Approimation of Signals (Functions) Belonging to the Weighted $W(L_p, \xi(t))$-Class by Linear Operators

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In memory of Professor Brian Kuttner (1908–1992)

Mittal and Rhoades (1999–2001) and Mittal et al. (2006) have initiated the studies of error estimates $E_n(f)$ through trigonometric Fourier approximations (TFA) for the situations in which the summability matrix $T$ does not have monotone rows. In this paper, we determine the degree of approximation of a function $\tilde{f}$, conjugate to a periodic function $f$ belonging to the weighted $W(L_p, \xi(t))$-class ($p \geq 1$), where $\xi(t)$ is nonnegative and increasing function of $t$ by matrix operators $T$ (without monotone rows) on a conjugate series of Fourier series associated with $f$. Our theorem extends a recent result of Mittal et al. (2005) and a theorem of Lal and Nigam (2001) on general matrix summability. Our theorem also generalizes the results of Mittal, Singh, and Mishra (2005) and Qureshi (1981–1982) for Nörlund ($N_p$)-matrices.

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1. Introduction

Let $f$ be $2\pi$-periodic function (signal) in $L_1[-\pi, \pi]$. The Fourier series associated with $f$ at a point $x$ is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x)$$

(1.1)

with partial sums $s_n(f;x)$. The conjugate series of Fourier series (1.1) of $f$ is given by

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x)$$

(1.2)

with partial sums $\tilde{s}_n(f;x)$. Throughout this paper, we will call (1.2) as conjugate Fourier series of function $f$.

Define for all $n \geq 0$,

$$t_n(f;x) = \sum_{k=0}^{n} a_{n,k} s_k(f;x),$$

(1.3)
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where $T \equiv (a_{n,k})$ is a linear operator represented by the infinite lower triangular matrix. The series (1.1) is said to be $T$-summable to $s$ if $t_n(f; x) \to s$ as $n \to \infty$. The $T$-operator reduces to the Nörlund $N_p$-operator if

$$a_{n,k} = \frac{p_{n-k}}{p_n}, \quad 0 \leq k \leq n,$$

$$= 0, \quad k > n,$$

(1.4)

where $p_n = \sum_{k=0}^{n} p_k \neq 0$ and $p_{-1} = 0 = p_{-1}$. In this case, the transform $t_n(f; x)$ reduces to the Nörlund transform $N_n(f; x)$.

A linear operator $T$ is said to be regular if it is limit-preserving over $c$, the space of convergent sequences. Each matrix $T$ in this paper has nonnegative entries with row sums one. If $\lim_{n \to \infty} a_{n,k} = 0$, for each $k$, then $T$ is regular.

The $L_p$-norm is defined by

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p dx\right)^{1/p}, \quad p \geq 1,$$

$$\|f\|_\infty = \sup_{x \in [0, 2\pi]} |f(x)|,$$

(1.5)

and the degree of approximation $E_n(f)$ is given by

$$E_n(f) = \min_n \|f(x) - T_n(x)\|_p,$$

(1.6)

in terms of $n$, where $T_n(x)$ is a trigonometric polynomial of degree $n$. This method of approximation is called trigonometric Fourier approximation (TFA).

A function $f \in \text{Lip} \alpha$ if

$$|f(x + t) - f(x)| = O(|t|^\alpha) \quad \text{for } 0 < \alpha \leq 1,$$

(1.7)

and $f(x) \in \text{Lip}(\alpha, p)$, for $0 \leq x \leq 2\pi$, if

$$\omega_p(t; f) = \left(\int_0^{2\pi} |f(x + t) - f(x)|^p dx\right)^{1/p} = O(|t|^\alpha), \quad 0 < \alpha \leq 1, \ p \geq 1.$$

(1.8)

Given a positive increasing function $\xi(t)$ and $p \geq 1$, $f(x) \in \text{Lip}(\xi(t), p)$ if

$$\omega_p(t; f) = \left(\int_0^{2\pi} |f(x + t) - f(x)|^p dx\right)^{1/p} = O(\xi(t)),$$

