A NEWTON-TYPE METHOD AND ITS APPLICATION

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We prove an existence and uniqueness theorem for solving the operator equation \( F(x) + G(x) = 0 \), where \( F \) is a continuous and Gâteaux differentiable operator and the operator \( G \) satisfies Lipschitz condition on an open convex subset of a Banach space. As corollaries, a recent theorem of Argyros (2003) and the classical convergence theorem for modified Newton iterates are deduced. We further obtain an existence theorem for a class of nonlinear functional integral equations involving the Urysohn operator.

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1. Introduction

This paper considers the problem of approximating a locally unique solution \( x^* \) of the equation \( F(x) + G(x) = 0 \), where \( F \) and \( G \) are continuous operators defined on an open convex subset \( D \) of a Banach space \( X \) with values in a Banach space \( Y \). The solution is obtained as the limit of a sequence of Newton-type iterates given by

\[
x_{n+1} = x_n - (F'_{x_n})^{-1} [F(x_n) + G(x_n)].
\] (1.1)

It may be noted that for \( G \equiv 0 \), (1.1) reduces to Newton’s method of iterates whose convergence is proved under the usual hypotheses that \( F \) is Fréchet differentiable and \( F'_{x} \) is invertible. Recently, Argyros [3] obtained an interesting generalization of Newton’s method which is further extended in this paper along with an earlier result of Vijesh and Subrahmanyam [1]. More specifically, we prove the convergence of a sequence of Newton-type iterates under mild conditions on \( F \). In particular, the Gâteaux differentiable operator \( F \) is assumed to satisfy the inequality \( \| (F'_{x_0})^{-1} (F'_x - F'_{x_0}) \| \leq \epsilon \) in a certain neighbourhood \( N(x_0) \) of \( x_0 \) while \( G \) is required to be a contraction in \( N(x_0) \). Using the main theorem, the existence of a unique continuous solution to a nonlinear functional integral equation involving the Urysohn operator is deduced from a corollary to the main theorem. This seems to be the first application of Newton-type algorithm to such nonlinear equations. Illustrative examples are also discussed.
2. Convergence analysis

Theorem 2.1 is a general theorem on the convergence of Newton-type iterates, proved under mild assumptions. It generalizes the theorems in [1, 3].

**Theorem 2.1.** Let $F$ and $G$ be continuous operators defined on an open convex subset $D$ of a Banach space $X$ with values in a Banach space $Y$. Suppose that $F$ is Gâteaux differentiable at each point in some neighbourhood of $x_0 \in D$ and $G$ is Lipschitz on $D$. Assume further that

(i) $(F'_{x_0})^{-1} \in L(Y, X)$, the space of bounded linear operators from $Y$ to $X$;

(ii) for some $\eta \geq 0$, $\|(F'_{x_0})^{-1}[F(x_0) + G(x_0)]\| \leq \eta$;

(iii) for some $\eta \geq 0$, $\|(F'_{x_0})^{-1}[F'_x - F'_{x_0}]\| \leq \epsilon$ ($\epsilon > 0$) whenever $x \in U(x_0, r)$, where $U(x_0, r) = \{x \in B : \|x - x_0\| < r\}$ and $\|G(x) - G(y)\| \leq k\|x - y\|$ for $x, y \in D$ such that $3\epsilon + \epsilon^* < 1$ and $(1 + c_0/(1 - c))\eta < r$, where $\epsilon^* = k\|(F'_{x_0})^{-1}\|, c_0 = (\epsilon + \epsilon^*)/(1 - \epsilon)$, and $c = (2\epsilon + \epsilon^*)/(1 - \epsilon)$;

(iv) $F'_x$ is piecewise hemicontinuous for each $x \in U(x_0, r)$ and $\overline{U}(x_0, r) \subset D$.

