A characterization of the invertibility of a class of matrix Wiener-Hopf plus Hankel operators is obtained based on a factorization of the Fourier symbols which belong to the Wiener subclass of the almost periodic matrix functions. Additionally, a representation of the inverse, lateral inverses, and generalized inverses is presented for each corresponding possible case.

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1. Introduction

Operators of Wiener-Hopf plus Hankel type have been receiving an increasing attention in the last years (see [1, 2, 4, 6, 10, 12–16]). Some of the interest in their study arises directly from concrete applications where these kind of operators appear. This is the case in problems of wave diffraction by some particular rectangular geometries which originate specific boundary-transmission value problems that may be equivalently translated by systems of integral equations that lead to such kind of operators (see, e.g., [5, 7, 8]).

A great part of the study in this kind of operators is concentrated in the description of their Fredholm and invertibility properties. In particular, for some classes of the so-called Fourier symbols of the operators, their invertibility properties are already known (cf. the above references). Despite those advances, for some other classes of Fourier symbols, a complete description of the Fredholm and invertibility properties is still missing. In this way, some of the ongoing researches try to achieve the best possible factorization procedures of the involved Fourier symbols in such a way that a representation of the (generalized) inverses of the Wiener-Hopf plus Hankel operators will be possible to obtain when in the presence of a convenient factorization.

Within this spirit, the main aim of the present work is to provide an invertibility criterion for the Wiener-Hopf plus Hankel operators of the form

$$WH_{\Phi} = W_{\Phi} + H_{\Phi} : [L^2_+(\mathbb{R})]^N \longrightarrow [L^2(\mathbb{R}_+)]^N,$$

(1.1)
where $W_\Phi$ stands for the Wiener-Hopf operator defined by

$$W_\Phi = r_+ \mathcal{F}^{-1} \Phi : [L^2_+(\mathbb{R})]^N \longrightarrow [L^2(\mathbb{R})]^N,$$

(1.2)

and the Fourier symbol $\Phi$ belongs to the so-called $APW$ subclass of $[L^\infty(\mathbb{R})]^{N\times N}$. This will therefore extend some of the results of [15] to the matrix case.

As for the notations in (1.1)–(1.3), and in what follows, $[L^2_+(\mathbb{R})]^N$ denotes the subspace of $[L^2(\mathbb{R})]^N$ formed by all vectors of functions supported in the closure of $\mathbb{R}_+$, $r_+$ represents the operator of restriction from $[L^2(\mathbb{R})]^N$ into $[L^2(\mathbb{R}_+)]^N$, $\mathcal{F}$ stands for the Fourier transformation, and $J$ is the reflection operator given by the rule $J\Phi(x) = \tilde{\Phi}(x) := \Phi(-x)$, $x \in \mathbb{R}$.

In view of defining the $APW$ functions, let us first consider the algebra of almost periodic functions, usually denoted by $AP$. When endowed with the usual norm and multiplicative operation, $AP$ is the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions $e_\lambda$ ($\lambda \in \mathbb{R}$), where

$$e_\lambda(x) := e^{i\lambda x}, \quad x \in \mathbb{R}.$$

(1.4)

In this framework, it turns out that the elements of $APW$ are those from $AP$ which allow a representation by an absolutely convergent series. In fact, $APW$ is precisely the (proper) subclass of all functions $\varphi \in AP$ which can be written in an absolutely convergent series of the form

$$\varphi = \sum_j \varphi_j e_{\lambda_j}, \quad \lambda_j \in \mathbb{R}, \quad \sum_j |\varphi_j| < \infty.$$

(1.5)

Let us agree on the notation $\mathcal{G}B$ for the group of all invertible elements of a Banach algebra $B$. To end with the notation, we will say that a matrix function $\Phi$ belongs to $APW^{N\times N}$, and write $\Phi \in APW^{N\times N}$, if all entries of the matrix $\Phi$ belong to $APW$.

As mentioned above, the representation of the (generalized/lateral/both-sided) inverses of $WH_\Phi$ based on some factorization of the Fourier symbol $\Phi$ is an important goal, and will be obtained in the final part of the paper. In this way, the main contributions of the present work are described in Theorems 5.1, 4.1, and 3.3.

In the next section we will recall some useful particular known results which are anyway presented with complete proofs for the reader’s convenience.

2. Initial multiplicative decompositions

According to (1.1)–(1.3), we have

$$WH_\Phi = r_+ (\mathcal{F}^{-1} \Phi \cdot \mathcal{F} + \mathcal{F}^{-1} \Phi \cdot J) = r_+ \mathcal{F}^{-1} \Phi \cdot \mathcal{F}(I_{[L^2_+(\mathbb{R})]^N} + J),$$

(2.1)
where $I_{L^2_+(\mathbb{R})^N}$ denotes the identity operator in $[L^2_+(\mathbb{R})]^N$. Furthermore, since
\[ I_{L^2_+(\mathbb{R})^N} + J = \ell^e r_+ , \] (2.2)
where $\ell^e : [L^2(\mathbb{R}_+)]^N \to [L^2(\mathbb{R})]^N$ denotes the even extension operator, we may also rewrite the Wiener-Hopf plus Hankel operator in (1.1) as
\[ WH = r_+ \mathcal{F}^{-1} \Phi \cdot \mathcal{F} \ell^e r_+. \] (2.3)

