Making use of the Ruscheweyh derivatives, we introduce the subclasses $T(n, \alpha, \lambda) \ (n \in \{0, 1, 2, 3, \ldots\}, -\pi/2 < \alpha < \pi/2, \text{ and } 0 \leq \lambda \leq \cos^2 \alpha)$ of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic in $|z| < 1$. Subordination and inclusion relations are obtained. The radius of $\alpha$-spirallikeness of order $\rho$ is calculated. A convolution property and a special member of $T(n, \alpha, \lambda)$ are also given.

Copyright © 2006 Hindawi Publishing Corporation. All rights reserved.

1. Introduction

Let $A$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. Let $S \subset A$ consist of univalent functions in $U$. For $0 \leq \rho < 1$, a function $f \in S$ is said to be starlike of order $\rho$ if

$$\text{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > \rho \quad (z \in U). \quad (1.2)$$

The class of such functions we denote by $S^*(\rho) \ (0 \leq \rho < 1)$. A function $f \in S$ is said to be convex in $U$ if

$$\text{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > 0 \quad (z \in U). \quad (1.3)$$

We denote by $K$ the class of all convex functions in $U$. For $-\pi/2 < \alpha < \pi/2$ and $0 \leq \rho < 1$, a function $f \in S$ is said to be $\alpha$-spirallike of order $\rho$ in $U$ if

$$\text{Re} \left\{ e^{i\alpha} \frac{z f'(z)}{f(z)} \right\} > \rho \cos \alpha \quad (z \in U). \quad (1.4)$$
Subclasses of $\alpha$-spirallike functions

Further let $UCV \subset K$ be the class of functions introduced by Goodman \cite{2} called uniformly convex in $U$. It was shown in \cite{4, 7} that $f \in A$ is in $UCV$ if and only if
\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \left| \frac{zf''(z)}{f'(z)} \right| (z \in U). \tag{1.5}
\]

In \cite{7}, Ronning investigated the class $Sp$ defined by
\[
Sp = \{ f \in S^+(0) : f(z) = zg'(z), g \in UCV \}. \tag{1.6}
\]

The uniformly convex and related functions have been studied by several authors (see, e.g., \cite{1–4, 7, 6, 8, 12}).

If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in A$, then the Hadamard product or convolution of $f$ and $g$ is defined by $(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$. Let
\[
D^n f(z) = \frac{z}{(1-z)^{n+1}} \ast f(z), \tag{1.7}
\]
for $f \in A$ and $n \in N_0 = \{0,1,2,3,\ldots\}$. Then
\[
D^n f(z) = \frac{z^n - 1}{n!} f(z). \tag{1.8}
\]

This symbol $D^n f$ is called the Ruscheweyh derivative of order $n$ of $f$. It was introduced by Ruscheweyh \cite{9}.

In this paper we introduce and investigate the subclasses $T(n, \alpha, \lambda)$ of $A$ as follows.

Definition 1.1. A function $f \in A$ is said to be in $T(n, \alpha, \lambda)$ if
\[
\left( \text{Re} \left\{ e^{i\alpha} \frac{z(D^n f(z))'}{D^n f(z)} \right\} \right)^2 + \lambda \left| \frac{z(D^n f(z))'}{D^n f(z)} - 1 \right|^2 (z \in U), \tag{1.9}
\]
where $n \in N_0, -\pi/2 < \alpha < \pi/2$, and $0 \leq \lambda \leq \cos^2 \alpha$.

Note that, for $\lambda = 0$,
\[
T(n, \alpha, 0) = \left\{ f \in A : \text{Re} \left\{ e^{i\alpha} \frac{z(D^n f(z))'}{D^n f(z)} \right\} > \left| \frac{z(D^n f(z))'}{D^n f(z)} - 1 \right| (z \in U) \right\}. \tag{1.10}
\]

In particular, $T(0,0,0) = Sp$ and $T(1,0,0) = UCV$.

