THE ARMENDARIZ MODULE AND ITS APPLICATION TO THE IKEDA-NAKAYAMA MODULE

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A ring $R$ is called a right Ikeda-Nakayama (for short IN-ring) if the left annihilator of the intersection of any two right ideals is the sum of the left annihilators, that is, if $\ell(I \cap J) = \ell(I) + \ell(J)$ for all right ideals $I$ and $J$ of $R$. $R$ is called an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each $i, j$. In this paper, we show that if $R[x]$ is a right IN-ring, then $R$ is a right IN-ring in case $R$ is an Armendariz ring.

1. Introduction

Throughout this work, all rings will be associative with identity. Let $R$ be a ring. A right (or left) annihilator of a subset $U$ of $R$ is defined by $r_R(U) = \{a \in R : Ua = 0\}$ (or $\ell_R(U) = \{a \in R : aU = 0\}$).

Recall that, a ring $R$ is called a right Ikeda-Nakayama ring if the left annihilator of the intersection of any two right ideals is the sum of the left annihilators, that is, if $\ell(I \cap J) = \ell(I) + \ell(J)$ for all right ideals $I$ and $J$ of $R$ (cf. [6]). Let $sM_R$ be an $(S, R)$-bimodule. Extend the notion of an IN-ring to module such as $\ell_S(A \cap B) = \ell_S(A) + \ell_S(B)$ for any submodules $A, B$ of $M_R$ (cf. [10]).

For a module $M_R$, let $M[x]$ be the set of all formal polynomials in indeterminate $x$ with coefficients from $M$. Then $M[x]$ becomes a right $R[x]$-module under usual addition and multiplication of polynomials.

We prove that if $s[x]M[x]_R[x]$-bimodule and $U$ and $V$ are $R[x]$-submodules of $M[x]_R[x]$, then for any $t(x) \in \ell_{s[x]}(U \cap V)$, every $U + V \stackrel{a_{i,j}}{\rightarrow} M[x]$ extends commutatively to $M[x]$ by $\lambda(s(x))$ for some $s(x) \in S[x]$, where $\lambda : S[x] \rightarrow \text{End}(M[x]_R[x])$ if and only if $M[x]$ is an IN-module.

Following [1], $R$ is called Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each $i, j$. 

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A module $M$ is called $\alpha$-Armendariz if

(i) for any $m \in M$ and $a \in R$, $ma = 0$ if and only if $ma(a) = 0$;

(ii) for any $m(x) = \sum_{t=0}^{n} m_t x^t \in M[x]$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in R[x]$, $m(x)f(x) = 0$ implies $ma_j = 0$ for each $i, j$ (cf. [4, 6, 10]).

Professor Harmanci asked (private communication) for a

2.2 Lemma

In [5, Proposition 3.1], Hirano showed that

In particular, if $U$ is an Armendariz ring, then $R$ is a right IN-ring, in case $R$ is an Armendariz ring.

2. Ikeda-Nakayama modules

Let $S[x]$ and $R[x]$ be the polynomial rings over rings $S$ and $R$ and, for a module $sM_R$, let $M[x]$ be the set of all formal polynomials in indeterminate $x$ with coefficients from $M$. Then $M[x]$ becomes an $(S[x], R[x])$-bimodule under usual addition and multiplication of polynomials. Extend the notion of an IN-ring to module such as the following.

Definition 2.1. Recall that $M[x]$ is called an Ikeda-Nakayama module (IN-module) if

$$\ell_{S[x]}(U \cap V) = \ell_{S[x]}(U) + \ell_{S[x]}(V)$$

for any $R[x]$-submodules $U$ and $V$ of $M[x]_{R[x]}$. Such modules and rings were studied by many authors (cf. [4, 6, 10]). Professor Harmanci asked (private communication) for a description modules $M$ (rings $R$) such that $M[x] (R[x])$ are Ikeda-Nakayama modules (right Ikeda-Nakayama rings), respectively.

Note that there is a canonical ring homomorphism $\lambda : S[x] \to \text{End}(M[x]_{R[x]})$ given by $\lambda(s)(x)(m(x)) = s(x)m(x)$ for $m(x) \in M[x]$ and $s(x) \in S[x]$.

Let $U$ and $V$ be $R[x]$-submodules of $M[x]$. Then, for any $t(x) \in \ell_{S[x]}(U \cap V)$, $\alpha_{t(x)} : U + V \to M[x]$, $u + v \to t(x)u$ is well defined, where $u \in U$ and $v \in V$.