From the theory of Wiener-Hopf and Hankel operators, it is well known that
\[ W\Psi \Phi = W\Psi \ell_0 W\Phi + H\Psi \ell_0 H\Phi , \] (2.4)
\[ H\Psi \Phi = W\Psi \ell_0 H\Phi + H\Psi \ell_0 W\Phi , \] (2.5)
where $\ell_0 : [L^2(\mathbb{R}_+)]^N \to [L^2(\mathbb{R})]^N$ denotes the zero extension operator. Additionally, from the last two identities, it follows that
\[ WH\Psi \Phi = W\Psi \ell_0 WH\Phi + H\Psi \ell_0 WH\Phi , \] (2.6)
\[ WH\Psi \Phi = W\Psi \ell_0 WH\Phi + H\Psi \ell_0 WH\Phi - \Phi . \] (2.7)

Let $C_+ := \{z \in \mathbb{C} : \text{Im } z > 0\}$, $C_- := \{z \in \mathbb{C} : \text{Im } z < 0\}$, and $H^\infty(\mathbb{C}_{\pm})$ be the set of all bounded and analytic functions in $\mathbb{C}_{\pm}$. Fatou’s theorem ensures that functions in $H^\infty(\mathbb{C}_{\pm})$ have nontangential limits on $\mathbb{R}$ almost everywhere, and it is usually denoted by $H^\infty_\pm(\mathbb{R})$ the set of all elements in $L^\infty(\mathbb{R})$ that are nontangential limits of functions in $H^\infty(\mathbb{C}_{\pm})$. Below, we will use the matrix versions $[H^\infty(\mathbb{C}_{\pm})]^{N \times N}$ and $[H^\infty_\pm(\mathbb{R})]^{N \times N}$ of those Hardy spaces.

It is already interesting to observe that, due to (2.6), if we consider $\Phi$ being an even function or $\Psi \in [H^\infty_-(\mathbb{R})]^{N \times N}$, we will then obtain the multiplicative relation
\[ WH\Psi \Phi = WH\Psi \ell_0 WH\Phi \] (2.7)
of two corresponding Wiener-Hopf plus Hankel operators. A more general property in this direction is formulated in the next result.

**Proposition 2.1.** If $\Psi \in [H^\infty(\mathbb{R})]^{N \times N}$ and $\Phi, \Xi \in [L^\infty(\mathbb{R})]^{N \times N}$, such that $\Xi = \tilde{\Xi}$, then
\[ WH\Psi \Phi \Xi = WH\Psi \ell_0 WH\Phi \ell_0 WH\Xi = W\Psi \ell_0 WH\Phi \ell_0 WH\Xi . \] (2.8)

**Proof.** Since $\Psi \in [H^\infty(\mathbb{R})]^{N \times N}$, we may apply the first presented multiplicative relation for Wiener-Hopf plus Hankel operators; see (2.7). This leads us to
\[ WH\Psi \Phi \Xi = WH\Psi \ell_0 WH\Phi \Xi . \] (2.9)

In addition, since $\Xi = \tilde{\Xi}$, it also follows from (2.7) that
\[ WH\Phi \Xi = WH\Phi \ell_0 WH\Xi . \] (2.10)
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From (2.9) and (2.10), we have that

\[ WH_{\Psi\Xi} = WH_{\Psi} \ell_0 WH_{\Phi} \ell_0 WH_{\Xi}. \]  

(2.11)

Since \( \Psi \in [H^\infty(\mathbb{R})]^{N \times N} \), we have \( H_{\Psi} = 0 \) due to the structure of the Hankel operators. Therefore \( WH_{\Psi} = W_{\Psi} \), and it follows from (2.11) that \( WH_{\Psi\Xi} = W_{\Psi} \ell_0 WH_{\Phi} \ell_0 WH_{\Xi}. \)

**Proposition 2.2.** If \( \Phi_e \in \mathcal{G}[L^\infty(\mathbb{R})]^{N \times N} \) and \( \tilde{\Phi}_e = \Phi_e \), then \( WH_{\Phi_e} \) is invertible and its inverse is the operator \( \ell_0 WH_{\Phi_e^{-1}} \ell_0 : [L^2(\mathbb{R}_+)]^N \to [L^2(\mathbb{R}_+)]^N. \)

**Proof.** Since both \( \Phi_e \) and \( \Phi_e^{-1} \) are even functions, we directly obtain from the above multiplicative relations of Wiener-Hopf plus Hankel operators that

\[
WH_{\Phi_e} \ell_0 WH_{\Phi_e^{-1}} \ell_0 = WH_{\Phi_e} \ell_0 = WH_{\Phi_e} \ell_0 = W_1 \ell_0 = I_{[L^2(\mathbb{R}_+)]^N},
\]

\[
\ell_0 WH_{\Phi_e^{-1}} \ell_0 WH_{\Phi_e} = \ell_0 W_1 WH_{\Phi_e} = \ell_0 W_1 = I_{[L^2(\mathbb{R}_+)]^N}.
\]

