LERAY-SCHAUDER RESULTS FOR MULTIVALUED NONLINEAR CONTRACTIONS DEFINED ON CLOSED SUBSETS OF A FRÉCHET SPACE

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New Leray-Schauder results are presented for multivalued contractions defined on subsets of a Fréchet space $E$. The proof relies on fixed point results in Banach spaces and on viewing $E$ as the projective limit of a sequence of Banach spaces.

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1. Introduction

In this paper, we present new fixed point results for nonlinear contractions (both single and multivalued) defined on subsets $X$ (which may have empty interior) of a Fréchet space $E$. Some results for single-valued maps were presented in [2, 3] and the approach in these papers was based on constructing a specific map $F_n$ (for each $n \in \mathbb{N} = \{1, 2, \ldots \}$) whose fixed points converge to a fixed point of the original operator $F$. In the approach in this paper, the maps $\{F_n\}_{n \in \mathbb{N}}$ only need to satisfy a closure property and are specified in a completely different way. The advantage of this approach is that multivalued maps can also be discussed. Our theory is based on results in Banach spaces and on viewing a Fréchet space $E$ as a projective limit of a sequence of Banach spaces $\{E_n\}_{n \in \mathbb{N}}$.

For the remainder of this section, we present some definitions and some known facts. Let $(X,d)$ be a metric space and $S$ a nonempty subset of $X$. For $x \in X$, let $d(x,S) = \inf_{y \in S} d(x,y)$. Also $\text{diam}S = \sup \{d(x,y) : x, y \in S\}$. We let $B(x,r)$ denote the open ball in $X$ centered at $x$ of radius $r$ and by $B(S,r)$ we denote $\bigcup_{x \in S} B(x,r)$. For two nonempty subsets $S_1$ and $S_2$ of $X$, we define the generalized Hausdorff distance $H$ to be

$$H(S_1,S_2) = \inf \{\varepsilon > 0 : S_1 \subseteq B(S_2,\varepsilon), S_2 \subseteq B(S_1,\varepsilon)\}.$$  \hspace{1cm} (1.1)

Now suppose $G : S \to 2^X$; here $2^X$ denotes the family of nonempty subsets of $X$. Then $G$ is said to be hemicompact if each sequence $\{x_n\}_{n \in \mathbb{N}}$ in $S$ has a convergent subsequence whenever $d(x_n, G(x_n)) \to 0$ as $n \to \infty$.

We now recall a result from the literature.
2 Multivalued nonlinear contractions

Theorem 1.1. Let $(X, d)$ be a complete metric space, $C \subseteq X$ closed, and $F : C \to X$ with $F(C)$ bounded (i.e., there exists $M > 0$ with $d(z,w) \leq M$ for $z,w \in F(C)$). Suppose the following condition is satisfied:

there exists a continuous nondecreasing function

$$\phi : [0, \infty) \to [0, \infty)$$

satisfying $\phi(z) < z$ for $z > 0$ \hspace{1cm} (1.2)

such that $d(Fx,Fy) \leq \phi(d(x,y))$ for $x, y \in C$.

Then $F$ is hemicompact.

Now let $I$ be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha \in I, \beta \in I$ for which $\alpha \leq \beta$, let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha : x_\alpha = \pi_{\alpha,\beta}(x_\beta) \ \forall \alpha, \beta \in I, \ \alpha \leq \beta \right\} \tag{1.3}$$

is a closed subset of $\prod_{\alpha \in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha \in I}$ and is denoted by $\lim_\rightarrow E_\alpha$ (or $\lim_\nearrow \{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection $[5, \text{page 439}] \cap_{\alpha \in I} E_\alpha$).

Existence in Section 2 is based on the following fixed point results in the literature [1, 6].

Theorem 1.2 [6, Theorem 3.9]. Let $U$ be an open subset in a Banach space $(X, \| \cdot \|)$ and $F : \overline{U} \to X$. Assume $0 \in U$ and suppose there exists a continuous nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that $\|Fx - Fy\| \leq \phi(\|x - y\|)$ for all $x, y \in \overline{U}$. In addition, assume $F(\overline{U})$ is bounded and $x \neq \lambda Fx$ for $x \in \partial U$ and $\lambda \in (0, 1)$. Then $F$ has a fixed point in $\overline{U}$.

