We give a different proof of a lemma by Phelps (1960) which asserts, roughly speaking, that if two norm-one functionals \( f \) and \( g \) have their hyperplanes \( f^{-1}(0) \) and \( g^{-1}(0) \) sufficiently close together, then either \( \|f - g\| \) or \( \|f + g\| \) must be small. We also extend this result to a complex Banach space.

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In 1960 in [2], Phelps proved the following lemma.

**Lemma 1.** Suppose that \( E \) is a real normed linear space and that \( \epsilon > 0 \). If \( f, g \in S^* \) are such that \( f^{-1}(0) \cap \mathcal{U} \subset g^{-1}[-\epsilon/2, \epsilon/2] \), then either \( \|f - g\| \leq \epsilon \) or \( \|f + g\| \leq \epsilon \). (Here, \( \mathcal{U} \) represents the unit ball of \( E \) and \( S^* \) is the unit sphere of \( E^* \).)

This lemma was then used the following year as a crucial step in the proof of the well-known Bishop-Phelps theorem [1] that every Banach space is subreflexive; in other words, every functional on a Banach space \( E \) can be approximated by a norm-attaining functional on the same space. The original proof of this lemma uses the Hahn-Banach theorem and is therefore fairly abstract.

In this note, we present an alternate proof for Lemma 1 stated above. This proof gives a geometric argument while extending the lemma to a complex Banach space. Lemma 1 is shown to be a special case when the bound of \( \epsilon \) on \( \|f \pm g\| \) is replaced by \( 5\epsilon \). This replacement does not affect the fundamental conclusion of Lemma 1.

We now state the extended lemma.

**Lemma 2.** Let \( X \) be a complex Banach space and let \( \epsilon \) be such that \( 0 < \epsilon < 1/2 \). Let \( \varphi, \psi \in X^* \), \( \|\varphi\| = \|\psi\| = 1 \). Suppose that for all \( x \in X \) with \( \|x\| \leq 1 \) and \( \varphi(x) = 0 \), it holds that \( |\psi(x)| \leq \epsilon \). Then there is some complex number \( \alpha \) such that \( |\alpha| = 1 \) and \( \|\varphi - \alpha \psi\| \leq 5\epsilon \).

It will be shown that if \( \varphi \) and \( \psi \) are real-valued functionals on a real Banach space \( X \), then \( \alpha \) will in fact be either 1 or \( -1 \), thus proving the amended original result.

We now prove Lemma 2.
2 A new proof of a lemma by Phelps

Proof. Let \( e \in X \) be such that \( \|e\| = 1 \) and \( |\varphi(e)| \geq 1 - \varepsilon/4 \). We will first show that \( |\psi(e)| \geq 1 - (5/2)\varepsilon \). To see this, let \( f \in X \) such that \( \|f\| = 1 \) and \( |\varphi(f)| \geq 1 - \varepsilon/4 \). Let \( k = 1 - \varepsilon/4 \) and let \( t = \varphi(f)/\varphi(e) \). Then \( 0 \leq |t| \leq 1/(1 - \varepsilon/4) = 1/k \leq 8/7 \) and if we take \( w = (k/(k + 1))(f - te) \), then \( \|w\| = (k/(k + 1))(\|f\| + |t|\|e\|) \leq (k/(k + 1))(1 + 1/k) = 1 \). Moreover,

\[
\varphi(w) = \frac{k}{k + 1} \left( \varphi(f) - \frac{\varphi(f)}{\varphi(e)} \varphi(e) \right) = 0
\]

so we have

\[
\varepsilon \geq |\psi(w)| = \frac{k}{k + 1} |\varphi(f) - t\varphi(e)| \geq \frac{k}{k + 1} \left( |\varphi(f)| - |t|\left| \varphi(e) \right| \right) \geq \frac{k}{k + 1} \left( \left| \varphi(f) \right| - \left| t \right| \left| \varphi(e) \right| \right).
\]

Thus

\[
\frac{1}{k} \left| \varphi(e) \right| \geq |t| \left| \varphi(e) \right| \geq |\varphi(f)| - \frac{k + 1}{k} \varepsilon \geq \left( 1 - \frac{\varepsilon}{4} \right) - \frac{k + 1}{k} \varepsilon = k - \frac{k + 1}{k} \varepsilon.
\]

This gives

\[
|\varphi(e)| \geq k^2 - (k + 1)\varepsilon = \left( 1 - \frac{\varepsilon}{4} \right)^2 - \left( 2 - \frac{\varepsilon}{4} \right) \varepsilon = 1 - \varepsilon + \frac{\varepsilon^2}{16} - 2\varepsilon + \frac{\varepsilon^2}{4} \geq 1 - \frac{5}{2} \varepsilon
\]

as required. Notice that, if \( \varphi \) and \( \psi \) are real valued, the above still holds.

Now, there exist \( \beta, \gamma \in \mathbb{C} \) such that \( |\beta| = |\gamma| = 1 \), \( \beta \varphi(e) \in [1 - \varepsilon/4, 1] \subset \mathbb{R} \), and \( \gamma \varphi(e) \in [1 - 5\varepsilon/2, 1] \subset \mathbb{R} \); and so \( |\beta \varphi(e) - \gamma \varphi(e)| \leq 5\varepsilon/2 \).

Let \( x \in X \) be such that \( \|x\| \leq 1 \) and write \( x = \lambda e + y \), where \( \lambda = \varphi(x)/\varphi(e) \) and \( y = x - \lambda e \). Then \( |\lambda| \leq |\varphi(x)|/|\varphi(e)| \leq 1/(1 - \varepsilon/4) \leq 8/7 \), \( \|y\| \leq \|x\| + |\lambda|\|e\| \leq 15/7 \), and \( \varphi(y) = \varphi(x) - (\varphi(x)/\varphi(e)) \varphi(e) = 0 \). So, by hypothesis, \( |\psi(7/15)y| \leq \varepsilon \), that is, \( |\psi(y)| \leq (15/7)\varepsilon \). Then, if we take \( \alpha = \gamma/\beta \), we have \( |\alpha| = 1 \) and

\[
|\varphi(x) - \alpha \psi(x)| = \frac{1}{|\beta|} \left| \beta \varphi(x) - \gamma \varphi(e) \right| = \left| \beta \lambda \varphi(e) + \beta \varphi(y) - \gamma \lambda \varphi(e) - \gamma \varphi(y) \right| \leq |\lambda| \left| \beta \varphi(e) - \gamma \varphi(e) \right| + |\gamma| \left| \psi(y) \right| \leq \frac{8}{7} \cdot \frac{5}{2} \varepsilon + 1 \cdot \frac{15}{7} \varepsilon = 5\varepsilon.
\]

But \( x \) was an arbitrary element of the unit ball of \( X \), so we have \( \|\varphi - \alpha \psi\| \leq 5\varepsilon \).

Notice that if \( X \) is a real Banach space, the above argument still holds. Also, if \( \varphi \) and \( \psi \) are real valued, we can choose \( e \in X \) such that \( \varphi(e) \geq 1 - \varepsilon/4 \), giving \( \beta = 1 \), and then from the claim, either \( \gamma = 1 \) or \( \gamma = -1 \). So either \( \alpha = 1 \) or \( \alpha = -1 \), yielding Phelps’ result, up to a constant. \( \square \)
References


Antonia E. Cardwell: Department of Mathematics, Millersville University of Pennsylvania, Millersville, PA 17551-0302, USA
E-mail address: antonia.cardwell@millersville.edu