(2.12)

This obviously shows that \( WH_{\Phi_e} \) is invertible and its inverse is \( \ell_0 WH_{\Phi_e^{-1}} \ell_0. \)

**3. Matrix APW asymmetric factorization**

The **(Bohr) mean value** of \( \phi \in AP \) is defined by

\[
M(\phi) = \lim_{\alpha \to \infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} \phi(x) dx,
\]

(3.1)

where \( \{I_\alpha\}_{\alpha \in A} = \{(x_\alpha, y_\alpha)\}_{\alpha \in A} \) is a family of intervals \( I_\alpha \subset \mathbb{R} \) such that \( |I_\alpha| = y_\alpha - x_\alpha \to \infty \) as \( \alpha \to \infty \) (for an unbounded set \( A \subset \mathbb{R}_+ \)). The mean value of an element in \( AP \) always exists, is finite, and is independent of the particular choice of the family \( \{I_\alpha\}_{\alpha \in A}. \)

Let \( \Omega(\psi) := \{\lambda \in \mathbb{R} : M(\psi_{-\lambda}) \neq 0\} \) be the **Bohr-Fourier spectrum** of \( \psi \). We will denote by \( AP^- \) (\( AP^+ \)) the smallest closed subalgebra of \( L^\infty(\mathbb{R}) \) that contains all the functions \( e_\lambda, \lambda \leq 0 \) (\( \lambda \geq 0 \)), and consider \( APW^- \) (\( APW^+ \)) to be the set of all functions \( \psi \in APW \) such that \( \Omega(\psi) \subset (-\infty, 0] \) (\( \Omega(\psi) \subset [0, +\infty) \), resp.). It is therefore clear that \( APW^- \subset AP^- \) and \( APW^+ \subset AP^+. \)

**Definition 3.1.** Say that a matrix function \( \Phi \in \mathcal{G}[APW_{N \times N}] \) admits an **APW asymmetric factorization** if it can be represented in the form

\[
\Phi = \Phi_{-\diag[e_{\lambda_1}, \ldots, e_{\lambda_N}]} \Phi_e,
\]

(3.2)

where \( \lambda_k \in \mathbb{R}, e_{\lambda_k}(x) = e^{i\lambda_k x}, x \in \mathbb{R}, \Phi_{-} \in \mathcal{G}[APW_{N \times N}], \Phi_e \in \mathcal{G}[L^\infty(\mathbb{R})]^{N \times N}, \) and \( \tilde{\Phi}_e = \Phi_e. \)

**Remark 3.2.** We would like to remark that an APW asymmetric factorization, if it exists, is not unique. Anyway, the partial indices of two APW asymmetric factorizations of the same matrix function are unique up to a change in their order (cf. Theorem 3.3). Consequently the \( \lambda_k \) partial indices can be rearranged in any desired way. Namely, if (3.2) is an APW asymmetric factorization of \( \Phi \) and \( \Pi \) is a permutation constant matrix, then by considering \( \Pi^{-1} \diag[e_{\lambda_1}, \ldots, e_{\lambda_N}] \Pi =: \tilde{\diag[e_{\lambda_1}, \ldots, e_{\lambda_N}]}, \tilde{\Phi}_e := \Phi_{-\Pi}, \) and \( \tilde{\Phi}_e := \Pi^{-1} \Phi_e, \)
we obtain a second asymmetric APW factorization of $\Phi$ given by

$$\Phi = \overleftarrow{\Phi} \overrightarrow{\text{diag}}[e_{\lambda_1}, \ldots, e_{\lambda_N}] \Phi_e. \tag{3.3}$$

Besides this last fact, we have the following general result about the uniqueness of these factorizations.

**Theorem 3.3.** Let $\Phi \in \mathcal{G}_{\text{APW}}^{N \times N}$. Suppose that

$$\Phi = \Phi_{-}^{(1)} D^{(1)} \Phi_{e}^{(1)}, \tag{3.4}$$

with $D^{(1)} = \text{diag}[e_{\lambda_1}, \ldots, e_{\lambda_N}]$ and $\lambda_1 \geq \cdots \geq \lambda_N$, is an APW asymmetric factorization of $\Phi$ and assume additionally that

$$\Phi = \Phi_{-}^{(2)} D^{(2)} \Phi_{e}^{(2)}, \tag{3.5}$$

with $D^{(2)} = \text{diag}[e_{\mu_1}, \ldots, e_{\mu_N}]$ and $\mu_1 \geq \cdots \geq \mu_N$, represents any other APW asymmetric factorization of $\Phi$. Then

$$\Phi_{-}^{(2)} = \Phi_{-}^{(1)} \Psi^{-1}, \quad D^{(1)} = D^{(2)} = : D, \quad \Phi_{e}^{(2)} = D^{-1} \Psi D \Phi_{e}^{(1)},$$

where $\Psi(x) = (\psi_{jk}(x))_{j,k=1}^{N}$ is a matrix function with nonzero and constant determinant, having entries which are entire functions, and

$$\psi_{jk}(z) = \begin{cases} 0, & \text{if } \lambda_j > \lambda_k, \\ c_{jk} = \text{const} \neq 0, & \text{if } \lambda_j = \lambda_k. \end{cases} \tag{3.7}$$