Theorem 1.3 [1, Theorem 2.3 (and Remark 2.1)]. Let $U$ be an open subset in a Banach space $(X, \| \cdot \|)$ and $F : \overline{U} \to C(X)$ a closed map (i.e., has closed graph); here $C(X)$ denotes the family of nonempty closed subsets of $X$. Assume $0 \in U$ and suppose there exists a continuous strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that $H(Fx,Fy) \leq \phi(\|x - y\|)$ for all $x, y \in \overline{U}$. In addition, assume the following conditions hold:

$$\Phi : [0, \infty) \to [0, \infty), \text{ given by } \Phi(x) = x - \phi(x), \text{ is strictly increasing,} \tag{1.4}$$

$$\Phi^{-1}(a) + \Phi^{-1}(b) \leq \Phi^{-1}(a + b) \quad \text{for } a, b \geq 0, \tag{1.5}$$

$$\sum_{i=0}^\infty \phi^i(t) < \infty \quad \text{for } t > 0, \tag{1.6}$$

$$\sum_{i=1}^\infty \phi^i(x - \phi(x)) \leq \phi(x) \quad \text{for } x > 0, \tag{1.7}$$

$$F(\overline{U}) \text{ is bounded}, \tag{1.8}$$

$$x \notin \lambda Fx \quad \text{for } x \in \partial U, \lambda \in (0, 1). \tag{1.9}$$

Then $F$ has a fixed point in $\overline{U}$.
Remark 1.4. In fact, the assumption that $F$ is closed can be removed in Theorem 1.3. In [1, Theorem 2.3], we assume a more general contractive condition and the map $G : \mathcal{U} \times [0, 1] \rightarrow \mathbb{C}(X)$ (given by $G(x, \lambda) = \lambda Fx$ in our situation) was assumed to be closed in order to guarantee that if $\{x_n\}_1^n \subseteq \mathcal{U}$, $\{\lambda\}_1^n \subseteq [0, 1]$ with $x_n \in G(x_n, \lambda_n)$ and $(x_n, \lambda_n) \rightarrow (x, \lambda)$, then $x \in G(x, \lambda)$. However, this is automatically true in Theorem 1.3 since the contractive condition and (1.8) guarantee that $G$ is continuous in the Hausdorff metric and as a result,

$$\text{dist}(x, G(x, \lambda)) \leq d(x, x_n) + H(G(x_n, \lambda_n), G(x, \lambda)).$$ (1.10)

Remark 1.5. If $\phi(t) = kt$, $0 \leq k < 1$, then trivially (1.2)–(1.7) hold.

2. Fixed point theory in Fréchet spaces

Let $E = (E, \{\cdot | n\}_n \in \mathbb{N})$ be a Fréchet space with the topology generated by a family of seminorms $\{\cdot | n : n \in \mathbb{N}\}$. We assume that the family of seminorms satisfies

$$|x|_1 \leq |x|_2 \leq |x|_3 \leq \cdots \quad \text{for every } x \in E. \quad (2.1)$$

A subset $X$ of $E$ is bounded if for every $n \in \mathbb{N}$ there exists $r_n > 0$ such that $|x|_n \leq r_n$ for all $x \in X$. To $E$ we associate a sequence of Banach spaces $\{\langle E_n, \cdot | n \rangle \}$ described as follows. For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \iff |x - y|_n = 0. \quad (2.2)$$

We denote by $E^n = (E/\sim_n, \cdot | n)$ the quotient space, and by $\langle E_n, \cdot | n \rangle$ the completion of $E^n$ with respect to $\cdot | n$ (the norm on $E^n$ induced by $\cdot | n$ and its extension to $E_n$ are still denoted by $\cdot | n$). This construction defines a continuous map $\mu_n : E \rightarrow E_n$. Now since (2.1) is satisfied, the seminorm $\cdot | n$ induces a seminorm on $E_m$ for every $m \geq n$ (again this seminorm is denoted by $\cdot | n$). Also (2.2) defines an equivalence relation on $E_m$ from which we obtain a continuous map $\mu_{n,m} : E_m \rightarrow E_n$ since $E_m/\sim_n$ can be regarded as a subset of $E_n$. We now assume the following condition holds: for each $n \in \mathbb{N}$, there exists a Banach space $(E_n, \cdot | n)$ and an isomorphism (between normed spaces) $j_n : E_n \rightarrow E_n$.

Remark 2.1. (i) For convenience, the norm on $E_n$ is denoted by $\cdot | n$.