2. Properties of $T(n, \alpha, \lambda)$

Let $f$ and $g$ be analytic in $U$. Then we say that $f$ is subordinate to $g$ in $U$, written $f \prec g$, if there exists an analytic function $w$ in $U$ such that $|w(z)| \leq |z|$ and $f(z) = g(w(z))$ for $z \in U$. If $g$ is univalent in $U$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$. 
Theorem 2.1. Let $n \in \mathbb{N}_0, \alpha \in (-\pi/2, \pi/2)$, and $\lambda \in [0, \cos^2 \alpha]$. A function $f \in A$ belongs to $T(n, \alpha, \lambda)$ if and only if
\begin{equation}
  e^{ia} \frac{z(D^n f(z))'}{D^n f(z)} < h(z) \cos \alpha + i \sin \alpha \quad (z \in U),
\end{equation}
where
\begin{equation}
  h(z) = 1 - \frac{\lambda}{2 \cos^2 \alpha} + \frac{2}{\pi^2} \left( \log \frac{1 + \sqrt{(z + \beta)/(1 + \beta z)}}{1 - \sqrt{(z + \beta)/(1 + \beta z)}} \right)^2,
\end{equation}
with
\begin{equation}
  \beta = \left( \frac{e^\mu - 1}{e^\mu + 1} \right)^2, \quad \mu = \frac{\sqrt{\lambda} \pi}{2 \cos \alpha}.
\end{equation}

Proof. Let us define $w = u + iv$ by
\begin{equation}
  e^{ia} \frac{z(D^n f(z))'}{D^n f(z)} = w(z) \cos \alpha + i \sin \alpha \quad (z \in U).
\end{equation}
Then $w(0) = 1$ and the inequality (1.9) can be rewritten as
\begin{equation}
  u > \frac{1}{2} \left( v^2 + 1 - \frac{\lambda}{\cos^2 \alpha} \right).
\end{equation}
Thus
\begin{equation}
  w(U) \subset G = \{ w = u + iv : u \text{ and } v \text{ satisfy (2.5)} \}.
\end{equation}
It follows from (2.2) that
\begin{equation}
  h(0) = 1 - \frac{\lambda}{2 \cos^2 \alpha} + \frac{2}{\pi^2} \left( \log \frac{1 + \sqrt{\beta}}{1 - \sqrt{\beta}} \right)^2 = 1.
\end{equation}
In order to prove the theorem, it suffices to show that the function $w = h(z)$ defined by (2.2) maps $U$ conformally onto the parabolic region $G$.

Note that
\begin{equation}
  0 \leq \frac{1}{2} \left( 1 - \frac{\lambda}{\cos^2 \alpha} \right) < 1 - \frac{\lambda}{2 \cos^2 \alpha} \leq 1,
\end{equation}
for $0 \leq \lambda \leq \cos^2 \alpha$. Consider the transformations
\begin{equation}
  w_1 = \sqrt{w - \left( 1 - \frac{\lambda}{2 \cos^2 \alpha} \right)}, \quad w_2 = e^{\sqrt{\pi} w_1}, \quad t = \frac{1}{2} \left( w_2 + \frac{1}{w_2} \right).
\end{equation}
4 Subclasses of $\alpha$-spirallike functions

It is easy to verify that the composite function

$$t = \varphi(w) = \text{ch} \left( \pi \sqrt{2w - \left(2 - \frac{\lambda}{\cos^2 \alpha}\right)} \right)$$  \hspace{1cm} (2.10)

maps $G^+ = G \cap \{w = u + iv : v > 0\}$ conformally onto the upper half plane $\text{Im}(t) > 0$ so that $w = (1/2)(1 - \lambda/\cos^2 \alpha)$ corresponds to $t = -1$ and $w = 1 - \lambda/2\cos^2 \alpha$ to $t = 1$. Applying the symmetry principle, the function $t = \varphi(w)$ maps $G$ conformally onto $\Omega = \{t : |\text{arg}(t+1)| < \pi\}$. Since $t = 2((1 + \zeta)/(1 - \zeta))^2 - 1$ maps the unit disk $|\zeta| < 1$ onto $\Omega$, we see that

$$w = \varphi^{-1}(t) = 1 - \frac{\lambda}{2\cos^2 \alpha} + \frac{1}{2\pi^2} \left( \log \left( t + \sqrt{t^2 - 1} \right) \right)^2$$