**Proof.** If $\Phi$ admits the above-mentioned two APW asymmetric factorizations, then we can write

$$\Phi = \Phi_{-}^{(1)} D^{(1)} \Phi_{e}^{(1)} = \Phi_{-}^{(2)} D^{(2)} \Phi_{e}^{(2)}, \tag{3.8}$$

which leads to

$$\left(\Phi_{-}^{(2)}\right)^{-1} \Phi_{e}^{(1)} D^{(1)} = D^{(2)} \Phi_{e}^{(1)} \Phi_{e}^{(1)} \left(\Phi_{-}^{(1)}\right)^{-1}. \tag{3.9}$$

We now define $\Phi_{-} := (\Phi_{-}^{(2)})^{-1} \Phi_{e}^{(1)}$ and $\Phi_{e} := \Phi_{e}^{(2)} (\Phi_{e}^{(1)})^{-1}$. Thus, we have $\Phi_{-} \in \mathcal{G}_{\text{APW}}^{N \times N}$, $\Phi_{e} \in \mathcal{G}_{L^\infty(\mathbb{R})}^{N \times N}$, and $\Phi_{e} = \Phi_{e}$. From (3.9), we obtain the following identity for each $(j, k)$ element of that matrix:

$$\left(\Phi_{-}\right)_{jk}(x) e^{i \lambda_k x} = e^{i \mu_j x} \left(\Phi_{e}\right)_{jk}(x); \tag{3.10}$$

whence

$$\left(\Phi_{e}\right)_{jk}(x) = \left(\Phi_{-}\right)_{jk}(x) e^{i(\lambda_k - \mu_j) x}, \tag{3.11}$$
and recall that $\Phi_e$ is an even function. Thus
\[(\Phi_-)_{jk}(x)e^{i(\lambda_k - \mu_j)x} = (\widetilde{\Phi_-})_{jk}(x)e^{i(\mu_j - \lambda_k)x},\] (3.12)
and finally we infer from (3.12) that
\[(\Phi_-)_{jk}(x) = e^{2i(\mu_j - \lambda_k)x}(\widetilde{\Phi_-})_{jk}(x).\] (3.13)
If $\mu_j \geq \lambda_k$, then the element in the left-hand side of (3.13) is in the class $APW^-$, and the function in the right-hand side belongs to $APW^+$, which implies that there exist constants $c_{jk}$ such that
\[(\Phi_-)_{jk}(x) = c_{jk} = (\widetilde{\Phi_-})_{jk}(x)e^{2i(\mu_j - \lambda_k)x}.\] (3.14)
Therefore, $c_{jk} = c_{jk}e^{2i(\mu_j - \lambda_k)x}$. Thus, if $\mu_j > \lambda_k$, we obtain $c_{jk} = 0$, and in the case where $\mu_j = \lambda_k$, we conclude that $c_{jk}$ are nonzero constants. Altogether, we have
\[(\Phi_-)_{jk}(x) = \begin{cases} 0, & \text{if } \mu_j > \lambda_k, \\ c_{jk} = \text{const} \neq 0, & \text{if } \mu_j = \lambda_k. \end{cases}\] (3.15)
Let us now assume that $\mu_j < \lambda_k$. By the hypothesis, we know that $(\Phi_-)_{jk} \in APW^-$ and so $(\Phi_-)_{jk}$ can be represented in the following form:
\[(\Phi_-)_{jk}(x) = \sum_m (a_m)_{jk}e^{i(y_m)_{jk}x},\] (3.16)
with $\sum_m |(a_m)_{jk}| < \infty$ for all $j, k = 1, N$. From (3.16) we directly have
\[\widetilde{(\Phi_-)}_{jk}(x) = \sum_m (a_m)_{jk}e^{-i(y_m)_{jk}x}.\] (3.17)
Combining (3.13), (3.16), and (3.17) we obtain
\[\sum_m (a_m)_{jk}e^{i(y_m)_{jk}x} = e^{2i(\mu_j - \lambda_k)x}\sum_m (a_m)_{jk}e^{-i(y_m)_{jk}x},\] (3.18)
or equivalently
\[\sum_m (a_m)_{jk}e^{i(y_m)_{jk}x} = \sum_m (a_m)_{jk}e^{i(2(\mu_j - \lambda_k) - (y_m)_{jk})x},\] (3.19)
and this leads us to the following identity:
\[(y_m)_{jk} = 2(\mu_j - \lambda_k) - (y_m)_{jk}.\] (3.20)
In conclusion, we have in the present case
\[(y_m)_{jk} = \mu_j - \lambda_k < 0.\] (3.21)
So, for any couple \((j,k)\), we will obtain only one real number \((\nu_m)_{jk}\), which is precisely the difference \(\mu_j - \lambda_k\) and this means that in the representation of \((\Phi_-)_{jk}\) (cf. (3.16)) we need to have \((\Phi_-)_{jk}(x) = c_{jk}e^{(\nu_m)_{jk}x}\), with some constant \(c_{jk}\) = const, for all \(j, k = 1, N\). Thus, we arrive at the conclusion that \((\Phi_-)_{jk}\) are entire functions when \(\mu_j < \lambda_k\).