(1.9)

and $f(x) \in W(L_p, \xi(t))$ if

$$\left(\int_0^{2\pi} |[f(x + t) - f(x)] \sin^\beta x|^p dx\right)^{1/p} = O(\xi(t)), \quad (\beta \geq 0).$$

(1.10)

If $\beta = 0$, our newly defined class $W(L_p, \xi(t))$ coincides with the class $\text{Lip}(\xi(t), p)$. We observe that

$$\text{Lip} \alpha \subseteq \text{Lip}(\alpha, p) \subseteq \text{Lip}(\xi(t), p) \subseteq W(L_p, \xi(t)) \quad \text{for } 0 < \alpha \leq 1, \ p \geq 1.$$
We write
\[
\psi(t) = \psi_x(t) = 2^{-1}[f(x+t) - f(x-t)], \quad W_n = |P_n^{-1}| \sum_{k=1}^{n} k |p_k - p_{k-1}|,
\]

\[
W_n(r) = \sum_{k=0}^{r} (k+1) |\Delta_k a_{n,k}|, \quad 0 \leq r \leq n, \quad J(n,t) = \sum_{k=0}^{n} a_{n,k-1} \cos\left(n - k + \frac{1}{2}\right)t,
\]

\[
\pi \tilde{f}(x) = \int_{0}^{\pi} \psi(t) \cot\left(\frac{t}{2}\right) dt = \lim_{h \to 0} \int_{h}^{\pi} \psi(t) \cot\left(\frac{t}{2}\right) dt, \quad A_{n,k} = \sum_{r=k}^{n} a_{n,r},
\]

\[
V_{n,k} = \frac{(n-k+1)a_{n,k}}{A_{n,k}}, \quad K(n,t) = \frac{J(n,t)}{\sin(t/2)},
\]

\[
\tau = \left[ \frac{1}{t} \right], \text{ the integral part of } \frac{1}{t}, \quad \Delta_k a_{n,k} = a_{n,k} - a_{n,k+1}.
\] (1.12)

Furthermore, \( C \) denotes an absolute positive constant, not necessarily the same at each occurrence.

2. Previous results

Qureshi [12] has proved a theorem on the degree of approximation of a function \( \tilde{f}(x) \), conjugate to a periodic function \( f(x) \) with period \( 2\pi \) and belonging to the class \( \text{Lip}_\alpha \), for \( 0 < \alpha < 1 \), by \( N_p \)-means of its conjugate Fourier series. He has proved the following theorem.

**Theorem 2.1** [12]. If the sequence \( \{p_n\} \) satisfies the conditions
\[
\begin{align*}
(i) & \quad n |p_n| < C |P_n|, \\
(ii) & \quad W_n < C,
\end{align*}
\] (2.1)

then the degree of approximation of a function \( \tilde{f}(x) \), conjugate to a periodic function \( f(x) \) with period \( 2\pi \) and belonging to the class \( \text{Lip}_\alpha \), for \( 0 < \alpha < 1 \), by \( N_p \)-means of its conjugate Fourier series, is given by
\[
|\tilde{f}(x) - \tilde{t}_n(x)| = O\left(P_n^{-1} \sum_{k=1}^{n} \frac{P_k}{k^{\alpha+1}}\right),
\] (2.2)

where \( \tilde{t}_n(x) \) is the \( N_p \)-mean of conjugate Fourier series (1.2).

**Remark 2.2.** Qureshi [12] has taken \( p_n \geq 0 \), so conditions (2.1) can be stated without modulus sign.

Generalizing Theorem 2.1 of Qureshi [12], many interesting results have been proved by various investigators such as Qureshi [13, 14], Lal and Nigam [2], Mittal et al. [8, 9] for functions of various classes (defined above) using \( N_p \)-matrices and general summability matrices.
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Qureshi [13, 14] has extended his Theorem 2.1 for the functions of classes Lip($\alpha, p$) and Weighted, that is, $W(L_p, \xi(t))(p \geq 1)$ respectively, using monotonicity on the elements of $N_p$-matrix. For the $W(L_p, \xi(t))$-class, he proved the following theorem.