(2.11)

maps $|\zeta| < 1$ conformally onto $G$ so that $\zeta = \beta$ ($0 \leq \beta < 1$) corresponds to $w = 1$. Therefore the function

$$w = h(z) = g \left( \frac{z + \beta}{1 + \beta z} \right) \quad (z \in U)$$

(2.12)

maps $U$ conformally onto $G$ and the proof of the theorem is complete. \hfill \Box

**Corollary 2.2.** Let $f \in T(n, \alpha, \lambda)$, $n \in \mathbb{N}_0$, $\alpha \in (-\pi/2, \pi/2)$, $\lambda \in [0, \cos^2 \alpha]$, and $h$ be given by (2.2). Then

$$\left| \frac{D^n f(z)}{z} \right| < \exp \left( e^{-ia \cos \alpha} \int_0^z \frac{h(t) - 1}{t} dt \right),$$

(2.13)

$$\exp \left( \int_0^1 \frac{h(-\rho|z|) - 1}{\rho} d\rho \right) \leq \left| \left( \frac{D^n f(z)}{z} \right)^{\text{eis sec} \alpha} \right| \leq \exp \left( \int_0^1 \frac{h(\rho|z|) - 1}{\rho} d\rho \right),$$

(2.14)

for $z \in U$. The bounds in (2.14) are sharp with the extremal function $f_0 \in A$ defined by

$$D^n f_0(z) = z \exp \left( e^{-ia \cos \alpha} \int_0^z \frac{h(t) - 1}{t} dt \right).$$

(2.15)

**Proof.** From Theorem 2.1 we have

$$\frac{e^{ia}}{\cos \alpha} \left( \frac{z(D^n f(z))'}{D^n f(z)} - 1 \right) < h(z) - 1,$$

(2.16)
for $f \in T(n,\alpha,\lambda)$. Since the function $h-1$ is univalent and starlike (with respect to the origin) in $U$, using (2.16) and the result of Suffridge [11, Theorem 3], we obtain

$$\frac{e^{i\alpha}}{\cos \alpha} \log \frac{D^n f(z)}{z} = \frac{e^{i\alpha}}{\cos \alpha} \int_0^z \left( \frac{(D^n f(t))'}{D^n f(t)} - \frac{1}{t} \right) dt \prec \int_0^z h(t) - \frac{1}{t} dt. \quad (2.17)$$

This implies (2.13).

Noting that the univalent function $h$ maps the disk $|z| < \rho$ ($0 < \rho \leq 1$) onto a region which is convex and symmetric with respect to the real axis, we get

$$h(-\rho |z|) \leq \Re h(\rho z) \leq (\rho |z|) \quad (z \in U). \quad (2.18)$$

Now, (2.17) and (2.18) lead to

$$\int_0^1 h(-\rho |z|) - \frac{1}{\rho} d\rho \leq \log \left| \frac{D^n f(z)}{z} \right|^e \leq \int_0^1 h(\rho |z|) - \frac{1}{\rho} d\rho, \quad (2.19)$$

for $z \in U$, which yields (2.14).

