$ by semi-preopen subsets of $X$ admits a countable subfamily the union of the closures of whose members (resp., the closure of the union of whose members) contains $A$.

(iii) Every cover of $A$ by $b$-open subsets of $X$ admits a countable subfamily the union of the closures of whose members (resp., the closure of the union of whose members) contains $A$.

(iv) Every cover of $A$ by semiopen subsets of $X$ admits a countable subfamily the union of the closures of whose members (resp., the closure of the union of whose members) contains $A$.

(v) Every cover of $A$ by semiregular subsets of $X$ admits a countable subfamily the union of the closures of whose members (resp., the closure of the union of whose members) contains $A$.

Proof. (i) $\Rightarrow$ (ii): follows since the closure of a semi-preopen set is regular closed.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i): follows from the following implications: regular closed $\Rightarrow$ semiregular $\Rightarrow$ semiopen $\Rightarrow b$-open $\Rightarrow$ semi-preopen.

The following theorem also characterizes rc-Lindelöf sets and almost rc-Lindelöf sets, it is a direct consequence of Theorem 2.5 and the definition of rc-Lindelöf (almost rc-Lindelöf) sets.

Theorem 2.6. Let $A$ be a subset of a space $X$. Then the following are equivalent.

(i) $A$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$.

(ii) If $U_{\sim} = \{U_\alpha : \alpha \in \Lambda\}$ is a family of regular open subsets of $X$ satisfying that for any countable subcollection $U_\sim^*\alpha$ of $U_{\sim}$, $A \cap (\cap U_\sim^*\alpha) \neq \phi$ (resp., $A \cap \text{int}(\cap U_\sim^*\alpha) \neq \phi$), then $A \cap (\cap U_{\sim}) \neq \phi$.

(iii) If $U_{\sim} = \{U_\alpha : \alpha \in \Lambda\}$ is a family of semi-preclosed subsets of $X$ satisfying that for any countable subcollection $U_\sim^*\alpha$ of $U_{\sim}$, $A \cap (\cap \{\text{int} U : U \in U_\sim^*\alpha\}) \neq \phi$ (resp., $A \cap \text{int}(\cap U_\sim^*\alpha) \neq \phi$), then $A \cap (\cap U_{\sim}) \neq \phi$.

(iv) If $U_{\sim} = \{U_\alpha : \alpha \in \Lambda\}$ is a family of $b$-closed subsets of $X$ satisfying that for any countable subcollection $U_\sim^*\alpha$ of $U_{\sim}$, $A \cap (\cap \{\text{int} U : U \in U_\sim^*\alpha\}) \neq \phi$ (resp., $A \cap \text{int}(\cap U_\sim^*\alpha) \neq \phi$), then $A \cap (\cap U_{\sim}) \neq \phi$.

(v) If $U_{\sim} = \{U_\alpha : \alpha \in \Lambda\}$ is a family of semiclosed subsets of $X$ satisfying that for any countable subcollection $U_\sim^*\alpha$ of $U_{\sim}$, $A \cap (\cap \{\text{int} U : U \in U_\sim^*\alpha\}) \neq \phi$ (resp., $A \cap \text{int}(\cap U_\sim^*\alpha) \neq \phi$), then $A \cap (\cap U_{\sim}) \neq \phi$.

(vi) If $U_{\sim} = \{U_\alpha : \alpha \in \Lambda\}$ is a family of semiregular subsets of $X$ satisfying that for any countable subcollection $U_\sim^*\alpha$ of $U_{\sim}$, $A \cap (\cap \{\text{int} U : U \in U_\sim^*\alpha\}) \neq \phi$ (resp., $A \cap \text{int}(\cap U_\sim^*\alpha) \neq \phi$), then $A \cap (\cap U_{\sim}) \neq \phi$.

3. Invariance properties

In this section, we mainly study several types of functions that preserve the property of being an rc-Lindelöf (almost rc-Lindelöf) set.
Definition 3.1 [19]. A function \( f \) from a space \( X \) into a space \( Y \) is said to be slightly continuous if \( f(U) \subset f(U) \) whenever \( U \) is open in \( X \).