**Theorem 2.3** [14]. If a $2\pi$-periodic function $f$ belongs to the class $W(L_p, \xi(t))$, then its degree of approximation by Nörlund means of a conjugate Fourier series is given by

$$||\hat{f}(x) - \tilde{t}_n(x)||_p = O\left(n^{\delta + 1/p} \xi\left(\frac{1}{n}\right)\right)$$

provided that $\xi(t)$ satisfies the conditions

$$\int_0^{\pi/n} \left(\frac{t|\psi(t)|}{\xi(t)}\right)^p \sin^{\beta_p} t \, dt = O\left(\frac{1}{n}\right), \quad (2.4)$$

$$\int_{\pi/n}^{\pi} \left(\frac{t-\delta |\psi(t)|}{\xi(t)}\right)^p \, dt = O(n^\delta), \quad (2.5)$$

where $\delta$ is an arbitrary number such that $q(1 - \delta) - 1 > 0$. Conditions (2.4) and (2.5) hold uniformly in $x$ and

$$\int_0^{\pi/n} \left(\frac{\xi(t)}{t^{2+\beta}}\right)^q \, dt = O\left(n^{\delta + 1/p + 1/q} \xi\left(\frac{1}{n}\right)\right), \quad (2.6)$$

where $p^{-1} + q^{-1} = 1$ and $1 \leq p \leq \infty$.

**Remark 2.4.** Qureshi [12–14] has used monotonicity on the generating sequence $\{p_n\}$ in the proof of his theorems but has not mentioned it explicitly in the statement of these theorems.

**Remark 2.5.** Condition (2.1)(i) can be dropped in Theorem 2.1, as condition (2.1)(ii) implies condition (2.1)(i) [1, page 16].

Recently Mittal et al. [8] have generalized Theorem 2.3 by taking semimonotonicity on the generating sequence $\{p_n\}$ and also by dropping the condition (2.6). They proved the following theorem.

**Theorem 2.6** [8]. The degree of approximation of a function $\tilde{f}$, conjugate to a $2\pi$-periodic function $f$ belonging to weighted class $W(L_p, \xi(t))(p \geq 1)$ by $N_p$-means of its conjugate Fourier series (1.2), is given by

$$||\tilde{f}(x) - \tilde{t}_n(x)||_p = O\left(n^{\delta + 1/p} \xi\left(\frac{1}{n}\right)\right), \quad (2.7)$$

provided that $\{p_n\}$ satisfies (2.1)(ii) and $\xi(t)$ satisfies conditions (2.4) and (2.5) uniformly in $x$ and

$$\frac{\xi(t)}{t} \text{ is nonincreasing in } t, \quad (2.8)$$

where $\delta$ is an arbitrary number such that $0 \neq \delta q + 1 < q$, such that $p^{-1} + q^{-1} = 1$, $1 \leq p \leq \infty$, and $\tilde{t}_n(x)$ is the same as in Theorem 2.1.
Recently, Mittal et al. [9] extended Theorem 2.1 [12] for functions of Lip(ξ(t), p)(p ≥ 1)-class to matrix summability using semimonotonicity on the sequence \{a_{n,k}\}, which in turn generalizes a result of Lal and Nigam [2]. They proved the following theorem.

**Theorem 2.7 [9].** Let \( T \equiv (a_{n,k}) \) be an infinite regular triangular matrix with nonnegative entries such that

\[
W_n(r) = O(A_{n,n-r}), \quad 0 \leq r \leq n, \tag{2.9}
\]

then the degree of approximation of a function \( \tilde{f}(x) \), conjugate to a 2\(\pi\)-periodic function \( f(x) \) belonging to Lip(\(\xi(t), p\))-class, by using a matrix operator on its conjugate Fourier series, is given by

\[
||\tilde{f}(x) - \tilde{t}_n(x)||_p = O\left(n^{\frac{1}{p}}\xi\left(\frac{1}{n}\right)\right), \tag{2.10}
\]

provided that \( \xi(t) \) is nonnegative, increasing, and satisfies conditions (2.4), (2.5) uniformly in \( x \) and (2.8), where \( \delta \) is an arbitrary positive number such that \( q(1 - \delta) - 1 > 0 \) and \( \tilde{t}_n(x) \) are the matrix means of the conjugate Fourier series (1.2).