The bounds in (2.14) are best possible since the equalities are attained for the function $f_0$ in $T(n,\alpha,\lambda)$ defined by (2.15). \[ \square \]

**Theorem 2.3.** Let $f \in T(n,\alpha,\lambda)$, $n \in N_0$, $\alpha \in (-\pi/2,\pi/2)$, $\lambda \in [0, \cos^2 \alpha]$. Then $D^n f$ is $\alpha$-spirallike of order $\rho$ in $|z| < r$, where

$$r = r(\rho,\alpha,\lambda) = \frac{\beta + \left( \tan \left( \frac{\pi}{4}\sqrt{2(1-\rho) - \lambda \cos^2 \alpha} \right) \right)^2}{1 + \beta \left( \tan \left( \frac{\pi}{4}\sqrt{2(1-\rho) - \lambda \cos^2 \alpha} \right) \right)^2} \cdot \left( \frac{1}{2} \left( 1 - \frac{\lambda}{\cos^2 \alpha} \right) \leq \rho < 1 - \frac{\lambda}{2 \cos^2 \alpha} \right) \quad (2.20)$$

and $\beta$ is given by (2.2). The result is sharp.

**Proof.** It follows from (2.20) and (2.2) that

$$0 < 2(1-\rho) - \frac{\lambda}{\cos^2 \alpha} \leq 1, \quad 0 \leq \beta < r \leq 1. \quad (2.21)$$

Let $h$ be given by (2.2). Then

$$h(-r) = 1 - \frac{\lambda}{2 \cos^2 \alpha} + \frac{2}{\pi^2} \left( \log \frac{1 + i\sqrt{(r-\beta)/(1-\beta r)}}{1 - i\sqrt{(r-\beta)/(1-\beta r)}} \right)^2 \quad (2.22)$$

$$= 1 - \frac{\lambda}{2 \cos^2 \alpha} - \frac{8}{\pi^2} \left( \arctan \sqrt{\frac{r-\beta}{1-\beta r}} \right)^2$$
Subclasses of $\alpha$-spirallike functions

and hence

$$\inf_{|z|<r} \Re h(z) = h(-r) = \rho. \quad (2.23)$$

If $f \in T(n, \alpha, \lambda)$, then from Theorem 2.1 and (2.23) we have

$$\Re \left\{ e^{ia} \frac{z(D^n f(z))'}{D^n f(z)} \right\} > \rho \cos \alpha \quad (|z| < r), \quad (2.24)$$

that is, $D^n f$ is $\alpha$-spirallike of order $\rho$ in $|z| < r$. Further, the result is sharp with the extremal function $f_0$ defined by (2.15).

Taking $\rho = \frac{1}{2}(1 - \lambda/\cos^2 \alpha)$, Theorem 2.3 yields.

**Corollary 2.4.** Let $f \in T(n, \alpha, \lambda)$, $n \in \mathbb{N}_0$, $\alpha \in (-\pi/2, \pi/2)$, $\lambda \in [0, \cos^2 \alpha]$. Then $D^n f$ is $\alpha$-spirallike of order $(1 - \lambda/2\cos^2 \alpha)$ in $U$ and the result is sharp.

**Theorem 2.5.** Let $f \in T(n, \alpha, \lambda)$, $n \in \mathbb{N}_0$, $\alpha \in (-\pi/2, \pi/2)$, $\lambda \in [0, \cos^2 \alpha]$. Then $D^n f \in S^*(1 - \lambda/2)$ and the order $(1 - \lambda/2)$ is sharp.

**Proof.** Let $h$ be given by (2.2). Then it follows from the proof of Theorem 2.1 that

$$\partial h(U) = \left\{ w = u + iv : u = \frac{1}{2} \left( v^2 + 1 - \frac{\lambda}{\cos^2 \alpha} \right) \right\}. \quad (2.25)$$

Hence

$$\min_{|z|=1 (z \neq 1)} \Re \left\{ e^{-ia} (h(z) \cos + i \sin \alpha) \right\} = \min_{u \geq \left( \frac{1}{2} \left( 1 - \frac{\lambda}{\cos^2 \alpha} \right) \right)} \ g(u) \cos \alpha + \sin^2 \alpha, \quad (2.26)$$

where

$$g(u) = u \cos \alpha - |\sin \alpha| \sqrt{2u - 1 + \frac{\lambda}{\cos^2 \alpha}} \left( u \geq \frac{1}{2} \left( 1 - \frac{\lambda}{\cos^2 \alpha} \right) \right). \quad (2.27)$$

