We give a new Hilbert-type integral inequality with the best constant factor by estimating the weight function. And the equivalent form is considered.

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1. Introduction

If $f, g$ are real functions such that $0 < \int_0^\infty f^2(x)\,dx < \infty$ and $0 < \int_0^\infty g^2(x)\,dx < \infty$, then we have (see [1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi \left\{ \int_0^\infty f^2(x)\,dx \int_0^\infty g^2(x)\,dx \right\}^{1/2}, \quad (1.1)$$

where the constant factor $\pi$ is the best possible. Inequality (1.1) is the well-known Hilbert’s inequality. And inequality (1.1) had been generalized by Hardy in 1925 as follows.

If $f, g \geq 0, p > 1, 1/p + 1/q = 1, 0 < \int_0^\infty f^p(x)\,dx < \infty$, and $0 < \int_0^\infty g^q(x)\,dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)} \left\{ \int_0^\infty f^p(x)\,dx \right\}^{1/p} \left\{ \int_0^\infty g^q(x)\,dx \right\}^{1/q}, \quad (1.2)$$

$$\int_0^\infty \left( \int_0^\infty \frac{f(x)}{x+y}\,dx \right)^p \,dy < \left[ \frac{\pi}{\sin(\pi/p)} \right]^p \int_0^\infty f^p(x)\,dx, \quad (1.3)$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. When $p = q = 2$, (1.2) reduces to (1.1), inequality (1.2) is named of Hardy-Hilbert integral inequality, which is important in analysis and its applications. It has been studied and generalized in many directions by a number of mathematicians.
A new Hilbert-type integral inequality

In this paper, we give a new type of Hilbert’s integral inequality as follows:

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} \, dx \, dy < c \left\{ \int_0^\infty f^2(x) \, dx \int_0^\infty g^2(x) \, dx \right\}^{1/2},
\]

(1.4)

where \( c = \sqrt{2}(\pi - 2 \arctan \sqrt{2}) = 1.7408 \ldots \).

2. Main results

Lemma 2.1. Suppose \( \varepsilon > 0 \), then

\[
\int_1^{x^{-\varepsilon-1}} \int_0^{x^{-1}} \frac{1}{1 + t + \max\{1,t\}} t^{(-1-\varepsilon)/2} \, dt \, dx = O(1) (\varepsilon \to 0^+). \tag{2.1}
\]

Proof. There exists \( n \in \mathbb{N} \) which is large enough, such that \( 1 + (1 - \varepsilon)/2 > 0 \) for \( \varepsilon \in (0,1/n] \), we have

\[
\int_0^{x^{-1}} \frac{1}{1 + t + \max\{1,t\}} t^{(-1-\varepsilon)/2} \, dt < \int_0^{x^{-1}} t^{(-1-\varepsilon)/2} \, dt = \frac{1}{1 + (1 - \varepsilon)/2} \left( \frac{1}{x} \right)^{1+(1-\varepsilon)/2}. \tag{2.2}
\]

Since for \( a \geq 1 \) the function \( g(y) = (1/ya^y) (y \in (0,\infty)) \) is decreasing, we find

\[
\frac{1}{1 + (1 - \varepsilon)/2} \left( \frac{1}{x} \right)^{1+(1-\varepsilon)/2} \leq \frac{1}{1 + ((1 - 1)/n)/2} \left( \frac{1}{x} \right)^{1+((1-1)/n)/2}, \tag{2.3}
\]

so

\[
0 < \int_1^{x^{-\varepsilon-1}} \int_0^{x^{-1}} \frac{1}{1 + t + \max\{1,t\}} t^{(-1-\varepsilon)/2} \, dt \, dx
\]

\[
< \int_1^{x^{-1}} \frac{1}{1 + ((1 - 1)/n)/2} \left( \frac{1}{x} \right)^{1+((1-1)/n)/2} \, dx \tag{2.4}
\]

\[
= \left( \frac{1}{1 + ((1 - 1)/n)/2} \right)^2.
\]

Hence the relation (2.1) is valid. The lemma is proved.

Now we study the following inequality.