(ii) Usually in applications, $E_n = E^n$ for each $n \in \mathbb{N}$.

(iii) Note that if $x \in E_n$ (or $E^n$), then $x \in E$. However, if $x \in E_n$, then $x$ is not necessarily in $E$ and in fact, $E_n$ is easier to use in applications (even though $E_n$ is isomorphic to $E_n$). For example, if $E = \mathbb{C}[0, \infty)$, then $E^n$ consists of the class of functions in $E$ which coincide on the interval $[0, n]$ and $E_n = \mathbb{C}[0, n]$.

Finally, we assume

$$E_1 \supseteq E_2 \supseteq \cdots \quad \text{and for each } n \in \mathbb{N}, \quad |x|_n \leq |x|_{n+1} \forall x \in E_{n+1}. \quad (2.3)$$

Let $\lim_{n} E_n$ (or $\bigcap_{n=1}^{\infty} E_n$ where $\bigcap_{n=1}^{\infty}$ is the generalized intersection [5]) denote the projective limit of $\{E_n\}_{n \in \mathbb{N}}$ (note that $\pi_{n,m} = j_n \mu_{n,m} j_m^{-1} : E_m \rightarrow E_n$ for $m \geq n$) and note that $\lim_{n} E_n \equiv E$, so for convenience, we write $E = \lim_{n} E_n$. 


For each $X \subseteq E$ and each $n \in \mathbb{N}$, we set $X_n = j_n \mu_n(X)$ and we let $\overline{X_n}$ and $\partial X_n$ denote, respectively, the closure and the boundary of $X_n$ with respect to $| \cdot |_n$ in $E_n$. Also the pseudo-interior of $X$ is defined by [4]

$$\text{pseudo-intt}(X) = \{ x \in X : j_n \mu_n(x) \in \overline{X_n} \setminus \partial X_n \text{ for every } n \in \mathbb{N} \}. \quad (2.4)$$

Also, here $H_n$ and $\text{diam}_n$ denote the Hausdorff metric and the diameter induced by $| \cdot |_n$ on $E_n$.

We begin with single-valued maps and present two results. The first was motivated by Volterra type operators.

**Theorem 2.2.** Let $E$ and $E_n$ be as described above and let $F : X \to E$ with $X \subseteq E$ and for each $n \in \mathbb{N}$ assume that $F : \overline{X_n} \to E_n$. Suppose the following conditions are satisfied:

(a) $0 \in \text{pseudo-intt}(X)$,
(b) for each $n \in \mathbb{N}$, $F(\overline{X_n})$ is bounded,
(c) for each $n \in \mathbb{N}$, $F : \overline{X_n} \to E_n$ and there exists a continuous nondecreasing function $\phi_n : [0, \infty) \to [0, \infty)$ satisfying $\phi_n(z) < z$ for $z > 0$ such that $|Fx - Fy|_n \leq \phi_n(|x - y|_n)$ for all $x, y \in \overline{X_n}$ for each $n \in \mathbb{N}$, $y \neq \lambda Fy$, in $E_n$ for all $\lambda \in (0, 1)$, $y \in \partial X_n$,
(d) for each $n \in \{2, 3, \ldots\}$, if $y \in \overline{X_n}$ solves $y = Fy$ in $E_n$, then $y \in \overline{X_k}$ for $k \in \{1, \ldots, n - 1\}$. Then $F$ has a fixed point in $E$.