### 3. Main result

Mittal [3], Mittal and Rhoades [4–6], and Mittal et al. [7] have obtained many interesting results on TFA (these approximations have assumed important new dimensions due to their wide application in signal analysis [10] in general, and in digital signal processing [11] in particular, in view of the classical Shannon sampling theorem), using summability methods without monotonicity on the rows of the matrix \( T \). In this paper, we extend Theorem 2.6 to matrix (linear) operators and generalize Theorem 2.7 for functions of the weighted class \( W(L_p, \xi(t)) \). We prove the following theorem.

**Theorem 3.1.** Let \( T \equiv (a_{n,k}) \) be an infinite regular triangular matrix with nonnegative entries satisfying (2.9), then the degree of approximation of function \( \tilde{f}(x) \), conjugate to a 2\(\pi\)-periodic function \( f(x) \) belonging to class \( W(L_p, \xi(t)) \), \( p \geq 1 \), by using a matrix operator on its conjugate Fourier series, is given by

\[
||\tilde{f}(x) - \tilde{t}_n(x)|| = O\left(n^{\frac{1}{p}}\xi\left(\frac{1}{n}\right)\right) \tag{3.1}
\]

provided that \( \xi(t) \) satisfies (2.4) and (2.5) uniformly in \( x \), in which \( \delta \) is an arbitrary positive number with \( q(1 - \delta) - 1 > 0 \), where \( p^{-1} + q^{-1} = 1 \), \( 1 \leq p \leq \infty \) and condition (2.8) holds.

**Note 3.2.** In the case of the \( N_p \)-transform, condition (2.9), for \( r = n \), reduces to (2.1)(ii) and thus Theorem 3.1 extends Theorem 2.6 to matrix summability for the weighted class functions.

**Note 3.3.** Also for \( \beta = 0 \), Theorem 3.1 reduces to Theorem 2.7, and thus generalizes the theorem of Lal and Nigam [2].
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4. Lemmas

**Lemma 4.1** [3]. Let $T \equiv (a_{n,k})$ be an infinite triangular matrix satisfying (2.9). Then

$$V_{n,k} = O(1), \quad 0 \leq k \leq n. \quad (4.1)$$

For $k = 0$,

$$V_{n,0} = O(1), \quad (4.2)$$

that is,

$$(n + 1)a_{n,0} = O(1). \quad (4.3)$$

**Lemma 4.2** [9]. Let $T \equiv (a_{n,k})$ be an infinite triangular matrix satisfying (2.9). Then

$$\left| J(n,t) \right| = O(A_{n,n-\tau}) + O(t^{-1}) \left( \sum_{k=\tau}^{n-1} \left| \Delta_k a_{n,n-k} \right| + a_{n,0} \right). \quad (4.4)$$

5. Proof of Theorem 3.1

It is well known that

$$\tilde{s}_{n-k}(x) - \tilde{f}(x) = -\frac{1}{2\pi} \int_0^\pi \psi(t) \left( \frac{\cos(n-k+1/2)t}{\sin t/2} \right) dt,$$

$$\tilde{f}(x) - \tilde{\tau}_n(x) = \sum_{k=0}^n a_{n,n-k}\left\{ \tilde{f}(x) - \tilde{s}_{n-k}(x) \right\}$$

$$= \frac{1}{2\pi} \int_0^\pi \psi(t) \left( \sum_{k=0}^n a_{n,n-k} \frac{\cos(n-k+1/2)t}{\sin t/2} \right) dt. \quad (5.1)$$