Theorem 2.2. Suppose \( f(x), g(x) \geq 0, 0 < \int_0^\infty f^2(x) \, dx < \infty, 0 < \int_0^\infty g^2(x) \, dx < \infty \). Then

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} \, dx \, dy < c \left\{ \int_0^\infty f^2(x) \, dx \int_0^\infty g^2(x) \, dx \right\}^{1/2}, \tag{2.5}
\]

where the constant factor \( c = \sqrt{2}(\pi - 2 \arctan \sqrt{2}) = 1.7408 \ldots \) is the best possible.
Proof. By Hölder’s inequality, we have

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y + \max\{x, y\}} \, dx \, dy
\]

\[
= \int_0^\infty \int_0^\infty \left[ \frac{f(x)}{(x + y + \max\{x, y\})^{1/2}} \left( \frac{x}{y} \right)^{1/4} \right] \times \left[ \frac{g(y)}{(x + y + \max\{x, y\})^{1/2}} \left( \frac{y}{x} \right)^{1/4} \right] \, dx \, dy
\]

\[
\leq \int_0^\infty \int_0^\infty \frac{f^2(x)}{x + y + \max\{x, y\}} \left( \frac{x}{y} \right)^{1/2} \, dx \, dy
\]

\[
\times \int_0^\infty \int_0^\infty \frac{g^2(y)}{x + y + \max\{x, y\}} \left( \frac{y}{x} \right)^{1/2} \, dx \, dy.
\]

(2.6)

Define the weight function \( \varpi(u) \) as

\[
\varpi(u) := \int_0^\infty \frac{1}{u + v + \max\{u, v\}} \left( \frac{u}{v} \right)^{1/2} \, dv,
\]

(2.7)

then the above inequality yields

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y + \max\{x, y\}} \, dx \, dy
\]

\[
\leq \left[ \int_0^\infty \varpi(x) f^2(x) \, dx \right]^{1/2} \left[ \int_0^\infty \varpi(y) g^2(y) \, dy \right]^{1/2}.
\]

(2.8)

For fixed \( u \), let \( v = ut \), we have

\[
\varpi(u) := \int_0^\infty \frac{1}{1 + t + \max\{1, t\}} \left( \frac{1}{t} \right)^{1/2} \, dt
\]

\[
= \int_0^1 \frac{1}{2t + \left( \frac{1}{t} \right)} \left( \frac{1}{t} \right)^{1/2} \, dt + \int_1^\infty \frac{1}{1 + 2t} \left( \frac{1}{t} \right)^{1/2} \, dt
\]

\[
= \sqrt{2}(\pi - 2\arctan \sqrt{2}).
\]

(2.9)

Thus

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y + \max\{x, y\}} \, dx \, dy
\]

\[
\leq \sqrt{2}(\pi - 2\arctan \sqrt{2}) \left\{ \int_0^\infty f^2(x) \, dx \right\}^{1/2} \left\{ \int_0^\infty g^2(x) \, dx \right\}^{1/2}.
\]

(2.10)
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If (2.10) takes the form of the equality, then there exist constants \(a\) and \(b\), such that they are not all zero and

\[
a \frac{f^2(x)}{x + y + \max\{x, y\}} \left(\frac{x}{y}\right)^{1/2} = b \frac{g^2(y)}{x + y + \max\{x, y\}} \left(\frac{y}{x}\right)^{1/2}
\]

a.e. on \((0, \infty) \times (0, \infty)\).

Then we have

\[
ax^2(x) = byg^2(y) \quad \text{a.e. on } (0, \infty) \times (0, \infty).
\]

Hence we have

\[
ax^2(x) = byg^2(y) = \text{constant} = d \quad \text{a.e. on } (0, \infty) \times (0, \infty).
\]

Without losing the generality, suppose \(a \neq 0\), then we obtain \(f^2(x) = d/ax\), a.e. on \((0, \infty)\), which contradicts the fact that \(0 < \int_0^\infty f^2(x)dx < \infty\). Hence (2.10) takes the form of strict inequality; we get (2.5).

For \(0 < \varepsilon < 1\), set \(f_\varepsilon(x) = x^{(-\varepsilon-1)/2}\), for \(x \in [1, \infty)\); \(f_\varepsilon(x) = 0\), for \(x \in (0, 1)\). \(g_\varepsilon(y) = y^{(-\varepsilon-1)/2}\), for \(y \in [1, \infty)\); \(g_\varepsilon(y) = 0\), for \(y \in (0, 1)\). Assume that the constant factor \(c = \sqrt{2(\pi - 2 \arctan \sqrt{2})}\) in (2.2) is not the best possible, then there exists a positive number \(K\) with \(K < c\), such that (2.5) is valid by changing \(c\) to \(K\). We have