Since

$$g'(u) = \cos \alpha - \frac{|\sin \alpha|}{\sqrt{2u - 1 + \lambda/\cos^2 \alpha}} \left( u > \frac{1}{2} \left( 1 - \frac{\lambda}{\cos^2 \alpha} \right) \right), \quad (2.28)$$

the function $g$ attains its minimum value at $u = (1 - \lambda)/2\cos^2 \alpha$. Thus

$$\min_{|z|=1 (z \neq 1)} \Re \left\{ e^{-ia} (h(z) \cos + i \sin \alpha) \right\} = g \left( \frac{1 - \lambda}{2\cos^2 \alpha} \right) \cos \alpha + \sin^2 \alpha = \frac{1 - \lambda}{2}. \quad (2.29)$$

Let $f \in T(n, \alpha, \lambda)$. Then, by Theorem 2.1 and (2.29), we conclude that $D^n f$ is starlike of order $(1 - \lambda)/2$ in $U$, and the function $f_0$ defined by (2.15) shows that the order $(1 - \lambda)/2$ is sharp.

**Theorem 2.6.** $T(n + 1, \alpha, \lambda) \subset T(n, \alpha, \lambda)$, where $n \in \mathbb{N}_0$, $\alpha \in (-\pi/2, \pi/2)$, $\lambda \in [0, \cos^2 \alpha]$. 

Proof. It follows from (1.7) that
\[
z(D^n f(z))' = (n + 1)D^{n+1} f(z) - nD^n f(z) \quad (z \in U),
\]
for \( f \in A \). Set
\[
p(z) = e^{i\alpha} \frac{z(D^n f(z))'}{D^n f(z)} \quad (z \in U).
\]
Then (2.30) and (2.31) lead to
\[
\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{e^{-i\alpha} p(z) + n}{n + 1} \quad (z \in U).
\]
Differentiating both sides of (2.32) logarithmically and using (2.31), we get
\[
e^{i\alpha} \frac{z(D^{n+1} f(z))'}{D^{n+1} f(z)} = p(z) + \frac{z p'(z)}{e^{-i\alpha} p(z) + n} \quad (z \in U).
\]
If \( f \in T(n+1, \alpha, \lambda) \), then by Theorem 2.1 and (2.33) we have
\[
p(z) + \frac{z p'(z)}{e^{-i\alpha} p(z) + n} < h(z) \cos \alpha + i \sin \alpha \quad (z \in U),
\]
where \( h \) is given by (2.2). The function \( Q(z) = e^{-i\alpha} (h(z) \cos \alpha + i \sin \alpha) + n \) is univalent and convex in \( U \) and
\[
\text{Re} Q(z) > \frac{1 - \lambda}{2} + n \geq 0 \quad (z \in U)
\]
because of (2.29). Hence an application of the result of Miller and Mocanu [5, Corollary 1.1] yields
\[
p(z) = e^{i\alpha} \frac{z(D^n f(z))'}{D^n f(z)} < h(z) \cos \alpha + i \sin \alpha \quad (z \in U).
\]
Now, by Theorem 2.1, we know that \( f \in T(n, \alpha, \lambda) \) and the theorem is proved. \( \square \)

Remark 2.7. Combining Theorem 2.6 with Corollary 2.4, we see that each function in \( T(n, \alpha, \lambda) \) is \( \alpha \)-spirallike of order \((1/2)(1 - \lambda / \cos^2 \alpha)\) in \( U \). In view of Theorems 2.5 and 2.6 we have \( T(n, \alpha, \lambda) \subset S^*(1 - \lambda / 2) \).