Therefore,

$$\left| \tilde{f}(x) - \tilde{\tau}_n(x) \right| \leq \frac{1}{2\pi} \int_0^\pi \left| \psi(t) \right| \left| \tilde{K}(n,t) \right| dt$$

$$= \frac{1}{2\pi} \left( \int_0^{\pi/n} + \int_{\pi/n}^\pi \right) \left| \psi(t) \right| \left| \tilde{K}(n,t) \right| dt = \frac{1}{2\pi} [I_1 + I_2], \quad \text{say.} \quad (5.2)$$
Using Hölder’s inequality, condition (2.4), and the fact that 
\( (\sin t)^{-1} \leq \pi/2t \), for \( 0 < t \leq \pi/2 \), and the second mean value theorem for integrals,

\[
I_1 = \int_0^{\pi/n} |\psi(t)| |\mathcal{K}(n,t)| \, dt \\
= O\left( \int_0^{\pi/n} \left\{ \frac{t \psi(t) \sin^\beta t}{\xi(t)} \right\}^p \, dt \right)^{1/p} \left\{ \int_0^{\pi/n} \frac{\xi(t)|\mathcal{K}(n,t)|}{t \sin^\beta t} \, dt \right\}^{1/q}
\]

\begin{equation}
= O\left( \int_0^{\pi/n} \left\{ \frac{\xi(t)}{t^{2+\beta}} \right\}^q \, dt \right)^{1/q}.
\end{equation}

Since \( \xi(t) \) is nondecreasing with \( t \) and also using condition (2.8),

\[
I_1 = O\left( \frac{1}{n} \right) O\left( \frac{\pi}{n} \right) \left( \int_0^{\pi/n} t^{-(2+\beta)q} \, dt \right)^{1/q} \\
= O\left( n^{-1} \xi\left( \frac{1}{n} \right) \right) O(n^{2+\beta-1/q}) = O\left( n^{\beta+1/p} \xi\left( \frac{1}{n} \right) \right).
\end{align}

Using Hölder’s inequality, condition (2.5), Lemma 4.2, Minkowski’s inequality, and condition (2.8),

\[
I_2 = \int_0^{\pi/n} |\psi(t)| |\mathcal{K}(n,t)| \, dt \\
\leq \left\{ \int_0^{\pi/n} \left\{ \frac{t \psi(t) \sin^\beta t}{\xi(t)} \right\}^p \, dt \right\}^{1/p} \left\{ \int_0^{\pi/n} \frac{\xi(t)|\mathcal{K}(n,t)|}{t \sin^\beta t} \, dt \right\}^{1/q}
\]

\[
= O\left( n^\delta \right) \left\{ \int_0^{\pi/n} \left\{ \frac{\xi(t)}{t^{\delta+1+\beta}} \right\}^q \, dt \right\} \left\{ \int_0^{\pi/n} \frac{A_{n,n-k} + t^{-1} \left( a_{n,0} + \sum_{k=\tau}^{n-1} \Delta_k a_{n,n-k} \right)}{t \sin^\beta t \sin(t/2)} \, dt \right\}^{1/q}
\]

\begin{equation}
= O\left( n^{\delta+1} \xi\left( \frac{\pi}{n} \right) \right) O\left[ \left\{ \int_0^{\pi/n} \left( t^{\delta-1} A_{n,n-k} \right)^q \, dt \right\}^{1/q} + \left\{ \int_0^{\pi/n} \left( t^{\delta-1} a_{n,0} \right)^q \, dt \right\}^{1/q} \right. \\
\left. + \left\{ \int_0^{\pi/n} \left( t^{\delta-1-\beta} \sum_{k=\tau}^{n-1} \Delta_k a_{n,n-k} \right)^q \, dt \right\}^{1/q} \right]
\end{equation}

\[
= O\left( n^{\delta+1} \xi\left( \frac{1}{n} \right) \right) [I_{2,1} + I_{2,2} + I_{2,3}], \text{ say.}
\]
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Since $A$ has nonnegative entries and row sums one,

$$I_{2,1} = \left\{ \int_{\pi/n}^{\pi} (t^{\delta-\beta} A_{n,n-\tau})^q dt \right\}^{1/q} = O\left( \int_{\pi/n}^{\pi} t^{(\delta-\beta)q} dt \right)^{1/q} = O(n^{\beta-\delta-1/q}). \quad (5.6)$$