We will now prove that \(D^{(1)} = D^{(2)}\), that is, \(\mu_j = \lambda_j\) for all \(j\). Let us first assume that \(\mu_j > \lambda_j\), for some \(j\). Then \(\mu_l > \lambda_k\) for \(l \leq j \leq k\) and from (3.15) we infer that \((\Phi_-)_{jk}(x) = 0\) for \(x \in \mathbb{R}\), which is impossible simply because \(\Phi_-\) is invertible. If for some \(j\) we would assume \(\mu_j < \lambda_j\), we can repeat the above reasoning starting from (3.8) with \(D^{(1)}\Phi_e^{(1)}(\Phi_e^{(2)})^{-1} = (\Phi_-^{(1)})^{-1}\Phi_-^{(2)}D^{(2)}\) instead of (3.9) and obtain once again a contradiction. Thus, \(\mu_j = \lambda_j\) for all \(j\).

Letting \(\Psi := \Phi_-\) we immediately have that \(\Psi\) is an entire function. Additionally, by virtue of the equality \(D^{(1)} = D^{(2)} =: D\) and (3.15), \(\Psi\) satisfies (3.7). The block-triangular structure of \(\Psi\) implies that \(\det \Psi\) is a constant, and since \(\Psi = (\Phi_e^{(2)})^{-1}\Phi_e^{(1)}\) this constant cannot be zero. Finally, identity (3.9) gives that \(\Phi_e = D^{-1}\Psi D\), and therefore \(\Phi_e^{(2)} = D^{-1}\Psi D\Phi_e^{(1)}\). This together with the identity \(\Phi_e^{(2)} = \Phi_e^{(1)}\Psi^{-1}\) concludes the proof. \(\square\)

4. Invertibility characterization

Let \(S : X \rightarrow Y\) be a bounded linear operator acting between Banach spaces. If \(\text{Im } S\) is closed, the cokernel of \(S\) is defined as \(\text{Coker } S = Y/\text{Im } S\). Then, in this case, \(S\) is said to be properly \(d\)-normal if \(\dim \text{Coker } S\) is finite and \(\dim \text{Ker } S\) is infinite, properly \(n\)-normal if \(\dim \text{Ker } S\) is finite and \(\dim \text{Coker } S\) is infinite, and Fredholm if both \(\dim \text{Ker } S\) and \(\dim \text{Coker } S\) are finite. An operator is called semi-Fredholm if it is properly \(n\)-normal, or properly \(d\)-normal, or Fredholm.

For further purposes let us also recall that two linear operators \(T\) and \(S\) are said to be equivalent operators if there exist two bounded invertible operators \(E\) and \(F\) such that \(T = ESF\).

**Theorem 4.1.** Let \(\Phi\) have an APW asymmetric factorization, with partial indices \(\lambda_1, \ldots, \lambda_N\).

(a) If there exist positive and negative partial indices, then WH\(_{\Phi}\) is not semi-Fredholm.
(b) If \(\lambda_i \leq 0\), \(i = 1, N\), and if for at least one \(i\), \(\lambda_i < 0\), then WH\(_{\Phi}\) is properly \(d\)-normal and right-invertible.
(c) If \(\lambda_i \geq 0\), \(i = 1, N\), and if for at least one index \(i\), \(\lambda_i > 0\), then WH\(_{\Phi}\) is properly \(n\)-normal and left-invertible.
(d) If \(\lambda_i = 0\), \(i = 1, N\), then WH\(_{\Phi}\) is invertible.

*Proof.* Since by hypothesis \(\Phi\) admits an APW asymmetric factorization, we have

\[
\Phi = \Phi_- D\Phi_e,
\]

where \(\Phi_- \in \mathcal{G}\text{APW}^{N \times N}\), \(D = \text{diag}[e_{\lambda_1}, \ldots, e_{\lambda_N}]\), and \(\Phi_e\) is an invertible even element. Without lost of generality (cf. Remark 3.2), we will assume that \(\lambda_1 \geq \cdots \geq \lambda_N\). As
previously observed, from (4.1) we therefore obtain the operator factorization

\[ WH_\Phi = W_\Phi \ell_0 W H_D \ell_0 W H_\Phi. \]  

(4.2)

We know that \( W_{\Phi_+} \) is invertible because \( \Phi_+ \in \mathcal{AP} W_{N \times N}^+ \) (and its inverse is given by \( \ell_0 W_{\Phi_+}^{-1} \ell_0 \)). Additionally, \( WH_{\Phi_+} \) is also invertible because \( \Phi_+ \) is an even element (cf. Proposition 2.2). Thus, (4.2) shows us an operator equivalence relation between \( WH_\Phi \) and \( WH_D \) (note that \( \ell_0 : [L^2(\mathbb{R}^+)]^N \to [L^2_+(\mathbb{R})]^N \) is invertible by \( r_+: [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}^+)]^N \)). We will therefore analyze the regularity properties of \( WH_D \).