Theorem 2.8. A function \( f \in A \) is in \( T(n, \alpha, \lambda) \) if and only if
\[
F(z) = \frac{n + 1}{z^n} \int_0^z t^{n-1} f(t) dt
\]
is in \( T(n+1, \alpha, \lambda) \), where \( n \in N_0, \alpha \in (-\pi / 2, \pi / 2), \lambda \in [0, \cos^2 \alpha] \).
Subclasses of \(\alpha\)-spirallike functions

Proof. It follows from (2.37) that \(F \in A\) and

\[(n + 1)f(z) = nF(z) + zF'(z) \quad (z \in U),\]  

for \(f \in A\). By using (2.30) and (2.38), we obtain

\[D^n f(z) = \frac{nD^n F(z) + z(D^n F(z))'}{n + 1} = D^{n+1} F(z) \quad (z \in U),\]

which proves the assertions of the theorem. \(\square\)

Let \(R(\rho)\) be the class of prestarlike functions of order \(\rho\) in \(U\) consisting of functions \(f \in A\) satisfying

\[\frac{z}{(1 - z)^{2 - 2\rho}} \ast f(z) \in S^*(\rho),\]

for some \(\rho\) \((0 \leq \rho < 1)\). The following lemma is due to Ruscheweyh [10].

Lemma 2.9. If \(f \in S^*(\rho)\) and \(g \in R(\rho)\) \((0 \leq \rho < 1)\), then for any analytic function \(F\) in \(U\),

\[\frac{g \ast (Ff)}{g \ast f}(U) \subset \mathfrak{C}(F(U)),\]

where \(\mathfrak{C}(F(U))\) stands for the convex hull of \(F(U)\).

Applying the lemma, we derive the following.

Theorem 2.10. Let \(f \in T(n, \alpha, \lambda)\) and \(g \in R((1 - \lambda)/2)\). Then

\[f \ast g \in T(n, \alpha, \lambda),\]

where \(n \in N_0\), \(\alpha \in (-\pi/2, \pi/2)\), \(\lambda \in [0, \cos^2 \alpha]\).

Proof. Let \(f \in T(n, \alpha, \lambda)\). Making use of Theorems 2.1 and 2.5, we have

\[F(z) = \frac{z(D^n f(z))'}{D^n f(z)} < e^{-i\alpha}(h(z)\cos \alpha + i\sin \alpha), \quad D^n f \in S^*\left(\frac{1 - \lambda}{2}\right).\]

If we put \(\varphi = f \ast g\), then for \(z \in U\),

\[\frac{z(D^n \varphi(z))'}{D^n \varphi(z)} = \frac{z(D^n f(z))'}{g(z) \ast D^n f(z)} = \frac{g(z) \ast (z(D^n f(z))')}{g(z) \ast D^n f(z)} = \frac{g(z) \ast (F(z)D^n f(z))}{g(z) \ast D^n f(z)}.\]
Since the univalent function $e^{-ia}(h(z) \cos \alpha + i \sin \alpha)$ is convex in $U$ and $g \in R((1 - \lambda)/2)$, from (2.43), (2.44), and the lemma we deduce that

$$\frac{z(D^n \varphi(z))'}{D^n \varphi(z)} \prec e^{-ia}(h(z) \cos \alpha + i \sin \alpha).$$

Therefore, by using Theorem 2.1, $\varphi \in T(n, \alpha, \lambda)$ and the proof is complete. \hfill \Box

Note that $R(1/2) = S^\ast(1/2)$. Since $R(\rho_1) \subset R(\rho_2)$ for $0 \leq \rho_1 < \rho_2 < 1$ (see [10]), we have $K = R(0) \subset R((1 - \lambda)/2)$. Thus Theorem 2.10 yields the following.

**Corollary 2.11.** (i) If $f \in T(n, \alpha, 0)$, $n \in \mathbb{N}_0$, $\alpha \in (-\pi/2, \pi/2)$, and $g \in S^\ast(1/2)$, then $f \ast g \in T(n, \alpha, 0)$.