Then the sequence $x_n$ ($n \geq 0$) generated recursively by (1.1) is well defined, remains in $U(x_0, r)$ for all $n \geq 0$, and converges to a unique solution $x^* \in \overline{U}(x_0, r)$ of the equation $F(x) + G(x) = 0$. Moreover, the following error bounds hold for all $n \geq 2$:

\[
\|x_{n+1} - x_n\| \leq c^{n-1}c_0 \eta, \\
\|x_n - x^*\| \leq \frac{c^{n-1}}{1 - c} c_0 \eta. 
\]

**Proof.** Clearly by (ii) $\|x_1 - x_0\| = \|(F'_{x_0})^{-1}[F(x_0) + G(x_0)]\| \leq \eta < r$, and hence $x_1 \in U(x_0, r)$. It follows from the choice of $\epsilon$ and the well-known Banach lemma (see Rall [4, page 50]) that $(F'_x)^{-1}$ exists and $\|(F'_{x_0})^{-1}F'_{x_0}\| \leq 1/(1 - \epsilon)$. Moreover (1.1) gives for $n = 1$ that

\[
x_2 - x_1 = - (F'_{x_1})^{-1} [F(x_1) + G(x_1)]
= - (F'_{x_1})^{-1} (F'_{x_0} F'_{x_0})^{-1} [F(x_1) + G(x_1) - F(x_0) - G(x_0)]
= - (F'_{x_1})^{-1} (F'_{x_0})^{-1}
\times \left\{ \int_0^1 (F'_{\theta x_1 + (1 - \theta)x_0} - F'_{x_0}) (x_1 - x_0) d\theta + G(x_1) - G(x_0) \right\} \quad \text{(using (iv))}
= - (F'_{x_1})^{-1} F'_{x_0}
\times \left\{ \int_0^1 (F'_{x_0})^{-1} [F'_{\theta x_1 + (1 - \theta)x_0} - F'_{x_0}] (x_1 - x_0) d\theta + (F'_{x_0})^{-1} [G(x_1) - G(x_0)] \right\}.
\]

So

\[
\|x_2 - x_1\| \leq \|(F'_{x_0})^{-1}F'_{x_0}\|
\times \left\{ \int_0^1 \|(F'_{x_0})^{-1} [F'_{\theta x_1 + (1 - \theta)x_0} - F'_{x_0}] (x_1 - x_0) d\theta\| + \|(F'_{x_0})^{-1} [G(x_1) - G(x_0)]\| \right\}
\leq \frac{1}{1 - \epsilon} \left\{ \epsilon \|x_1 - x_0\| + \|(F'_{x_0})^{-1} k \|x_1 - x_0\| \right\}
\]