Lemma 2.2. Let $S[x]M[x]_{R[x]}$-bimodule and $U$ and $V$ be $R[x]$-submodules of $M[x]_{R[x]}$. Then, for any $t(x) \in \ell_{S[x]}(U \cap V)$, every $U + V \xrightarrow{\alpha_{t(x)}} M[x]$ extends commutatively to $M[x]$ by $\lambda(s(x))$ for some $s(x) \in S[x]$ if and only if $M[x]$ is an IN-module.

In particular, if $U \cap V = 0$, then every $U + V \xrightarrow{\alpha_{t(x)}} M[x]$ extends commutatively to $M[x]$ by $\lambda(s(x))$ for some $s(x) \in S[x]$ if and only if $S[x] = \ell_{S[x]}(U) + \ell_{S[x]}(V)$.

Proof. Let $t(x) \in \ell_{S[x]}(U \cap V)$. Then $\alpha_{t(x)} : U + V \to M[x]$, $u + v \to t(x)u$ is a well defined $R[x]$-module homomorphism, where $u \in U$ and $v \in V$. By assumption, there exists $s(x) \in S[x]$ such that $\lambda(s(x))$ extends to $\alpha_{t(x)}$. Thus, for all $u \in U$ and $v \in V$, $t(x)u = \alpha_{t(x)}(u + v) = \lambda(s(x))(u + v) = s(x)(u + v)$ and so $t(x) - s(x)u + (-s(x))v = 0$. It follows that $t(x) - s(x) \in \ell_{S[x]}(U)$ and $-s(x) \in \ell_{S[x]}(V)$. Hence $t(x) = (t(x) - s(x)) + (-s(x)) \in \ell_{S[x]}(U) + \ell_{S[x]}(V)$. The other inclusion is clear.

As a result of Lemma 2.2, we have the following proposition.
Proposition 2.3. Let \( R[x] \) be the set of all polynomials in indeterminate \( x \) with coefficients from \( R \). If \( I \) and \( J \) are right ideals of \( R[x] \) such that every \( R[x] \)-linear map \( I + J \to R[x] \) extends to \( R[x] \), then

\[
\ell_{R[x]}(I \cap J) = \ell_{R[x]}(I) + \ell_{R[x]}(J). \tag{2.2}
\]

In particular, this holds if \( I + J = R[x] \), in which case \( \ell_{R[x]}(I \cap J) = \ell_{R[x]}(I) \oplus \ell_{R[x]}(J) \).

Let \( N \) be an \( R[x] \)-submodule of \( M[x] \) and \( N_C = \{ m_i \in M : \exists n \in N \text{ with } n = m_0 + m_1x + \cdots + m_ix^i \} \).

Theorem 2.4. Let \( M \) be an Ikeda-Nakayama module and let \( N \) and \( K \) be \( R[x] \)-submodules of \( M[x] \) such that \( \ell_S((N \cap K)C) = \ell_S(N_C \cap K_C) \). Then \( M[x] \) is an IN-module.

Proof. Let \( U \) and \( V \) be \( R[x] \)-submodules of \( M[x] \). Let \( t(x) \in \ell_{S[x]}(U \cap V) \). Then \( \alpha_{t(x)} : U + V \to M[x], u + v \to t(u)v \) is a well defined \( R[x] \)-homomorphism, where \( u \in U \) and \( v \in V \). Similarly, for all \( t \in \ell_S(U_C \cap V_C) \), the \( \alpha_t : U_C + V_C \to M, u' + v' \to tu' \) is a well defined \( R \)-homomorphism, where \( u' \in U_C \) and \( v' \in V_C \). Since \( M \) is an IN-module, we have \( \ell_S(U \cap V)_C = \ell_S(N_C \cap K_C) = \ell_S(U_C + V_C) \) by assumption and definition. Hence there exists a homomorphism \( h_1 : M \to M \) such that \( h_1i = \alpha_i \), where \( i : U_C + V_C \to M \) is the inclusion map by \([10, \text{Lemma 1}]\). We define \( h' : M[x] \to M[x] \) such that \( h_i(k_0 + k_1x + \cdots + k_nx^n) = h_i(k_0) + h_i(k_1)x + \cdots + h_i(k_n)x^n \). It is clear that \( h'_i \) is well defined. Let \( t(x) = t_0 + t_1x + t_2x^2 + \cdots + t_nx^n \in \ell_{S[x]}(U \cap V) \). Then \( t_0, t_1, \ldots, t_n \in \ell_S(U \cap V)_C = \ell_S(U_C + V_C) \). For each \( t_j, \alpha_{t_j} : U_C + V_C \to M, u' + v' \to tu' \) is a well defined \( R \)-homomorphism, and then we define a map \( h_j : M \to M \) such that \( h_ji = \alpha_{t_j} \), where \( i : U_C + V_C \to M \) is the inclusion map. We extend it by defining \( h'_j : M[x] \to M[x] \) such that, for \( j = 0, 1, 2, \ldots, n \),

\[
h'_j(k_0 + k_1x + \cdots + k_nx^n) = (h_0(k_0) + h_1(k_1)x + \cdots + h_n(k_n)x^n). \]