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y + \max\{x, y\}} dxdy < K \left\{ \int_0^\infty f^2(x)dx \right\}^{1/2} \left\{ \int_0^\infty g^2(y)dx \right\}^{1/2} = \frac{K}{\varepsilon},
\]

since

\[
\int_0^\infty \frac{1}{1 + t + \max\{1, t\}} t^{(-1-\varepsilon)/2} dt = \sqrt{2(\pi - 2 \arctan \sqrt{2})} + o(1) \quad (\varepsilon \to 0^+).
\]

Setting \(y = tx\), by (2.1), we find

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y + \max\{x, y\}} dxdy
\]

\[
= \int_1^\infty \int_1^\infty \frac{x^{(-\varepsilon-1)/2} y^{(-\varepsilon-1)/2}}{x + y + \max\{x, y\}} dxdy
\]

\[
= \int_1^\infty \int_1^\infty \frac{x^{(-\varepsilon-1)/2} (tx)^{(-\varepsilon-1)/2}}{1 + t + \max\{1, t\}} dxdt
\]

\[
= \int_1^\infty x^{(-\varepsilon-1)} \left( \int_0^\infty \frac{1}{1 + t + \max\{1, t\}} t^{(-1-\varepsilon)/2} dt - \int_0^{x^{-1}} 1 \frac{1}{1 + t + \max\{1, t\}} t^{(-1-\varepsilon)/2} dt \right) dx
\]

\[
= \frac{1}{\varepsilon} \left[ \sqrt{2(\pi - 2 \arctan \sqrt{2})} + o(1) \right].
\]
Since, for $\varepsilon > 0$ small enough, we have
\[ \sqrt{2}(\pi - 2 \arctan \sqrt{2}) + o(1) < K, \] (2.17)
thus we get $\sqrt{2}(\pi - 2 \arctan \sqrt{2}) \leq K$, then $c \leq K$, which contradicts the hypothesis. Hence the constant factor $c$ in (2.5) is the best possible. \(\square\)

**Theorem 2.3.** Suppose $f \geq 0$ and $0 < \int_0^\infty f^2(x) \, dx < \infty$. Then
\[ \int_0^\infty \left[ \int_0^\infty \frac{f(x)}{x + y + \max\{x,y\}} \, dx \right]^2 \, dy < c^2 \int_0^\infty f^2(x) \, dx, \] (2.18)
where the constant factor $c^2 = 2(\pi - 2 \arctan \sqrt{2})^2 = 3.0305\ldots$ is the best possible. Inequality (2.18) is equivalent to (2.5).

**Proof.** Setting $g(y)$ as
\[ \int_0^\infty \frac{f(x)}{x + y + \max\{x,y\}} \, dx, \quad y \in (0,\infty), \] (2.19)
then by (2.5), we find
\[
0 < \int_0^\infty g^2(y) \, dy = \int_0^\infty \left[ \int_0^\infty \frac{f(x)}{x + y + \max\{x,y\}} \, dx \right]^2 \, dy \\
= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y + \max\{x,y\}} \, dx \, dy \\
\leq \sqrt{2}(\pi - 2 \arctan \sqrt{2}) \left\{ \int_0^\infty f^2(x) \, dx \right\}^{1/2} \left\{ \int_0^\infty g^2(y) \, dy \right\}^{1/2}. \] (2.20)
Hence we obtain
\[ 0 < \int_0^\infty g^2(y) \, dy \leq 2(\pi - 2 \arctan \sqrt{2})^2 \int_0^\infty f^2(x) \, dx < \infty. \] (2.21)
By (2.5), both (2.20) and (2.21) take the form of strict inequality, so we have (2.18).

On the other hand, suppose that (2.18) is valid. By Hölder’s inequality, we find
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y + \max\{x,y\}} \, dx \, dy = \int_0^\infty \left[ \int_0^\infty \frac{f(x)}{x + y + \max\{x,y\}} \, dx \right] g(y) \, dy \\
\leq \left\{ \int_0^\infty \left[ \int_0^\infty \frac{f(x)}{x + y + \max\{x,y\}} \, dx \right]^2 \, dy \right\}^{1/2} \left\{ \int_0^\infty g^2(y) \, dy \right\}^{1/2}. \] (2.22)
Then by (2.18), we have (2.5). Thus (2.5) and (2.18) are equivalent.
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If the constant \( c^2 = 2(\pi - 2 \arctan \sqrt{2})^2 \) in (2.18) is not the best possible, by (2.22), we may get a contradiction that the constant factor \( c \) in (2.5) is not the best possible. Thus we complete the proof of the theorem.

Acknowledgment

The authors would like to thank the anonymous referees for their suggestions and corrections.

References