Thus, the sequence $x_n$ converges to a unique solution $x^* \in \overline{U}(x_0, r)$ of the equation $F(x) + G(x) = 0$.
In view of hypotheses (iii) and (iv), it follows as before from Banach’s lemma that
\[
\frac{1}{1 - \epsilon} \{\epsilon \|x_1 - x_0\| + \epsilon^* \|x_1 - x_0\|\} = \frac{\epsilon + \epsilon^*}{1 - \epsilon} \|x_1 - x_0\|.
\]
Thus \(x_2 \in U(x_0, r)\). Again by Banach’s lemma (see [4]) \((F'_{x_2})^{-1}\) exists and \(\|(F'_{x_2})^{-1}F'_{x_0}\| \leq 1/(1 - \epsilon)\). Assume that
\[
x_k \in U(x_0, r), \quad ||x_{k+1} - x_k|| \leq c^k c_0 \eta \quad \text{for} \quad k = 2, 3, \ldots, n - 1.
\]
In view of hypotheses (iii) and (iv), it follows as before from Banach’s lemma that \((F'_{x_n})^{-1}\) exists and
\[
\|(F'_{x_n})^{-1}F'_{x_0}\| \leq \frac{1}{1 - \epsilon}.
\]
By hypotheses (iii) and (iv) and (2.5), we obtain
\[
\begin{align*}
x_{n+1} - x_n &= -(F'_{x_n})^{-1}(F(x_n) + G(x_n)) \\ &= -(F'_{x_n})^{-1}F'_{x_0}(F'_{x_n})^{-1}(F(x_n) + G(x_n)) \\ &= -(F'_{x_n})^{-1}F'_{x_0}(F'_{x_0})^{-1} \\ &\quad \times [F(x_n) + G(x_n) - F(x_{n-1}) - G(x_{n-1}) + F(x_{n-1}) + G(x_{n-1})] \\ &= -(F'_{x_n})^{-1}F'_{x_0}(F'_{x_0})^{-1} \\ &\quad \times \left\{\int_0^1 ([F'_{x_0} + (1-\theta)x_{n-1} - F'_{x_0}] + [F'_{x_0} - F'_{x_{n-1}}]) (x_n - x_{n-1}) d\theta + G(x_n) - G(x_{n-1})\right\} \\ &= -(F'_{x_n})^{-1}F'_{x_0}\left\{\int_0^1 (F'_{x_0})^{-1}([F'_{0x_0} + (1-\theta)x_{n-1} - F'_{0x_0}] + [F'_{0x_0} - F'_{x_{n-1}}]) (x_n - x_{n-1}) d\theta \\ &\quad + (F'_{x_0})^{-1}[G(x_n) - G(x_{n-1})]\right\}, \\ ||x_{n+1} - x_n|| &\leq ||(F'_{x_n})^{-1}F'_{x_0}||\left\{\int_0^1 ||(F'_{x_0})^{-1}([F'_{0x_0} + (1-\theta)x_{n-1} - F'_{0x_0}] + [F'_{0x_0} - F'_{x_{n-1}}]) (x_n - x_{n-1})|| d\theta \\ &\quad + \int_0^1 ||F'_{x_0}||^2 ||x_n - x_{n-1}|| ||(x_n - x_{n-1})|| d\theta \\ &\quad + ||(F'_{x_0})^{-1}||k||x_n - x_{n-1}||\right\}, \\ ||x_{n+1} - x_n|| &\leq \frac{2\epsilon + \epsilon^*}{1 - \epsilon} ||x_n - x_{n-1}||.
\end{align*}
\]
Thus by induction hypothesis (2.4), \( \|x_{n+1} - x_n\| \leq c^{n-1}c_0\eta. \) Since
\[
\|x_{n+1} - x_0\| \leq \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| + \cdots + \|x_1 - x_0\|
\leq c^{n-1}c_0\eta + c^{n-2}c_0\eta + \cdots + c_0\eta + \eta,
\]
\[
\|x_{n+1} - x_0\| \leq \eta \left[ 1 + c_0 \frac{1 - c^n}{1 - c} \right] \leq \eta \left[ 1 + \frac{c_0}{1 - c} \right] < r. \tag{2.7}
\]

Hence \( x_n \in U(x_0, r) \) for all \( n \geq 0. \) For \( k \geq 2, \ m \geq 0. \)
\[
\|x_{k+m} - x_k\| \leq \|x_{k+m} - x_{k+m-1}\| + \|x_{k+m-1} - x_{k+m-2}\| + \cdots + \|x_{k+1} - x_k\|
\leq c^{k+m-2}c_0\eta + c^{k+m-3}c_0\eta + \cdots + c^{k-1}c_0\eta
\leq \eta c_0 c^{k-1} \left[ 1 + c + \cdots + c^{m-1} \right]
\leq \frac{1 - c^m}{1 - c} c^{k-1} c_0\eta \leq \frac{c^{k-1}}{1 - c} c_0\eta. \tag{2.8}
\]