To complete the proof, we show that \( hi = \alpha_{t(x)} \), where \( i' : U + V \to M[x] \) is the inclusion map. Let \( h = \sum_{j=0}^n h'_j \) and \( u = u_0 + u_1x + \cdots + u_nx^n \in U \) and \( v(x) = v_0 + v_1x + \cdots + v_nx^n \in V \). Then \( u_0, u_1, \ldots, u_n \in U_C \) and \( v_0, v_1, \ldots, v_n \in V_C \). So \( h'_j(u + v) = (h_i(u_0) + h_i(u_1)x + \cdots + h_i(u_n)x^n)x^j = t_jx^j(u_0 + u_1x + \cdots + u_nx^n) + h(u + v) = \sum_{j=0}^n h'_j(u + v) = t(x)(u + v) \). Hence \( M[x] \) is an IN-module by Lemma 2.2.

Let \( \alpha \) be an endomorphism of \( R \), that is, \( \alpha \) is a ring homomorphism from \( R \) to \( R \) with \( \alpha(1) = 1 \). Following \([9]\), a module \( M \) is called \( \alpha \)-Armendariz if

1. For any \( m \in M \) and \( a \in R \), \( ma = 0 \) if and only if \( ma(a) = 0 \);
2. For any \( m(x) = \sum_{i=0}^n m_ix^i \in M[x] \) and \( f(x) = \sum_{j=0}^na_jx^j \in R[x] \), \( m(x)f(x) = 0 \) implies \( m_ia_j = 0 \) for each \( i, j \).

Note that 1-Armendariz module is called Armendariz module.

We denote \( r\text{Ann}_R(2M) = \{ r_R(U) \mid U \subseteq M \} \) and \( \ell\text{Ann}_R(2M) = \{ \ell_R(U) \mid U \subseteq M \} \). If \( U \) is a subset of \( M \), then \( \ell_R[x](U) = r_R(U)[x] \) and \( r_R[x](U) = r_R(U)[x] \). Hence we have the maps

\[
\Phi : r\text{Ann}_R(2M) \to r\text{Ann}_R[x](2M[x]) \tag{2.3}
\]
defined by $\Phi(r_R(U)) = r_{R[x]}(U) = r_R(U)[x]$ for every $r_R(U) \in r\text{Ann}_R(2^M)$ and

$$\Phi' : r\text{Ann}_R(2^M) \rightarrow r\text{Ann}_{R[x]}(2^{M[x]})$$

(2.4)
defined by $\Phi'(\ell_R(U)) = r_{R[x]}(U) = \ell_R(U)[x]$ for every $\ell_R(U) \in \ell\text{Ann}_R(2^M)$.

For a polynomial $m(x) \in M[x]$, $C_m(x)$ denotes the set of coefficients of $m(x)$ and for a subset $V$ of $M[x]$, $C_V$ denotes the set $\bigcup_{m(x) \in V} C_m(x)$. Then

$$r_{R[x]}(V) \cap R = r_R(V) = r_R(C_V), \quad \ell_{R[x]}(V) \cap R = \ell_R(V) = \ell_R(C_V).$$

(2.5)

Hence we also have the maps

$$\Psi : r\text{Ann}_{R[x]}(2^{M[x]}) \rightarrow r\text{Ann}_R(2^M)$$

(2.6)
defined by $\Psi(r_{R[x]}(V)) = r_{R[x]}(V) \cap R$ for every $r_{R[x]}(V) \in r\text{Ann}_{R[x]}(2^{M[x]})$ and

$$\Psi' : \ell\text{Ann}_{R[x]}(2^{M[x]}) \rightarrow \ell\text{Ann}_R(2^M)$$

(2.7)
defined by $\Psi'(\ell_{R[x]}(V)) = \ell_{R[x]}(V) \cap R$ for every $\ell_{R[x]}(V) \in \ell\text{Ann}_{R[x]}(2^{M[x]})$.