**Remark 2.3.** If $F(X)$ is bounded, then clearly Theorem 2.2(b) holds.

**Proof.** Fix $n \in \mathbb{N}$. From Theorem 1.2, there exists $y_n \in \overline{X_n}$ with $y_n = Fy_n$ (note that $0 \in \overline{X_n} \setminus \partial X_n$ and $F(\overline{X_n})$ is bounded). Let us look at $\{y_n\}_{n \in \mathbb{N}}$. Notice that $y_1 \in \overline{X_1}$ and $y_k \in \overline{X_k}$ for $k \in \mathbb{N} \setminus \{1\}$ from Theorem 2.2(d). As a result, $y_n \in \overline{X_1}$ for $n \in \mathbb{N}$, $y_n = Fy_n$ in $E_n$ together with Theorem 1.1 implies there is a subsequence $\mathbb{N}_1^\star$ of $\mathbb{N}$ and a $z_1 \in \overline{X_1}$ with $y_n \to z_1$ in $E_1$ as $n \to \infty$ in $\mathbb{N}_1^\star$. Let $\mathbb{N}_1 = \mathbb{N}_1^\star \setminus \{1\}$. Now $y_n \in \overline{X_2}$ for $n \in \mathbb{N}_1$ together with Theorem 1.1 guarantees that there exists a subsequence $\mathbb{N}_2^\star$ of $\mathbb{N}_1$ and a $z_2 \in \overline{X_2}$ with $y_n \to z_2$ in $E_2$ as $n \to \infty$ in $\mathbb{N}_2^\star$. Note from (2.3) that $z_2 = z_1$ in $E_1$ since $\mathbb{N}_2^\star \subseteq \mathbb{N}_1$. Let $\mathbb{N}_2 = \mathbb{N}_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$\mathbb{N}_1^\star \supseteq \mathbb{N}_2^\star \supseteq \cdots,$$

$$\mathbb{N}_k^\star \subseteq \{k, k+1, \ldots\}, \quad (2.5)$$

and $z_k \in \overline{X_k}$ with $y_n \to z_k$ in $E_k$ as $n \to \infty$ in $\mathbb{N}_k^\star$. Note that $z_{k+1} = z_k$ in $E_k$ for $k \in \{1, 2, \ldots\}$. Also let $\mathbb{N}_k = \mathbb{N}_k^\star \setminus \{k\}$.

Fix $k \in \mathbb{N}$. Let $y = z_k$ in $E_k$. Notice that $y$ is well defined and $y \in \text{lim}_n E_n = E$. Now $y_n = Fy_n$ in $E_n$ for $n \in \mathbb{N}_k$ and $y_n \to y$ in $E_k$ as $n \to \infty$ in $\mathbb{N}_k$ (since $y = z_k$ in $E_k$) together with the fact that $F : \overline{X_k} \to E_k$ is continuous (note that $y_n \in \overline{X_k}$ for $n \in \mathbb{N}_k$) implies $y = Fy$ in $E_k$. We can do this for each $k \in \mathbb{N}$, so $y = Fy$ in $E$.

Our next result was motivated by contractions considered in [3]. In this case, the map $F_n$ will be related to $F$ by the closure property Theorem 2.4(f).
Theorem 2.4. Let $E$ and $E_n$ be as described in the beginning of Section 2 and let $F : X \to E$ with $X \subseteq E$. Also for each $n \in \mathbb{N}$ assume there exists $F_n : X_n \to E_n$. Suppose the following conditions are satisfied:

(a) $0 \in \text{pseudo-innt}(X)$,

(b) $X_1 \supseteq X_2 \supseteq \cdots$,

(c) for each $n \in \mathbb{N}$, $F_n(X_n)$ is bounded, for each $n \in \mathbb{N}$, $F_n : X_n \to E_n$ and there exists a continuous nondecreasing function $\phi_n : [0, \infty) \to [0, \infty)$ satisfying $\phi_n(z) < z$ for $z > 0$ such that $|F_n(x) - F_n(y)| \leq \phi_n(|x - y|)$ for all $x, y \in X_n$ for each $n \in \mathbb{N}$, $y \neq \lambda F_n y$ in $E_n$ for all $\lambda \in (0, 1)$, $y \in \partial X_n$.

(d) for each $n \in \mathbb{N}$, the map $\mathcal{K}_n : X_n \to 2^E$ given by $\mathcal{K}_n(y) = \bigcup_{m=n}^{\infty} F_m(y)$ (see Remark 2.5) satisfies $H_n(\mathcal{K}_n(x), \mathcal{K}_n(y)) \leq \psi_n(|x - y|)$ for all $x, y \in X_n$; here $\psi_n : [0, \infty) \to [0, \infty)$ is continuous, $\psi_n(z) < z$ for $z > 0$ with the map $\Psi_n : [0, \infty) \to [0, \infty)$, defined by $\Psi_n(x) = x - \psi_n(x)$, strictly increasing,

(e) for each $k \in \mathbb{N}$, for every $\epsilon > 0$, and sequence $\{x_n\}_{n \in \mathbb{N}}$, $S = \{k, k+1, k+2, \ldots\}$, with $x_n \in X_n$ and $x_n \in \mathcal{K}_n x_n$ in $E_n$, there exists $n_k \in S$ such that $\text{diam}_k(\mathcal{K}(x_n x_n)) < \epsilon$ for each $n \in S$ with $n \geq n_k$,

(f) if there exists $w \in E$ and a sequence $\{y_n\}_{n \in \mathbb{N}}$ with $y_n \in \mathcal{K}_n$ and $y_n = F_n y_n$ in $E_n$ such that for every $k \in \mathbb{N}$ with $y_n \to w$ in $E_k$ as $n \to \infty$ in $S = \{k+1, k+2, \ldots\}$, then $w = Fw$ in $E$.