(ii) If $f \in T(n, \alpha, \lambda)$, $n \in \mathbb{N}_0$, $\alpha \in (-\pi/2, \pi/2)$, $\lambda \in [0, \cos^2 \alpha]$, and $g \in K$, then $f \ast g \in T(n, \alpha, \lambda)$.

**Theorem 2.12.** Let $n \in \mathbb{N}_0$, $\alpha \in (-\pi/2, \pi/2)$, $\lambda \in [0, \cos^2 \alpha]$. The function $f \in A$ defined by

$$D^n f(z) = \frac{z}{(1 - bz)^2e^{-i\alpha}} \quad (z \in U) \quad (2.46)$$

is in $T(n, \alpha, \lambda)$, where $b$ is complex and

$$|b| = \begin{cases} \cos^2 \alpha + \lambda, & (0 \leq \lambda \leq (3 - 2\sqrt{2}) \cos^2 \alpha), \\ \frac{3\cos^2 \alpha - \lambda}{\sqrt{\lambda}} & (3 - 2\sqrt{2}) \cos^2 \alpha \leq \lambda \leq \cos^2 \alpha). \end{cases} \quad (2.47)$$

The result is sharp, that is, $|b|$ cannot be increased.

**Proof.** Let $f \in A$ be given by (2.46). Then

$$e^{ia}z(D^n f(z))' = \frac{1 + bz}{1 - bz} \cos \alpha + i \sin \alpha. \quad (2.48)$$

Hence, by Theorem 2.1, $f \in T(n, \alpha, \lambda)$ if and only if

$$\frac{1 + bz}{1 - bz} < h(z), \quad (2.49)$$

where $h$ is given by (2.2), or, equivalently, when

$$\left\{ w : \left| w - \frac{1 + |b|^2}{1 - |b|^2} \right| < \frac{2|b|}{1 - |b|^2} \right\} \subset h(U), \quad (2.50)$$

for $0 < |b| < 1$. 
Let $\delta$ denote the minimum distance from the point $(1 + |b|^2)/(1 - |b|^2)$ to the points on the parabola $\partial h(U)$ given by (2.25). Then

$$
\delta = \min_{u \geq (1/2)(1 - \lambda/\cos^2 \alpha)} \sqrt{g(u)}, \quad g(u) = \left( u - \frac{1 + |b|^2}{1 - |b|^2} \right)^2 + 2u - 1 + \frac{\lambda}{\cos^2 \alpha}.
$$

(2.51)

Note that

$$
\frac{1 + |b|^2}{1 - |b|^2} > \frac{1}{2} \left( 1 - \frac{\lambda}{\cos^2 \alpha} \right), \quad g'(u) = 2 \left( u - \frac{2|b|^2}{1 - |b|^2} \right).
$$

(2.52)

(i) If

$$
0 \leq \lambda \leq (3 - 2\sqrt{2}) \cos^2 \alpha, \quad |b| = \frac{\cos^2 \alpha + \lambda}{3 \cos^2 \alpha - \lambda},
$$

(2.53)

then $\lambda^2 - 6\lambda \cos^2 \alpha + \cos^4 \alpha \geq 0$. Thus

$$
|b|^2 = \left( \frac{\cos^2 \alpha + \lambda}{3 \cos^2 \alpha - \lambda} \right)^2 \leq \frac{\cos^2 \alpha - \lambda}{5 \cos^2 \alpha - \lambda}, \quad \frac{2|b|^2}{1 - |b|^2} \leq \frac{1}{2} \left( 1 - \frac{\lambda}{\cos^2 \alpha} \right).
$$

(2.54)

From (2.51), (2.52) and (2.54), we have $g'(u) \geq 0$ and hence

$$
\delta = \sqrt{g\left( \frac{1}{2} \left( 1 - \frac{\lambda}{\cos^2 \alpha} \right) \right)} = \frac{1 + |b|^2}{1 - |b|^2} - \frac{1}{2} \left( 1 - \frac{\lambda}{\cos^2 \alpha} \right) = \frac{2|b|}{1 - |b|^2}.
$$