As \( 0 < c < 1, \) \( x_n \) is a Cauchy sequence in the closed subset \( \overline{U}(x_0, r) \) of the Banach space \( X; \) it hence converges to an element \( x^* \) in \( \overline{U}(x_0, r). \) From hypothesis (iii) using triangle inequality, it follows that \( \|F_{x_n}'\| \leq M, \) where \( M = (\epsilon/\|F_{x_0}'\| + \|F_{x_0}'\|). \) Since
\[
x_{n+1} = x_n - (F_{x_n}')^{-1}(F(x_n) + G(x_n)) = F(x_n) + G(x_n) + F_{x_n}'(x_{n+1} - x_n). \tag{2.9}
\]

So
\[
\|F(x_n) + G(x_n)\| \leq \|F_{x_n}'\|\|x_{n+1} - x_n\| \leq M\|x_{n+1} - x_n\|. \tag{2.10}
\]

Proceeding to the limit in (2.10) as \( n \) tends to infinity and using the continuity of \( F \) and \( G \) it follows from the convergence of \( (x_n) \) to \( x^* \) that \( F(x^*) + G(x^*) = 0. \) If \( x^* \) and \( y^* \) are two solutions of \( F(x) + G(x) = 0 \) in \( \overline{U}(x_0, r), \) then
\[
\|x^* - y^*\| = \|x^* - y^* - (F_{x_0}')^{-1}[F(x^*) + G(x^*) - F(y^*) - G(y^*)]\| \leq \|\int_0^1 \left[ I - (F_{x_0}')^{-1}F_{x^*+(1-\theta)y^*} \right] (x^* - y^*) \, d\theta \|
\leq \|\int_0^1 (F_{x_0}')^{-1}(G(x^*) - G(y^*)) \, d\theta \|
\leq \epsilon \|x^* - y^*\| + \epsilon^* \|x^* - y^*\| \leq (\epsilon + \epsilon^*) \|x^* - y^*\| < \|x^* - y^*\| \quad \text{as} \quad 0 \leq \epsilon + \epsilon^* \leq 3\epsilon + \epsilon^* < 1. \tag{2.11}
\]

This contradiction implies that \( x^* = y^*. \) Hence the theorem holds.

\begin{corollary} \text{(see Argyros [3, Theorem 1]).} \ Let \( F \) be a continuous operator defined on an open convex subset \( D \) of a Banach space \( X \) with values in a Banach space \( Y \) and continuously Fréchet differentiable at some \( x_0 \in D. \) Assume that \( \)
\begin{itemize}
  \item[(i)] \( (F_{x_0}')^{-1} \in L(Y, X); \)
  \item[(ii)] there exists a parameter \( \eta \) such that \( 0 \leq \| (F_{x_0}')^{-1}F(x_0) \| \leq \eta; \)
\end{itemize}
\end{corollary}
(iii) for some \( \epsilon \in (0, 1/3) \), there exists \( \delta > 0 \) such that \( \|(F'_{x_n})^{-1}(F'_x - F'_{x_n})\| \leq \epsilon \) whenever \( x \in U(x_0, \delta) = \{x \in X : \|x - x_0\| < \delta\} \), with \( (c^2/(1 - c) + c_0 + 1)\eta < \delta \), where \( c_0 = \epsilon/(1 - \epsilon) \) and \( c = 2c_0 \).

Then the Newton iterates \( (x_n) \) generated by (1.1) are well defined, remain in \( U(x_0, \delta) \) for all \( n \geq 0 \), and converge to a solution \( x^* \in \overline{U}(x_0, \delta) \) of equation \( F(x) = 0 \). Moreover, for all \( n \geq 2 \), the following error bounds hold:

\[
\begin{align*}
\|x_{n+1} - x_n\| &\leq c^n\|x_1 - x_0\|, \\
\|x_n - x^*\| &\leq \frac{c^n}{1 - c}\|x_1 - x_0\|.
\end{align*}
\] (2.12)

Proof. Since \( F \) is Fréchet differentiable, it is Gâteaux differentiable. Take \( G \equiv 0 \) in Theorem 2.1. Let \( \delta = r \) and note that

\[
\delta > \left[ \frac{c^2}{1 - c} + c_0 + 1 \right] \eta \geq \left[ 1 + \frac{c_0}{1 - c} \right] \eta.
\] (2.13)

Since \( F \) is continuously Fréchet differentiable, \( F'_x \) is hemicontinuous at each \( x \in U(x_0, \delta) \) and thus all the conditions of Theorem 2.1 are satisfied. So \( F \) has a unique zero in \( \overline{U}(x_0, \delta) \).