Suppose that at least some of the partial indices are greater than zero, some of them may be equal to zero, and that some of them are less than zero; for instance, \( \lambda_1, \ldots, \lambda_i > 0, \lambda_{i+1} = \cdots = \lambda_j = 0 \), and \( \lambda_{j+1}, \ldots, \lambda_N < 0 \). This means that

\[
\ell_0 WH_D = \text{diag}[\ell_0 W H_{e_{i_1}}, \ldots, \ell_0 W H_{e_{i_1}}, \ell_0 W H_{e_{i_1+1}}, \ldots, \ell_0 W H_{e_{i_1}}, \ell_0 W H_{e_{i_{j+1}}}, \ldots, \ell_0 W H_{e_{i_{j+1}}}] = \text{diag}[\ell_0 W H_{e_{i_1}}, \ldots, \ell_0 W H_{e_{i_1}}, I, \ldots, I, \ell_0 W H_{e_{i_{j+1}}}, \ldots, \ell_0 W H_{e_{i_N}}],
\]

(4.3)

because \( WH_{e_{i_k}} = W_{e_{i_k}} \), for \( k = j+1, N \), due to the condition \( \lambda_{j+1} < 0, \ldots, \lambda_N < 0 \) and due to the structure of the Hankel operators (and also because \( \ell_0 W H_{e_{i_k}} = I, k = i+1, j \), due to the condition \( \lambda_{i+1} = \cdots = \lambda_j = 0 \)). The nonzero scalar operators in the diagonal matrix operator (4.3) are such that \( WH_{e_{i_1}}, \ldots, WH_{e_{i_j}} \) are properly \( n \)-normal and left-invertible (cf. [15, Theorem 6]); \( W_{e_{i_{j+1}}}, \ldots, W_{e_{i_N}} \) are \( d \)-normal and right-invertible (cf. the Gohberg-Feldman-Coburn-Douglas theorem [9, 11], [3, Theorem 2.28]). Therefore, \( WH_D \) cannot be semi-Fredholm, hence \( WH_\Phi \) cannot be semi-Fredholm. This proves part (a) of the theorem.

Suppose now that \( \lambda_i \leq 0 \), \( i = 1, N \). This implies that \( D \in \mathcal{AP} W_{N \times N}^+ \). Since \( \mathcal{AP} W_{N \times N}^+ = \mathcal{AP} W_{N \times N}^+ \cap [H^\infty(\mathbb{R})]^N \), it holds that \( D \in [H^\infty(\mathbb{R})]^N \) and hence \( WH_D = W_D \). So, in this case, \( WH_\Phi \) is equivalent to \( W_\Phi \). If we employ again the Gohberg-Feldman-Coburn-Douglas theorem to the each one of the operators in the main diagonal of the operator \( W_D \), it follows the assertion (b) of the theorem.

Part (c) can be deduced from the assertion (b) by passing to adjoints.

If all partial indices are zero, we have that \( \ell_0 WH_D \) is just the identity operator. This, together with the operator equivalence relation (4.2) presented in the first part of the proof, leads us to the last assertion (d).

\[ \square \]

5. Inverses representation

We now reach to our final goal: the representation of generalized/lateral/both-sided inverses of \( WH_\Phi \) based on a factorization of the Fourier symbol. This result extends the scalar version obtained in [15, Theorem 7].

Let us first recall that a bounded linear operator \( S^- : Y \to X \) (acting between Banach spaces) is called a reflexive generalized inverse of a bounded linear operator \( S : X \to Y \) if (i) \( S^- \) is a generalized inverse (or an inner pseudoinverse) of \( S \), that is, \( SS^- S = S \); (ii) \( S^- \) is an outer pseudoinverse of \( S \), that is, \( S^- SS^- = S^- \).
Theorem 5.1. Suppose $\Phi$ admits an APW asymmetric factorization and

$$T = \ell_0 r_+ F^{-1} \Phi^{-1}_e \cdot F \ell e r_+ F^{-1} \cdot \text{diag}[e_{-\lambda_1}, \ldots, e_{-\lambda_N}] \cdot F \ell e r_+ F^{-1} \Phi^{-1}_e$$

$$\cdot F \ell : [L^2(\mathbb{R}_+)]^N \rightarrow [L^2(\mathbb{R}_+)]^N,$$

(5.1)

where $\Phi^{-1}_e$ and $\Phi^{-1}$ are the inverses of the corresponding factors of an APW asymmetric factorization of $\Phi$, $\Phi = \Phi \cdot D \Phi_0$, and the operator $\ell : [L^2(\mathbb{R}_+)]^N \rightarrow [L^2(\mathbb{R}_+)]^N$ denotes an arbitrary extension operator (i.e., $T$ is independent of the particular choice of the extension $\ell$). Then the operator $T$ is a reflexive generalized inverse of $WH_\Phi$ and, in the following special cases, $T$ is additionally

(a) the right inverse of $WH_\Phi$, if $\lambda_i \leq 0$ for all $i = \overline{1,N}$;
(b) the left inverse of $WH_\Phi$, if $\lambda_i \geq 0$ for all $i = \overline{1,N}$;
(c) the both-sided inverse of $WH_\Phi$, if $\lambda_i = 0$ for all $i = \overline{1,N}$.