(2.55)

(ii) If $0 \leq \lambda < (3 - 2\sqrt{2}) \cos^2 \alpha$ and

$$
\frac{\cos^2 \alpha + \lambda}{3 \cos^2 \alpha - \lambda} < |b| < \sqrt{\frac{\cos^2 \alpha - \lambda}{5 \cos^2 \alpha - \lambda}},
$$

(2.56)

then $g'(u) > 0$ and

$$
\delta = \frac{1 + |b|^2}{1 - |b|^2} - \frac{1}{2} \left( 1 - \frac{\lambda}{\cos^2 \alpha} \right) < \frac{2|b|}{1 - |b|^2}.
$$

(2.57)

(iii) If

$$
(3 - 2\sqrt{2}) \cos^2 \alpha \leq \lambda \leq \cos^2 \alpha, \quad |b| = \frac{\sqrt{\lambda}}{2 \cos \alpha + \sqrt{\lambda}},
$$

(2.58)
then \( \lambda^2 - 6\lambda \cos^2 \alpha + \cos^4 \alpha \leq 0 \) and so

\[
|b|^2 = \frac{\sqrt{\lambda}}{2 \cos \alpha + \lambda} \geq \frac{\cos^2 \alpha - \lambda}{5 \cos^2 \alpha - \lambda}, \quad \frac{2|b|^2}{1 - |b|^2} \geq \frac{1}{2} \left( 1 - \lambda \cos^2 \alpha \right).
\] (2.59)

Thus we have

\[
\delta = \sqrt{g \left( \frac{2|b|^2}{1 - |b|^2} \right)} = \sqrt{\frac{4|b|^2}{1 - |b|^2} + \frac{\lambda}{\cos^2 \alpha}} = \frac{2|b|}{1 - |b|^2}.
\] (2.60)

(iv) If \((3 - 2\sqrt{2}) \cos^2 \alpha \leq \lambda \leq \cos^2 \alpha\) and \(\sqrt{\lambda/(2 \cos \alpha + \sqrt{\lambda})} < |b| < 1\), then

\[
\delta = \sqrt{\frac{4|b|^2}{1 - |b|^2} + \frac{\lambda}{\cos^2 \alpha}} < \frac{2|b|}{1 - |b|^2}.
\] (2.61)

By virtue of (2.49), (2.50), (2.55), (2.57), (2.60), and (2.61), the proof of the theorem is now complete. \(\square\)

Letting \(n = \alpha = 0\) in Theorem 2.12, we have the following.

**Corollary 2.13.** The function \( f(z) = z/(1 - bz)^2 \) is in \( T(0, 0, \lambda) \), where \( \lambda \in [0, 1] \), \( b \) is complex and

\[
|b| = \begin{cases} 
 1 + \lambda \\
 3 - \lambda \\
 \sqrt{\lambda/2 + \sqrt{\lambda}} \\
 \sqrt{\lambda/2 - \sqrt{\lambda}} 
\end{cases} \quad \begin{cases} 
 0 \leq \lambda \leq 3 - 2\sqrt{2}, \\
 3 - 2\sqrt{2} \leq \lambda \leq 1, \\
 0 \leq \lambda \leq 3 - 2\sqrt{2}, \\
 3 - 2\sqrt{2} \leq \lambda \leq 1.
\end{cases}
\] (2.62)

The result is sharp, that is, \( |b| \) cannot be increased.

**References**


12 Subclasses of $\alpha$-spirallike functions


Neng Xu: Department of Mathematics, Changshu Institute of Technology, Changshu, Jiangsu 215500, China

*E-mail address*: xuneng11@pub.sz.jsinfo.net

Dinggong Yang: Department of Mathematics, Suzhou University, Suzhou, Jiangsu 215006, China