Corollary 2.3. Let \( F \) be a continuous operator defined on an open convex subset \( D \) of a Banach space \( X \) with values in a Banach space \( Y \) having a Gâteaux derivative at each point in some neighbourhood of \( x_0 \in D \). Assume further that

(i) the inverse of \( F'_{x_0} \) exists and \( (F'_{x_0})^{-1} \in L(Y, X) \);

(ii) for some \( \eta > 0 \) \( \|(F'_{x_0})^{-1}F(x_0)\| \leq \eta \);

(iii) for some \( r > 0 \) \( \|(F'_{x_0})^{-1}[F'_x - F'_{x_0}]\| \leq \epsilon \) whenever \( x \in U(x_0, r) \). Suppose \( (1 + c_0/(1 - c))\eta < r \) and \( 0 < 3\epsilon < 1 \), where \( c_0 = \epsilon/(1 - \epsilon) \) and \( c = 2c_0 \);

(iv) \( F'_x \) is piecewise hemi continuous at each \( x \in U(x_0, r) \).

Then the modified Newton iterates \( (x_n), n \geq 0 \), generated by

\[
x_{n+1} = x_n - (F'_{x_n})^{-1}F(x_n)
\] (2.14)

are well defined, remain in \( U(x_0, r) \) for all \( n \geq 0 \), and converge to a solution \( x^* \in \overline{U}(x_0, r) \) of \( F(x) = 0 \).

To prove Corollary 2.3, set \( G \equiv 0 \) in Theorem 2.1 and proceed as in Theorem 2.1.

3. Solutions of a class of nonlinear functional integral equations

In this section, some existence theorems for an operator equation involving the Urysohn operator are proved. More specifically, given a compact subset \( \Omega \) of \( \mathbb{R} \), \( g \in C(\Omega \times \mathbb{R}) \) and \( K \in C(\Omega^2 \times \mathbb{R}) \), Theorem 3.2 gives sufficient conditions for the existence of a unique solution of the nonlinear functional integral equation of the form

\[
x(t) + \int_{\Omega} K(t, s, x(s)) \, ds + g(t, x(t)) = 0 \quad \forall t \in \Omega, \ x \in C(\Omega).
\] (3.1)

Hereafter we denote \( x(t) + \int_{\Omega} K(t, s, x(s)) \, ds \) by \( Fx(t) \).
For \( K \in C(\Omega^2 \times \mathbb{R}) \), suppose that \( K_3(t,s,u) = \partial K/\partial u \in C(\Omega^2 \times \mathbb{R}) \) satisfies the Lipschitz condition with respect to the third variable. Then the Fréchet derivative of \( F \) exists for \( x \in U(x_0,r) \) and \( x_0 \in C(\Omega) \) and the derivative at \( x \) is given by

\[
F'_x(h) = h(t) + \int_\Omega K_3(t,s,x(s)) \, h(s) \, ds.
\] (3.2)

For the proof, see [2].