Obviously $\Phi$ (or $\Phi'$) is injective and $\Psi$ (or $\Psi'$) is surjective. Also, $\Phi$ (or $\Phi'$) is surjective if and only if $\Psi$ (or $\Psi'$) is injective and in this case $\Phi$ and $\Psi$ (or $\Phi'$ and $\Psi'$) are the inverses of each other.

**Proposition 2.5.** Let $M_R$ be a module. Then the following are equivalent.

1. $M_R$ is an Armendariz module.
2. The map $\Phi : r\text{Ann}_R(2^M) \rightarrow r\text{Ann}_{R[x]}(2^{M[x]})$ defined by $\Phi(r_R(U)) = r_{R[x]}(U) = r_R(U)[x]$, for every $r_R(U) \in r\text{Ann}_R(2^M)$, is bijective.
3. The map $\Phi' : \ell\text{Ann}_R(2^M) \rightarrow \ell\text{Ann}_{R[x]}(2^{M[x]})$ defined by $\Phi'(\ell_R(U)) = \ell_{R[x]}(U) = \ell_R(U)[x]$, for every $\ell_R(U) \in \ell\text{Ann}_R(2^M)$, is bijective.

**Proof.** (1) $\Rightarrow$ (2). Assume $M$ is an Armendariz module. Obviously $\Phi'$ is injective. So it is enough to show $\Phi'$ is surjective. Let $\ell_{R[x]}(V) \in \ell\text{Ann}_{R[x]}(2^{M[x]})$ for some $V \subseteq M[x]$. Then for $\ell_R(C_V) \in \ell\text{Ann}_R(2^M)$, $\Phi'(\ell_R(C_V)) = \ell_{R[x]}(C_V) = \ell_{R[x]}(V)$. In fact, let $f(x) \in \ell_{R[x]}(C_V)$, where $f(x) = a_0 + a_1x + \cdots + a_nx^n$. Then $f(x)C_V = 0$. Thus for all $m \in C_V$, $f(x)m = a_0m + a_1mx + \cdots + a_nmx^n = 0$ and hence $a_jm = 0$ for all $j$. Let $n(x) = n_0 + n_1x + \cdots + n_mx^m \in V$ be arbitrary. Then $f(x)n(x) = 0$ since $n_i \in C_V$ for all $i$. Hence $f(x) \in \ell_{R[x]}(V)$. Conversely, let $g(x) = b_0 + b_1x + \cdots + b_kx^k \in \ell_{R[x]}(V)$. Then for all $m(x) \in V$, $g(x)m(x) = 0$, where $m(x) = m_0 + m_1x + \cdots + m_jx^j \in V$. Since $M_R$ is Armendariz, $b_jm_i = 0$ for all $i$ and $j$. Hence $g(x)m_i = 0$ for all $i$. So $g(x) \in \ell_{R[x]}(C_V)$ since $m(x) \in V$ is arbitrary. Consequently for each $\ell_{R[x]}(V) \in \ell\text{Ann}_{R[x]}(2^{M[x]})$ for some $V \subseteq M[x]$ there exists $\ell_{R}(C_V) \in \ell\text{Ann}_R(2^M)$ such that $\Phi'(\ell_R(C_V)) = \ell_{R[x]}(V)$, and therefore $\Phi'$ is surjective.
and \( J \) that 

\[ \ell R \]

So \( R \) the following addition and multiplication:

an IN-ring.

(\( i \)) Since \( J \) \( R \)

\[ (i) \quad \text{Let} \quad R \] \[ \text{be a trivial extension of} \quad Z \]

\[ \text{is an Armendariz module.} \]

(\( i \)) \[ \text{is Armendariz ring.} \]

\[ \text{for every right ideal} \quad I \]

\[ \text{and} \quad R \]

\[ \text{for} \quad j \]. \[ \text{Then} \]

\[ f(x) \in \ell_R(U)[x] \] and hence \( a_j \in \ell_R(U) \)

\[ \ell_R(U)[x] = \ell_{R[x]}(m(x)) \] then \( a_j m(x) = 0 \). Consequently, \( a_j m_i = 0 \) for all \( i \) and \( j \). Therefore \( M_R \) is an Armendariz module.

By Proposition 2.5, we can obtain \([5, \text{Proposition 3.1}]\).

**Proposition 2.6.** Let \( R \) be a ring. The following statements are equivalent.

(1) \( R \) is Armendariz ring.

(2) \( r\text{Ann}_R(2^R) \to r\text{Ann}_R(2^{R[x]}); A \to AR[x] \) is bijective, where \( r\text{Ann}_R(2^R) = \{ r_R(U) : U \subseteq R \} \).