Then $F$ has a fixed point in $E$.

Remark 2.5. The definition of $\mathcal{K}_n$ in Theorem 2.4(d) is as follows. If $y \in X_n$ and $y \notin X_{n+1}$, then $\mathcal{K}_n(y) = F_n(y)$, whereas if $y \in X_{n+1}$ and $y \notin X_{n+2}$, then $\mathcal{K}_n(y) = F_n(y) \cup F_{n+1}(y)$, and so on.

Proof. Fix $n \in \mathbb{N}$. From Theorem 1.2 there exists $y_n \in X_n$ with $y_n = F_n y_n$ in $E_n$. Let us look at $\{y_n\}_{n \in \mathbb{N}}$. From Theorem 2.4(b) we know that $y_n \in X_1$ for $n \in \mathbb{N}$. Note as well that $y_n \in \mathcal{H}_1 y_n$ for $n \in \mathbb{N}$ since $|x_1| \leq |x_n|$ for all $x \in E_n$ and $y_n = F_n y_n$ in $E_n$. We claim

$$\exists z_1 \in E_1 \quad \text{with } y_n \rightharpoonup z_1 \text{ in } E_1, \quad n \to \infty \text{ in } \mathbb{N}. \quad (2.6)$$

To see this, let $\epsilon > 0$ be given. Let $m, n \in \mathbb{N}$. It is easy to see, since $y_n \in \mathcal{H}_1 y_n$ and $y_m \in \mathcal{H}_1 y_m$, that

$$\left| y_n - y_m \right|_1 \leq H_1(\mathcal{H}_1 y_n, \mathcal{H}_1 y_m) + \text{diam}_1(\mathcal{H}_1 y_n) + \text{diam}_1(\mathcal{H}_1 y_m), \quad (2.7)$$

so Theorem 2.4(d) yields

$$\left| y_n - y_m \right|_1 \leq \Psi_1^{-1}(\text{diam}_1(\mathcal{H}_1 y_n) + \text{diam}_1(\mathcal{H}_1 y_m)). \quad (2.8)$$

Now Theorem 2.4(e) guarantees that there exists $n_1 \in \mathbb{N}$ such that

$$\left| y_n - y_m \right|_1 \leq \Psi_1^{-1}(2\epsilon) \quad \text{for } m, n \geq n_1. \quad (2.9)$$

Consequently, $\{y_n\}_{n \in \mathbb{N}}$ is Cauchy, so $(2.6)$ holds. Let $\mathbb{N}_1 = \mathbb{N} \setminus \{1\}$. 

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This together with Theorem 2.4(e) guarantees that \( \{y_n\}_{n \in \mathbb{N}} \) is Cauchy, so there exists a \( z_2 \in E_2 \) with \( y_n \to z_2 \) in \( E_2 \) as \( n \to \infty \) in \( \mathbb{N} \). Note that \( z_2 = z_1 \) in \( E_1 \) since \( \mathbb{N} \subseteq \mathbb{N} \). Let \( \mathbb{N}_2 = \mathbb{N} \{2\} \). Proceed inductively to obtain \( z_k \in E_k \) with \( y_n \to z_k \) in \( E_k \) as \( n \to \infty \) in \( \mathbb{N}_{k-1} = \{k, k+1, \ldots\} \). Note that \( z_{k+1} = z_k \) in \( E_k \) for \( k \in \mathbb{N} \). Also let \( \mathbb{N}_k = \mathbb{N}_{k-1} \{k\} \).

Fix \( k \in \mathbb{N} \). Let \( y = z_k \) in \( E_k \). Notice that \( y \) is well defined and \( y \in \text{lim} \ldots \lim E_n = E \). Now \( y_n = F_n y_n \) in \( E_n \) for \( n \in \mathbb{N} \) and \( y_n \to y \) in \( E_n \) as \( n \to \infty \) in \( \mathbb{N} \) (since \( y = z_k \) in \( E_k \)) together with Theorem 2.4(f) implies \( y = F y \) in \( E \).

\[ \Box \]

Our next two results are for multivalued maps.