In [19], it was shown that if a function \( f : X \rightarrow Y \) is slightly continuous and weakly \( \theta \)-irresolute, then \( f(A) \) is almost rc-Lindelöf in \( Y \) whenever \( A \) is almost rc-Lindelöf set in \( X \). The following theorem is analogous to this result; it has a similar proof that we will mention for the convenience of the reader.

Theorem 3.2. Let \( f : X \rightarrow Y \) be a slightly continuous and weakly \( \theta \)-irresolute function. If \( A \) is rc-Lindelöf set in \( X \), then \( f(A) \) is rc-Lindelöf in \( Y \).

Proof. Let \( \{ U_\alpha : \alpha \in \Lambda \} \) be a cover of \( f(A) \) by regular closed subsets of \( X \). Then \( \{ f^{-1}(U_\alpha) : \alpha \in \Lambda \} \) is a cover of \( A \) by semiopen subsets of \( X \) (as \( f \) is weakly \( \theta \)-irresolute). Since \( A \) is rc-Lindelöf in \( X \), it follows from Proposition 1.5 that there exist \( \alpha_1, \alpha_2, \ldots \in \Lambda \) such that \( A \subset \bigcup_{i=1}^{\infty} f^{-1}(U_{\alpha_i}) \). For each \( i \in \mathbb{N} \), there is an open subset \( V_i \) of \( X \) such that \( V_i \subset f^{-1}(U_{\alpha_i}) \subset \overline{V_i} \) and thus \( \bigcup_{i=1}^{\infty} f^{-1}(U_{\alpha_i}) = \bigcup_{i=1}^{\infty} \overline{V_i} \). Since \( f \) is slightly continuous, it follows that \( f(A) \subset \bigcup_{i=1}^{\infty} f(\overline{V_i}) \subset \bigcup_{i=1}^{\infty} \overline{U_{\alpha_i}} = \bigcup_{i=1}^{\infty} U_{\alpha_i} \). Hence \( f(A) \) is rc-Lindelöf in \( Y \). \( \square \)

Corollary 3.3. Let \( f : X \rightarrow Y \) be a slightly continuous, semicontinuous, and almost open function. If \( A \) is rc-Lindelöf (resp., almost rc-Lindelöf) in \( X \), then \( f(A) \) is rc-Lindelöf (resp., almost rc-Lindelöf) in \( Y \).

Corollary 3.4. Let \( f : X \rightarrow Y \) be a surjective, slightly continuous, semicontinuous, and almost open function. If \( X \) is rc-Lindelöf, then \( Y \) is rc-Lindelöf.

It will be seen later that the condition slightly continuous of Corollary 3.4 is not essential for preserving the almost rc-Lindelöf property.

Corollary 3.5 [2]. Let \( f : X \rightarrow Y \) be a surjective, continuous, and almost open function. If \( X \) is rc-Lindelöf, then \( Y \) is rc-Lindelöf.

Obviously, every continuous function is both semicontinuous and slightly continuous. However, the converse is not true as the following example tells.

Example 3.6. Let \( X = \{ a, b, c \}, \tau = \{ X, \phi, \{ a \} \}, \tau^* = \{ X, \phi, \{ a, b \} \} \). Then the identity function from \( (X, \tau) \) onto \( (X, \tau^*) \) is a semicontinuous, slightly continuous, and almost open surjection. However, it is not continuous.

Proposition 3.7. Let \( f : X \rightarrow Y \) be a semicontinuous function. If \( X \) is extremally disconnected (i.e., every regular closed subset of \( X \) is open), then \( f \) is slightly continuous.

Proof. Let \( U \) be open in \( X \). Then \( \text{scl}(U) = U \cup \text{int} U = U \) (as \( X \) is extremally disconnected). Since \( f \) is semicontinuous, it follows that \( f(\text{scl}(U)) = f(U) \subset f(U) \). Hence \( f \) is slightly continuous. \( \square \)

The following corollary is an immediate consequence of Corollary 3.4 and Proposition 3.7.