**Theorem 3.2.** Let \( K(t,s,u) \in C(\Omega^2 \times \mathbb{R}) \), let \( K_3(t,s,u) \in C(\Omega^2 \times \mathbb{R}) \), let \( g \in C(\Omega \times \mathbb{R}) \), and let \( \Omega \) be a compact subset of \( \mathbb{R} \) whose Lebesgue measure is equal to \( d > 0 \). Suppose that

(i) for some \( m \in (0,1) \), \( \|K_3(t,s,u_1) - K_3(t,s,u_2)\|_2 \leq m \|u_1 - u_2\| \) holds for all \( (t,s,u) \in \Omega^2 \times \mathbb{R}, i = 1,2 \);

(ii) \( |g(t,y_1) - g(t,y_2)| \leq \epsilon^* |y_1 - y_2| \) for all \( y_1, y_2 \in \mathbb{R}, t \in \Omega \), where \( \beta + \epsilon^* < 1 \beta = \sup_{t \in \Omega} |\int_\Omega K_3(t,s,x_0(s)) \, ds| \). Further let \( \eta \geq 0 \) be chosen such that \( \eta \geq \sup_{t \in \Omega} |F_{x_0}(t) + g(t,x_0(t))| \);

(iii) let \( \alpha_1 \) and \( \alpha_2 \) be the positive roots of \( p(x) = 4m^2d^2x^2 - 4md(1 - \beta)(2 - 2\beta - \epsilon^*)x + [(1 - \beta - \epsilon^*)(1 - \beta)]^2 \) and let \( r_1 \) and \( r_2 \) with \( r_1 \leq r_2 \) be the positive roots of \( q(x) = 3md(1 - \beta)x^2 - [(1 - \beta - \epsilon^*)(1 - \beta) + 2md\eta]x + (1 - \beta)\eta; \)

(iv) suppose further that \( \eta \leq \min \{\alpha_1, \alpha_2\} \) and \( \eta/(1 - \beta) < r_1 < (1 - \beta - \epsilon^*)/3md \).

Then the functional integral equation \( Fx(t) + g(t,x(t)) = 0 \) for all \( t \in \Omega \) has a unique solution in \( U(x_0,r_0) \) for all \( r_0 \in (r_1, \min \{r_2, (1 - \beta - \epsilon^*)/3md\}) \) and the sequence of iterates given in (1.1) converges to this solution.

**Proof.** Clearly \( F \) maps \( C(\Omega) \) into itself. Let \( G : C(\Omega) \rightarrow C(\Omega) \) be defined by \( Gx(t) = g(t,x(t)) \forall t \in \Omega \). Then \( G \) maps \( C(\Omega) \) into itself,

\[
\|Gx_1(t) - Gx_2(t)\|_2 \leq \epsilon^* \|x_1(t) - x_2(t)\|_2 \quad \text{by (ii)}
\] (3.3)

Thus \( \|Gx_1 - Gx_2\|_2 \leq \epsilon^* \|x_1 - x_2\|_2. \) Assumption (i) together with Lemma 3.1 implies that \( F'_x \) exists. Now,

\[
\| (I - F'_{x_0})h(t) \| = \| h(t) - h(t) - \int_\Omega K_3(t,s,x_0(s)) \, h(s) \, ds |)
\]

\[
\leq \int_\Omega |K_3(t,s,x_0(s)) \, h(s) \, ds| \leq \|h\| \int_\Omega \|K_3(t,s,x_0(s))\|_2 \, ds
\]

\[
\leq \|h\| \beta.
\] (3.4)

This implies that \( \| I - F'_{x_0} \| \leq \beta < 1 \), and hence \( F'_{x_0} \) is invertible and by Banach’s lemma, \( \| (F'_{x_0})^{-1} \| \leq 1/(1 - \beta) \). Since the roots \( \alpha_1 \) and \( \alpha_2 \) of \( p(x) \) are positive and