**Theorem 2.6.** Let \( E \) and \( E_n \) be as described above and let \( F : X \to 2^E \) with \( X \subseteq E \) and for each \( n \in \mathbb{N} \), assume \( F : \overline{X}_n \to \mathbb{C}(E_n) \). Suppose the following conditions are satisfied:

(a) \( 0 \in \text{pseudo-intt}(X) \),

(b) for each \( n \in \mathbb{N} \), \( F(\overline{X}_n) \) is bounded,

(c) for each \( n \in \mathbb{N} \), \( F : \overline{X}_n \to \mathbb{C}(E_n) \), and there exists a continuous strictly increasing function \( \phi_n : [0, \infty) \to [0, \infty) \) satisfying \( \phi_n(z) < z \) for \( z \) such that \( H_n(Fx, Fy) \leq \phi_n(|x - y|) \) for all \( x, y \in \overline{X}_n \),

(d) for each \( n \in \mathbb{N} \), the map \( \Phi_n : [0, \infty) \to [0, \infty) \) given by \( \Phi_n(x) = x - \phi_n(x) \) is strictly increasing, \( \Phi_n^{-1}(a) + \Phi_n^{-1}(b) \leq \Phi_n^{-1}(a + b) \) for \( a, b \geq 0 \), with \( \sum_{i=0}^{\infty} \phi_i(t) < \infty \) for \( t > 0 \) and \( \sum_{i=1}^{\infty} \phi_i(x - \phi(x)) \leq \phi_n(x) \) for \( x > 0 \),

(e) for each \( n \in \mathbb{N} \), \( y \notin F y \) in \( E_n \) for all \( \lambda \in (0,1) \), \( y \in \partial X_n \),

(f) for each \( n \in \{2, 3, \ldots\} \), if \( y \in \overline{X}_n \) solves \( y \in F y \) in \( E_n \), then \( y \in \overline{X}_k \) for \( k \in \{1, \ldots, n - 1\} \),

(g) for each \( k \in \mathbb{N} \), for every \( \epsilon > 0 \) and sequence \( \{x_n\}_{n \in \mathbb{N}}, S = \{k, k+1, k+2, \ldots\} \), with \( x_n \in \overline{X}_n \) and \( x_n \in F x_n \) in \( E_n \) there exists \( n_k \in S \) such that \( \text{diam}_k(Fx_n) < \epsilon \) for each \( n \in S \) with \( n \geq n_k \).

Then \( F \) has a fixed point in \( E \).

**Proof.** Fix \( n \in \mathbb{N} \). From Theorem 1.3 (and Remark 1.4) there exists \( y_n \in \overline{X}_n \) with \( y_n \in F y_n \) in \( E_n \). Let us look at \( \{y_n\}_{n \in \mathbb{N}} \). Notice that \( y_n \in \overline{X}_1 \) for \( n \in \mathbb{N} \) from Theorem 2.6(f). Let \( \epsilon > 0 \) be given and \( m, n \in \mathbb{N} \). Now since \( y_n \in F y_n \) and \( y_m \in F y_m \), we have

\[ |y_n - y_m|_1 \leq H_1(Fy_n, Fy_m) + \text{diam}_1(Fy_n) + \text{diam}_1(Fy_m) \quad (2.11) \]

so

\[ |y_n - y_m|_1 \leq \Phi_1^{-1}(\text{diam}_1(Fy_n) + \text{diam}_1(Fy_m)) \quad (2.12) \]

This, together with Theorem 2.6(g), guarantees that \( \{y_n\}_{n \in \mathbb{N}} \) is Cauchy, so there exists a \( z_1 \in E_1 \) with \( y_n \to z_1 \) in \( E_1 \) as \( n \to \infty \) in \( \mathbb{N} \). Let \( \mathbb{N}_1 = \mathbb{N} \{1\} \). Proceed inductively to obtain \( z_k \in E_k \) with \( y_n \to z_k \) in \( E_k \) as \( n \to \infty \) in \( \mathbb{N}_{k-1} = \{k, k+1, \ldots\} \). Note that \( z_{k+1} = z_k \) in \( E_k \) for \( k \in \mathbb{N} \). Also let \( \mathbb{N}_k = \mathbb{N}_{k-1} \{k\} \).
Fix \( k \in \mathbb{N} \). Let \( y = z_k \) in \( E_k \). Notice that \( y_n \in F y_n \in E_n \) for \( n \in \mathbb{N}_k \) and \( y_n \to y \) in \( E_k \) as \( n \to \infty \) in \( \mathbb{N}_k \) together with Remark 1.4 (note that \( F : \overline{X}_k \to C(E_k) \)) implies \( y \in F y \) in \( E_k \). We can do this for each \( k \in \mathbb{N} \), so \( y \in F y \) in \( E \).