Corollary 3.8 [2]. Let \( f : X \rightarrow Y \) be a semicontinuous, almost open surjection, where \( X \) is extremally disconnected. If \( X \) is rc-Lindelöf, then \( Y \) is rc-Lindelöf.
The following example shows that if \( X \) is extremally disconnected and \( f : X \rightarrow Y \) is slightly continuous, almost open surjection, then \( f \) need not be semicontinuous.

**Example 3.9.** Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a, b\}\} \), \( \tau^* = \{X, \phi, \{a\}\} \). Then \((X, \tau)\) is extremally disconnected, also the identity function from \((X, \tau)\) onto \((X, \tau^*)\) is slightly continuous and almost open; it is, however, not semicontinuous.

**Proposition 3.10** [10]. (i) Let \( f : X \rightarrow Y \) be a somewhat continuous and weakly \( \theta \)-irresolute function. If \( X \) is almost rc-Lindelöf, then \( Y \) is almost rc-Lindelöf.

(ii) Let \( f : X \rightarrow Y \) be a surjective, semicontinuous, and weakly \( \theta \)-irresolute function. If \( X \) is almost rc-Lindelöf, then \( Y \) is almost rc-Lindelöf.

**Corollary 3.11.** Let \( f : X \rightarrow Y \) be a surjective, semicontinuous, and almost open function. If \( X \) is almost rc-Lindelöf, then \( Y \) is almost rc-Lindelöf.

The following corollary is an immediate consequence of Corollary 3.11 and the fact that for a weak \( P \)-space, the concepts of being rc-Lindelöf and almost rc-Lindelöf coincide.

**Corollary 3.12** [2]. Let \( f : X \rightarrow Y \) be a surjective, semicontinuous, and almost open function, where \( Y \) is a weak \( P \)-space. If \( X \) is rc-Lindelöf, then \( Y \) is rc-Lindelöf.

**Definition 3.13.** A function \( f : X \rightarrow Y \) is said to be somewhat precontinuous if for each nonempty open set \( V \) in \( Y \), \( \text{pint} f^{-1}(V) \neq \phi \).

**Remark 3.14.** It was pointed out in [10] that every surjective semicontinuous function is somewhat continuous, a similar result that may be pointed out here asserts that every surjective semi-precontinuous function is somewhat precontinuous. However, the converses of these two facts are not true as the following two examples tell.

**Example 3.15.** Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a, b\}, \{c\}\} \), \( \tau^* = \{X, \phi, \{a, c\}\} \). Then the identity function from \((X, \tau)\) onto \((X, \tau^*)\) is somewhat continuous; it is, however, not semi-continuous.

**Example 3.16.** Let \( X = \{a, b, c, d\} \), \( \tau = \{X, \phi, \{b\}, \{d\}, \{b, d\}, \{a, d\}, \{a, b, d\}\} \), \( \tau^* = \{X, \phi, \{a, b\}\} \). Then the identity function from \((X, \tau)\) onto \((X, \tau^*)\) is even somewhat continuous and thus somewhat precontinuous; it is, however, not semi-precontinuous since \( \{a, b\} \) is not semi-preopen in \((X, \tau)\).

The following result is a slight improvement of Proposition 3.10(i), the similar proof follows from Theorem 2.5 and the fact that if \( A \) is a semiopen subset of a space \( X \), then \( \text{pcl}(A) = \overline{A} \).

**Proposition 3.17.** (i) Let \( f : X \rightarrow Y \) be a somewhat continuous and wrc-continuous function. If \( X \) is almost rc-Lindelöf, then \( Y \) is almost rc-Lindelöf.

(ii) Let \( f : X \rightarrow Y \) be a somewhat precontinuous and weakly \( \theta \)-irresolute function. If \( X \) is almost rc-Lindelöf, then \( Y \) is almost rc-Lindelöf.