\[
(x - \alpha_1)(x - \alpha_2) = x^2 - \left[ \frac{(1 - \beta)(2 - 2\beta - \epsilon^*)}{md} \right] x + \left[ \frac{(1 - \beta - \epsilon^*)(1 - \beta)}{2md} \right]^2,
\] (3.5)
by assumption (iv), \((\eta - \alpha_1)(\eta - \alpha_2) \geq 0\). This implies that
\[
\eta^2 - \left[ \frac{(1 - \beta)(2 - 2\beta - \epsilon^*)}{md} \right] \eta + \left[ \frac{(1 - \beta - \epsilon^*)(1 - \beta)}{2md} \right]^2 \geq 0. \tag{3.6}
\]
Thus
\[
[(1 - \beta - \epsilon^*)(1 - \beta)]^2 - 4md(1 - \beta)(2 - 2\beta - \epsilon^*)\eta + 4m^2d^2\eta^2 \geq 0. \tag{3.7}
\]
In other words, the discriminant of the quadratic equation \(q(x) = 0\) being
\[
[(1 - \beta - \epsilon^*)(1 - \beta) + 2md\eta] - 12md(1 - \beta)^2\eta
\]
is nonnegative by (3.6). So \(q(x)\) always has positive roots \(r_1\) and \(r_2\). Since by assumption (iv) \(\eta/(1 - \beta) < r_1 < (1 - \beta - \epsilon^*)/3md\), choose \(r_0\) with \(r_1 < r_0 < \min\{r_2, (1 - \beta - \epsilon^*)/3md\}\). Then
\[
q(r_0) = 3md(1 - \beta)r_0 - r_0[(1 - \beta - \epsilon^*)(1 - \beta) + 2md\eta] + (1 - \beta)\eta < 0. \tag{3.9}
\]
So
\[
[(1 - \beta - \epsilon^*)(1 - \beta) + 2md\eta]r_0 - 3md(1 - \beta)r_0^2 > (1 - \beta)\eta. \tag{3.10}
\]
Hence
\[
\left[ \frac{1 - \beta - 2mr_0 d}{1 - \beta - \epsilon^* - 3mr_0 d} \right] \eta \frac{1}{1 - \beta} < r_0. \tag{3.11}
\]
Now for \(t \in \Omega\),
\[
\left| (F_{x_0}')^{-1}(F_x' - F_{x_0}')h(t) \right| = \left| (F_{x_0}')^{-1} \int_{\Omega} (K_3(t,s,x(s)) - K_3(t,s,x_0(s)))h(s)ds \right|
\leq \left\| (F_{x_0}')^{-1} \right\| \int_{\Omega} \mid K_3(t,s,x(s)) - K_3(t,s,x_0(s)) \mid \mid h(s) \mid ds
\leq \frac{mr_0 d}{1 - \beta} \| h \| \quad \text{(using (i))}. \tag{3.12}
\]
This implies that \(\| (F_{x_0}')^{-1}(F_x' - F_{x_0}') \| \leq mr_0 d/(1 - \beta)\). Setting \(\epsilon = mr_0 d/(1 - \beta)\), \(\epsilon^*_1 = \epsilon^*/(1 - \beta)\), \(\eta_1 = \eta/(1 - \beta)\), \(c_0 = (\epsilon + \epsilon^*_1)/(1 - \epsilon)\), and \(c = (2\epsilon + \epsilon^*_1)/(1 - \epsilon)\), and using (3.11), all the hypotheses of Theorem 2.1 are readily verified. Hence \(Fx(t) + g(t,x(t)) = 0\) for all \(t \in \Omega\) has a unique solution in \(\mathcal{U}(x_0, r_0)\). Thus there is a unique continuous
function \( x^* \) satisfying

\[
x^*(t) + \int_\Omega K(t,s,x^*(s)) \, ds + g(t,x^*(t)) = 0, \quad \forall t \in \Omega,
\]

and this solution can obtained as the limit of the iterates in (1.1).

4. Illustrative examples

The example below illustrates Theorem 3.2.