(3) \( \ell\text{Ann}_R(2^R) \to \ell\text{Ann}_R(2^{R[x]}); B \to R[x]B \) is bijective, where \( \ell\text{Ann}_R(2^R) = \{ \ell_R(U) : U \subseteq R \} \).

Now, we give the main result of this work.

**Theorem 2.7.** Let \( R[x] \) is a right IN-ring, then \( R \) is a right IN-ring.

**Proof.** Let \( I \) and \( J \) be right ideals of \( R \). Since \( R \) is an Armendariz ring, we have \( \ell_{R[x]}(I) = \ell_R(I)[x] \) by Proposition 2.6, for every right ideal \( I \) of \( R \). Note that \( \ell_{R[x]}(I) = \ell_{R[x]}(I[x]) \). By assumption, \( \ell_{R[x]}(I) + \ell_{R[x]}(J) = \ell_{R[x]}(I[x]) + \ell_{R[x]}(J[x]) = \ell_{R[x]}((I \cap J)[x]) = \ell_{R[x]}(I \cap J) \). Then \( \ell_{R}(I \cap J)[x] = \ell_{R}(I[x]) + \ell_{R}(J[x]) = (\ell_{R}(I) + \ell_{R}(J))[x] \) implies that \( \ell_{R}(I \cap J) = \ell_{R}(I) + \ell_{R}(J) \). So \( R \) is a right IN-ring.

**Example 2.8.** (i) Since \( Z \) is an Armendariz ring, \( Z \) is a right IN-ring if and only if \( Z[x] \) is an IN-ring.

(ii) Let \( R \) be a trivial extension of \( Z \) and the \( Z \)-module \( Z_{2^n} \), that is, \( R = Z \oplus Z_{2^n} \) with the following addition and multiplication:

\[
(n, a) + (m, b) = (n + m, a + b),
\]

\[
(n, a)(m, b) = (nm, nb + ma).
\]

Also \( R \) is the subring \( \{ (\begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} : a \in Z, n \in Z_{2^n}) \} \). \( R \) is an IN-ring by \([10] \). As Lee and Zhou pointed out \([8, \text{Corollary 2.7}] \), \( R \) is an Armendariz ring. We consider the right ideals \( I \) and \( J \) of \( R[x] \):

\[
I = \left\{ \begin{pmatrix} px^2 & u(x) \\ 0 & px^2 \end{pmatrix} : u(x) \in Z_{2^n}, \ p \text{ is prime} \right\},
\]

\[
J = \left\{ \begin{pmatrix} qx + qx^2 & 0 \\ 0 & qx + qx^2 \end{pmatrix} : q \text{ is prime and } (p, q) = 1 \right\}.
\]

Clearly, \( \ell_{R[x]}(I \cap J) = R[x] \) since \( p \) and \( q \) are primes with \((p, q) = 1\) and so \( I \cap J = 0 \). But \( \ell_{R[x]}(I) \) and \( \ell_{R[x]}(J) \) do not contain constant. Therefore, \( \ell_{R[x]}(I) + \ell_{R[x]}(J) \neq \ell_{R[x]}(I \cap J) \).

So \( R[x] \) is not a right IN-ring by Proposition 2.3.
Recall that, a ring $R$ is called reduced ring if it has no nonzero nilpotent elements, a ring $R$ is called right $p.p.$-ring for all $a \in R$, $r_{R}(a) = eR$, where $e^{2} = e \in R$ and $R$ is called Baer ring, for all $I \leq_{R} R$, $r_{R}(I) = eR$, where $e^{2} = e \in R$.

As a result of Theorem 2.7, we can say the following corollary.

**Corollary 2.9.** Let $R[x]$ be a right IN-ring. Then $R$ is a right IN-ring in each of the following cases.

1. $R^{2} = 0$.
2. $R$ is a reduced ring.
3. $R$ is an Abelian (if every idempotent of $R$ is central) and von Neumann regular ring.
4. $R$ is an Abelian right (left) $p.p.$-ring.
5. $R$ is an Abelian Baer ring.

**Proof.** Assume $R[x]$ is a right IN-ring.

1. By [1], if $R^{2} = 0$, then $R$ is an Armendariz ring.
2. By [2], reduced rings are Armendariz.
3. Every Abelian von Neumann regular ring is a reduced ring.
4. By [1, Theorem 6] or [7, Lemma 7], if $R$ is an Abelian right (left) $p.p.$-ring, then $R$ is an Armendariz (a Reduced and so Armendariz) ring.
5. Every Abelian Baer ring is a reduced ring.

Hence $R$ is a right IN-ring by Theorem 2.7.

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