**Remark 3.18.** Clearly, every somewhat continuous function is somewhat precontinuous and every weakly \( \theta \)-irresolute function is wrc-continuous. However, the following two examples show that the property of being both somewhat continuous and wrc-continuous
and the property of being both somewhat precontinuous and weakly $\theta$-irresolute are independent.

Example 3.19. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a, b\}\}$, $\tau^* = \{X, \emptyset, \{a, c\}\}$. Then the identity function from $(X, \tau)$ onto $(X, \tau^*)$ is somewhat precontinuous and weakly $\theta$-irresolute; it is, however, not somewhat continuous.

Example 3.20. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{d\}, \{a, b, c\}, \{a, d\}, \{b, c, d\}\}$, $\tau^* = \{X, \emptyset, \{a, b\}, \{d\}, \{a, b, d\}\}$. Then the identity function from $(X, \tau)$ onto $(X, \tau^*)$ is somewhat continuous and wrc-continuous; it is, however, not weakly $\theta$-irresolute (observe that $\{d, c\}$ is regular closed in $(X, \tau^*)$ but not semiopen in $(X, \tau)$).

The following result is a slight improvement of Proposition 3.10(ii), it is a direct consequence of Remark 3.14 and Proposition 3.17.

Corollary 3.21. (i) Let $f : X \to Y$ be a surjective, semicontinuous, and wrc-continuous function. If $X$ is almost rc-Lindelöf, then $Y$ is almost rc-Lindelöf.

(ii) Let $f : X \to Y$ be a surjective, semi-precontinuous, and weakly $\theta$-irresolute function. If $X$ is almost rc-Lindelöf, then $Y$ is almost rc-Lindelöf.

Corollary 3.22 [2]. Let $f : X \to Y$ be a somewhat continuous and wrc-continuous surjection, where $Y$ is a weak $P$-space. If $X$ is rc-Lindelöf, then $Y$ is rc-Lindelöf.

Corollary 3.22 is still true even if the function $f$ is not surjective.

4. Product theorems

In this section, we study some types of functions that inversely preserve the property of being an rc-Lindelöf (almost rc-Lindelöf) set. We mainly obtain some product theorems concerning rc-Lindelöf spaces.

Definition 4.1 [19]. A function $f$ from a space $X$ into a space $Y$ is said to be regular open if it maps regular open subsets onto regular open subsets.

Definition 4.2 [19]. (i) A subset $A$ of a space $X$ is said to be an rc-$F_\sigma$ subset if $A$ is the countable union of regular closed subsets.

(ii) A function $f$ from a space $X$ into a space $Y$ is said to be weakly almost open if $f^{-1}(\overline{A}) \subset \overline{f^{-1}(A)}$ whenever $A$ is an rc-$F_\sigma$ subset of $Y$.

In [19], it was shown that every almost open function is weakly almost open, but not conversely.

Theorem 4.3 [19]. Let $f$ be a weakly almost open and regular open function from a space $X$ onto a space $Y$. Then the following hold.

(i) If for each $y \in Y$, $f^{-1}(y)$ is an $S$-set in $X$, then $X$ is almost rc-Lindelöf whenever $Y$ is almost rc-Lindelöf.

(ii) If for each $y \in Y$, $f^{-1}(y)$ is rc-Lindelöf in $X$, then $X$ is almost rc-Lindelöf whenever $Y$ is almost rc-Lindelöf provided that $X$ is a weak $P$-space.

We point out here that in the result of Theorem 4.3(ii), $X$ being almost rc-Lindelöf may be replaced by rc-Lindelöf since $X$ is a weak $P$-space.
Theorem 4.3 may be improved in the following form.

**Theorem 4.4.** Let $f$ be a weakly almost open and regular open function from a space $X$ onto a space $Y$. Then the following hold.