Using Lemma 4.1,

$$I_{2,2} = O\left( \int_{\pi/n}^{\pi} (t^{\delta-\beta} a_{n,0})^q dt \right)^{1/q} = O(a_{n,0}) \left\{ \int_{\pi/n}^{\pi} t^{(\delta-1-\beta)q} dt \right\}^{1/q} = O(n^{-1}) \left( n^{\beta-\delta-1-1/q} \right) = O(n^{\beta-\delta-1/q}). \quad (5.7)$$

Finally from condition (2.9),

$$I_{2,3} = \left\{ \int_{\pi/n}^{\pi} \left( t^{\delta-\beta} \sum_{k=\tau}^{n-1} |\Delta_k a_{n,n-k}| \right)^q dt \right\}^{1/q} = O\left( \int_{\pi/n}^{\pi} t^{\delta-\beta (\tau+1)} \sum_{k=\tau}^{n-1} |\Delta_k a_{n,n-k}| \right)^q dt \right\}^{1/q} = O\left( \int_{\pi/n}^{\pi} t^{\delta-\beta} W_n(n) \right)^q dt \right\}^{1/q} = O(n^{\beta-\delta-1/q}). \quad (5.8)$$

Combining (5.6), (5.7), and (5.8),

$$I_2 = O\left( n^{\beta+1/p} \xi(n^{1/p}) \right) \left( n^{\beta-\delta-1/q} \right) = O(n^{\beta+1/p} \xi(1/n)). \quad (5.9)$$

Combining $I_1$ and $I_2$ yields

$$| \tilde{f}(x) - \tilde{f}_n(x) | = O\left( n^{\beta+1/p} \xi(n^{1/p}) \right). \quad (5.10)$$

Now, using the $L_p$-norm, we get

$$||\tilde{f}(x) - \tilde{f}_n(x)||_p = \left\{ \int_0^{2\pi} |\tilde{f}(x) - \tilde{f}_n(x)|^p dx \right\}^{1/p} = O\left( \int_0^{2\pi} \left( n^{\beta+1/p} \xi(n^{1/p}) \right)^p dx \right)^{1/p} = O\left[ n^{\beta+1/p} \xi(1/n) \right]. \quad (5.11)$$
6. Applications

The following corollaries can be derived from Theorem 3.1.

**Corollary 6.1** [13]. If \( \xi(t) = t^\alpha \), \( 0 < \alpha \leq 1 \), then the weighted class \( W(L_p, \xi(t)) \), \( p \geq 1 \), reduces to the class \( \text{Lip}(\alpha, p) \) and the degree of approximation of a function \( \tilde{f}(x) \), conjugate to a \( 2\pi \)-periodic function \( f \) belonging to the class \( \text{Lip}(\alpha, p) \), is given by

\[
| \tilde{f}_n(x) - \tilde{f}(x) | = O\left( \frac{1}{n^{\alpha - 1/p}} \right).
\]

(6.1)

**Proof.** The result follows by setting \( \beta = 0 \) in (3.1).

**Corollary 6.2** [12]. If \( \xi(t) = t^\alpha \) for \( 0 < \alpha < 1 \) and \( p = \infty \) in Corollary 6.1, then \( f \in \text{Lip} \alpha \).

In this case, using (6.1), one has Theorem 2.1.

**Proof.** For \( p = \infty \), we get

\[
\| \tilde{f}(x) - \tilde{f}_n(x) \|_{\infty} = \sup_{0 \leq x \leq 2\pi} | \tilde{f}(x) - \tilde{f}_n(x) | = O\left( \frac{1}{n^\alpha} \right)
\]

(6.2)

that is,

\[
| \tilde{f}(x) - \tilde{f}_n(x) | = O\left( \frac{1}{n^{\alpha - 1}} \sum_{k=1}^{n} \frac{P_k}{k^{\alpha+1}} \right).
\]

(6.3)

**References**


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