In the case when there exist partial indices with different signs, the operator $WH_\Phi$ is not Fredholm but $T$ is still a (reflexive) generalized inverse of $WH_\Phi$.

Proof. We start with the cases (a), (b), and (c). Since $\Phi$ admits an APW asymmetric factorization, we can write

$$\Phi = \Phi \cdot \text{diag}[e_{\lambda_1}, \ldots, e_{\lambda_N}] \Phi_e$$

(5.2)

(with the corresponding factor properties). Consequently, from (2.3), it follows that

$$WH_\Phi = r_+ A_- E A_e \ell e r_+,$$

(5.3)

where $A_- = \overline{F}^{-1} \Phi_\cdot \overline{F}$, $E = \overline{F}^{-1} \cdot \text{diag}[e_{\lambda_1}, \ldots, e_{\lambda_N}] \cdot \overline{F}$ and $A_e = \overline{F}^{-1} \Phi_e \cdot \overline{F}$.

(a) If $\lambda_i \leq 0$ for all $i = \overline{1,N}$, consider

$$WH_\Phi T = r_+ A_- E A_e \ell e r_+ \ell_0 r_+ A_e^{-1} \ell e r_+ E^{-1} \ell e r_+ A_e^{-1} \ell$$

$$= r_+ A_- E A_e \ell e r_+ A_e^{-1} \ell e r_+ E^{-1} \ell e r_+ A_e^{-1} \ell,$$

(5.4)

where the term $\ell_0 r_+$ was omitted due to the fact that $r_+ \ell_0 r_+ = r_+$. Since $A_e^{-1}$ preserves the even property of its symbol, we may also drop the first $\ell e r_+$ term in (5.4), and obtain

$$WH_\Phi T = r_+ A_- E \ell e r_+ E^{-1} \ell e r_+ A_e^{-1} \ell.$$

(5.5)

Additionally, due to the definition of $E$ and $E^{-1}$ in the present case ($\lambda_i \leq 0$ for all $i = \overline{1,N}$), we have $\ell_0 r_+ E \ell e r_+ E^{-1} \ell e r_+ = \ell_0 r_+$; also because $A_-$ is a minus type factor it follows $r_+ A_- = r_+ A_- \ell_0 r_+$. Therefore, from (5.5), we have

$$WH_\Phi T = r_+ A_- \ell_0 r_+ A_e^{-1} \ell = r_\ell \ell = I_{[L^2(\mathbb{R}_+)]^N}.$$

(5.6)

(b) If $\lambda_i \geq 0$ for all $i = \overline{1,N}$, we will now analyze the composition

$$TWH_\Phi = \ell_0 r_+ A_e^{-1} \ell e r_+ E^{-1} \ell e r_+ A_e^{-1} \ell e r_+ A_- E A_e \ell e r_+.$$

(5.7)
In the present case, due to the definition of $E^{-1}$, it follows $\ell^e r_+ E^{-1} \ell^e r_+ = \ell^e r_+ E^{-1}$. The same reasoning applies to the minus type factor $A^{-1}$, and therefore the equality (5.7) takes the form

$$TWH_\Phi = \ell_0 r_+ A^{-1} \ell^e r_+ A \ell^e r_+ = \ell_0 r_+ \ell^e r_+ = \ell_0 r_+ = I_{L_2(\mathbb{R})^n},$$

(5.8)

where we have used the fact that $\ell^e r_+ A \ell^e r_+ = A \ell^e r_+$.

(c) From the last two cases (a) and (b), it directly follows that in the case of $\lambda_i = 0$ for all $i = 1, N$, the operator $T$ is the both-sided inverse of $WH_\Phi$ (cf. (5.6) and (5.8)).

Let us now turn to the more general case: assume now that there exist partial indices $\lambda_i$.

As about the generalized inverse, we will start by rewriting the operator $E$ in the following new form:

$$E = \text{diag}[\mathcal{F}^{-1} e_{\lambda_{11}} \cdot \mathcal{F}, \ldots, \mathcal{F}^{-1} e_{\lambda_{1N}} \cdot \mathcal{F}] \text{diag}[\mathcal{F}^{-1} e_{\lambda_{21}} \cdot \mathcal{F}, \ldots, \mathcal{F}^{-1} e_{\lambda_{2N}} \cdot \mathcal{F}]$$

$$=: E_1 E_2,$$

(5.9)

where

$$\lambda_{1j} = \begin{cases} \lambda_j, & \text{if } \lambda_j \leq 0, \\ 0, & \text{if } \lambda_j \geq 0, \end{cases} \quad \lambda_{2j} = \begin{cases} \lambda_j, & \text{if } \lambda_j \geq 0, \\ 0, & \text{if } \lambda_j \leq 0, \end{cases}$$

(5.10)

for $j = 1, N$.