(i) If for each $y \in Y$, $f^{-1}(y)$ is an $S$-set in $X$, then $f^{-1}(A)$ is almost rc-Lindelöf in $X$ whenever $A$ is rc-Lindelöf in $Y$.

(ii) If for each $y \in Y$, $f^{-1}(y)$ is rc-Lindelöf in $X$, then $f^{-1}(A)$ is rc-Lindelöf in $X$ whenever $A$ is rc-Lindelöf in $Y$ provided that $X$ is a weak P-space.

The following theorem shows that the assumption weakly almost open of Theorem 4.4 is not essential for the inverse preservation of the rc-Lindelöf set property.

**Theorem 4.5.** Let $f$ be a regular open function from a space $X$ onto a space $Y$. Then the following hold.

(i) If for each $y \in Y$, $f^{-1}(y)$ is an $S$-set in $X$, then $f^{-1}(A)$ is rc-Lindelöf in $X$ whenever $A$ is rc-Lindelöf in $Y$.

(ii) If for each $y \in Y$, $f^{-1}(y)$ is rc-Lindelöf in $X$, then $f^{-1}(A)$ is rc-Lindelöf in $X$ whenever $A$ is rc-Lindelöf in $Y$ provided that $X$ is a weak P-space.

The proof of the following proposition is straightforward and thus omitted.

**Proposition 4.6.** Let $X$ be a nearly Lindelöf space and $Y$ a weak P-space. Then the projection function $p : X \times Y \to Y$ sends regular closed sets onto closed sets.

**Corollary 4.7.** Let $X$, $Y$ be two spaces such that $Y$ is rc-Lindelöf and $X \times Y$ is extremally disconnected. Then the following hold.

(i) If $X$ is compact, then $X \times Y$ is rc-Lindelöf [2].

(ii) If $X$ is Lindelöf, then $X \times Y$ is rc-Lindelöf provided that $X \times Y$ is a weak P-space.

**Proof.** We will show (ii), the other part is similar. Consider the projection function $p : X \times Y \to Y$. Since $X \times Y$ is a weak P-space, it follows that $Y$ is a weak P-space, but $X$ is Lindelöf and thus nearly Lindelöf, so by Proposition 4.6, $p : X \times Y \to Y$ sends regular closed sets onto closed sets, but $X \times Y$ is extremally disconnected, so every regular open subset of $X \times Y$ is regular closed and thus $p : X \times Y \to Y$ sends regular open sets onto closed sets, but $p$ is an open function, so $p$ is regular open. Also for each $y \in Y$, $p^{-1}(y) = X \times \{ y \}$ is rc-Lindelöf in $X \times Y$ (as $X$ is Lindelöf and $X \times Y$ is extremally disconnected). Finally, since $Y$ is rc-Lindelöf, it follows immediately from Theorem 4.5(ii) that $X \times Y$ is rc-Lindelöf.

The following result is an improvement of Corollary 4.7, it follows from Theorem 1.2, Proposition 1.4, Corollary 4.7, and the fact that the properties of being extremally disconnected (a weak P-space) are hereditary with respect to open subsets.

**Corollary 4.8.** Let $X$, $Y$ be two rc-Lindelöf spaces such that $X \times Y$ is extremally disconnected. Then the following hold.

(i) If $X$ is locally compact, that is, for each $x \in X$, there exists an open set $U_x$ containing $x$ such that $\overline{U_x}$ is compact, then $X \times Y$ is rc-Lindelöf.

(ii) If $X$ is locally Lindelöf, that is, for each $x \in X$, there exists an open set $U_x$ containing $x$ such that $\overline{U_x}$ is Lindelöf, then $X \times Y$ is rc-Lindelöf provided that $X \times Y$ is a weak P-space.
Acknowledgment

The author is grateful to the referee for his/her careful reading of the manuscript and for the valuable suggestions.

References


Mohammad S. Sarsak: Department of Mathematics, Faculty of Science, The Hashemite University, P.O. Box 150459, Zarqa 13115, Jordan
E-mail address: sarsak@hu.edu.jo