**Theorem 2.7.** Let \( E \) and \( E_n \) be as described in the beginning of Section 2 and let \( F : X \to 2^E \) with \( X \subseteq E \). Also for each \( n \in \mathbb{N} \) assume there exists \( F_n : \overline{X}_n \to C(E_n) \). Suppose the following conditions are satisfied:

(a) 0 ∈ pseudo-intt(\( X \)),
(b) \( \overline{X}_1 \supseteq \overline{X}_2 \supseteq \cdots \),
(c) for each \( n \in \mathbb{N} \), \( F_n(\overline{X}_n) \) is bounded,
(d) for each \( n \in \mathbb{N} \), \( F_n : \overline{X}_n \to C(E_n) \) and there exists a continuous strictly increasing function \( \phi_n : [0, \infty) \to [0, \infty) \) satisfying \( \phi_n(z) \leq z \) for \( z > 0 \) such that \( H_n(F_n x, F_n y) \leq \phi_n(|x - y|) \) for all \( x, y \in \overline{X}_n \),
(e) for each \( n \in \mathbb{N} \), the map \( \Phi_n : [0, \infty) \to [0, \infty) \) given by \( \Phi_n(x) = x - \phi_n(x) \) is strictly increasing, \( \Phi_n^{-1}(a + b) \leq \Phi_n^{-1}(a) + \Phi_n^{-1}(b) \) for \( a, b \geq 0 \), with \( \sum_{i=0}^{\infty} \phi_i(t) < \infty \) for \( t > 0 \) and \( \sum_{i=1}^{\infty} \phi_i(x) - \phi_i(y) \leq \phi_i(x) \) for \( x > 0 \),
(f) for each \( n \in \mathbb{N} \), \( y \notin \lambda F_n y \) in \( E_n \) for all \( \lambda \in (0, 1) \) and \( y \in \partial X_n \),
(g) for each \( n \in \mathbb{N} \), the map \( \mathcal{H}_n : \overline{X}_n \to \mathcal{H}_n \) given by \( \mathcal{H}_n(y) = \bigcup_{m=n}^{\infty} F_m(y) \) satisfies \( H_n(\mathcal{H}_n(x), \mathcal{H}_n(y)) \leq \psi_n(|x - y|) \) for all \( x, y \in \overline{X}_n \); here \( \psi_n : [0, \infty) \to [0, \infty) \) is continuous, \( \psi_n(z) \leq z \) for \( z > 0 \) with the map \( \Psi_n : [0, \infty) \to \mathcal{H}_n \) given by \( \Psi_n(x) = x - \psi_n(x) \) is strictly increasing,
(h) for each \( k \in \mathbb{N} \), for every \( \epsilon > 0 \) and sequence \( \{x_n\}_{n \in S}, S = \{k, k+1, k+2, \ldots\} \), with \( x_n \in \overline{X}_n \) and \( x_n \in \mathcal{H}_k x_n \) in \( E_n \) there exists \( n_k \in S \) such that \( \text{diam}_{k}(\mathcal{H}_k x_n) < \epsilon \) for each \( n \in S \) with \( n \geq n_k \),
(i) if there exists a \( w \in E \) and a sequence \( \{y_n\}_{n \in \mathbb{N}} \) with \( y_n \in \overline{X}_n \) and \( y_n \in F_n y_n \) in \( E_n \) such that for each \( k \in \mathbb{N} \) with \( y_n \to w \) in \( E_k \) as \( n \to \infty \) in \( S = \{k+1, k+2, \ldots\} \), then \( w \in F w \) in \( E \).

Then \( F \) has a fixed point in \( E \).

**Proof.** The proof is essentially the same as in Theorem 2.4 (except that here we use Theorem 1.3 (and Remark 1.4) instead of Theorem 1.2). □

**References**


8 Multivalued nonlinear contractions


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