We will then directly compute $WH_\Phi TWH_\Phi$, in the following way:

$$WH_\Phi TWH_\Phi = (r_+ A_+ E_1 E_2 A \ell^e r_+) (\ell_0 r_+ A^{-1} \ell^e r_+ E_2^{-1} E_1^{-1} \ell^e r_+ A^{-1} \ell) (r_+ A_+ E_1 E_2 A \ell^e r_+)$$

$$= r_+ A_+ E_1 E_2 A \ell^e r_+ A^{-1} \ell^e r_+ E_2^{-1} E_1^{-1} \ell^e r_+ A^{-1} \ell r_+ A_+ E_1 E_2 A \ell^e r_+$$

$$= r_+ A_+ E_1 E_2 \ell^e r_+ E_2^{-1} E_1^{-1} \ell^e r_+ E_1 E_2 A \ell^e r_+$$

$$= r_+ A_+ E_1 E_2 A \ell^e r_+$$

$$= WH_\Phi,$$

(5.11)

(5.12)

(5.13)

(5.14)

(5.15)

(5.16)

where in (5.14) we omitted the first term $\ell^e r_+$ of (5.13) due to the factor (invariance) property of $A^{-1}$ that yields $A \ell^e r_+ A^{-1} \ell^e r_+ = \ell^e r_+$. Similarly we dropped the term $\ell r_+$ in $\ell^e r_+ A^{-1} \ell r_+ A_+$ due to a factor property of $A^{-1}$. Analogous arguments apply to the factors $E_2^{-1}$ and $E_1^{-1}$. In a more detailed way: (i) if one of the factors $E_1$ or $E_2$ equals $I$, then it is clear that $E_2 (\ell^e r_+ E_2^{-1} \ell^e r_+) E_2 = E_2 (\ell^e r_+ E_2^{-1}) E_2 = E_2 \ell^e r_+$ or $\ell_0 r_+ E_1 \ell^e r_+ E_1^{-1} \ell^e r_+ E_1 = \ell_0 r_+ E_1$ holds, respectively; (ii) in the general diagonal matrix case, the situation is identical just because in each place of the main diagonal we have the last situation. This justifies the simplification made in obtaining (5.15) from (5.14).
As about the composition $TWH_\Phi T$, it follows that

\[
TWH_\Phi T = (\ell_0 r_+ A_+^{-1} \ell^e r_+ E_2^{-1} E_1^{-1} \ell^e r_+ A_+^{-1} \ell)(r_+ A_+^{-1} \ell^e r_+ E_2^{-1} E_1^{-1} \ell^e r_+ A_+^{-1} \ell)
\]

\[
= \ell_0 r_+ A_+^{-1} \ell^e r_+ E_2^{-1} E_1^{-1} \ell^e r_+ A_+^{-1} \ell r_+ A_+^{-1} \ell^e r_+ A_+^{-1} \ell E_1^{-1} \ell^e r_+ A_+^{-1} \ell E_2^{-1} \ell^e r_+ A_+^{-1} \ell
\]

\[
= \ell_0 r_+ A_+^{-1} \ell^e r_+ E_2^{-1} E_1^{-1} \ell^e r_+ A_+^{-1} \ell
\]

\[
= T,
\]

(5.19)

where the third $\ell^e r_+$ is unnecessary in (5.18) due to the factor (invariance) property of $A_+$ that yields $A_+ \ell^e r_+ A_+^{-1} \ell^e r_+ = \ell^e r_+$, and we also can omit the term $\ell^e r_+$ in (5.18) since $A_+^{-1}$ is a minus type. Additionally, a similar reasoning as above was also used for obtaining equality (5.19) since due to the definitions of $E_1$ and $E_2$ it holds $\ell^e r_+ E_1 \ell^e r_+ = \ell^e r_+ E_1$, and $\ell^e r_+ E_2^{-1} \ell^e r_+ E_2 \ell^e r_+ E_2^{-1} = \ell^e r_+ E_2^{-1}$.

We end up by mentioning that almost all the above methods also work—without crucial changes—in the case of matrix Wiener-Hopf plus Hankel operators with almost periodic Fourier symbols. However, a corresponding version of Theorem 3.3 for invertible $APW_N \times N$ elements is an open problem. This has to do with the difficulties in substituting the arguments in the part of the proof of Theorem 3.3 where some representations of $APW$ elements are used.

**Acknowledgments**

This work was supported in part by Unidade de Investigação Matemática e Aplicações of University of Aveiro, through the Portuguese Science Foundation (FCT–Fundaçãoda para a Ciência e a Tecnologia).

**References**


Matrix Wiener-Hopf plus Hankel operators


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