**Example 4.1.** Consider the problem of solving the functional integral equation

\[
x(t) - \frac{1}{1000} \int_0^1 t \cos(st) \sin \left( \frac{x(s) - 1}{1000} \right) \, ds - \sin \left( \frac{|x(t)|}{1000} \right) = 0 \quad \text{in } C[0,1].
\]

For the choice \( F(x) = x(t) - 1/1000 \int_0^1 t \cos(st) \sin((x(s) - 1)/1000) \, ds \) and \( g(t,x(t)) = -\sin(|x(t)|/1000) \), where \( \Omega = [0,1] \) and \( x_0 \equiv 1/1000 \), \( \beta = \sup_{t \in \Omega} |F_{x_0}(t) + g(t,x_0(t))| = \sup_{t \in \Omega} |1/1000 - 0.17 \times 10^{-7} \sin t - 0.17 \times 10^{-7}| < 1/1000, \ F_{x}h(t) = h(t) - 1/1000^2 \int_0^1 t \cos(st) \cos((x(s) - 1)/1000)h(s) \, ds \). For \( \eta = 1/1000 \), \( \eta/(1 - \beta) = 0.001000000 \). Setting \( m = 10^{-6} \), \( \epsilon^* = 10^{-3} \), and \( d = 1 \), we have \( p(x) = 4 \times 10^{-12}x^2 - 0.000007993x + 0.997997006 \) and \( \alpha_1 \) the smaller positive root of \( p(x) = 0.000002999x^2 - 0.998999002x + 0.000009999 \), \( r_1 \) the smaller positive root of \( q(x) = (0.998999002 - 0.998998996)/0.000005998 = 0.001000333 > \eta/(1 - \beta) \). As the other positive root \( r_2 = 338110.7032 > 332999.667 = (1 - \beta - \epsilon^*)/3md \), Theorem 3.2(4) is satisfied. For \( r_0 \in (0.001000333,332999.667) \), all the conditions of Theorem 3.2 are fulfilled. So the functional integral equation has a unique solution in \( \mathcal{U}(x_0,r_0) \) and this solution can be obtained as the limit of the sequence of iterates in (1.1).

**Example 4.2.** The next example shows that Theorem 2.1 is more general than Corollary 2.2, the main theorem obtained by Argyros [3].

Let \( f : \mathbb{R} \to \mathbb{R} \) be the map defined by

\[
f(x) = \begin{cases} (\frac{x - 0.01}{25})(20 + x^2 \sin \frac{1}{x}), & x \neq 0, \\ -\frac{1}{125}, & x = 0, \end{cases}
\]

(4.2)

let \( g : \mathbb{R} \to \mathbb{R} \) be the map defined by \( g(x) = |x| - 0.01/1000 \). Choose \( x_0 = 0, \epsilon = 0.25 \), and \( r = 0.4 \). Clearly for \( y \in \mathbb{R} \),

\[
f_{x}'(y) = \begin{cases} (\frac{x - 0.01}{25})(2x \sin \frac{1}{x} - \cos \frac{1}{x}) + (\frac{20 + x^2 \sin(1/x)}{25})y, & \text{for } x \neq 0, \\ 0.8y, & \text{for } x = 0, \end{cases}
\]

(4.3)
\((f'_{x_0})^{-1}\) exists and \(||(f'_{x_0})^{-1}|| = 1/0.8\). As \(G\) satisfies the Lipschitz condition with \(M = 1/1000\), \(\epsilon^* = (M||(f'_{x_0})^{-1}||) = 0.00125\). For \(x \in (-0.4, 0.4)\),

\[
||f'_x - f'_x|| = \sup \left\{ \left| \left( f'_x - f'_x \right)(y) \right| : ||y|| \leq 1 \right\}
\]

\[
= \sup \left\{ \left| \left( \frac{x - 0.01}{25} \right) \left( 2x \sin \frac{1}{x} - \cos \frac{1}{x} \right) + \left( \frac{x^2 \sin (1/x)}{25} \right) \right| : ||y|| \leq 1 \right\}
\]

\[
\leq \sup \left\{ \left| \left( \frac{0.4 + 0.01}{25} (2 \times 0.4 + 1) + \frac{0.16}{25} \right) |y| : ||y|| \leq 1 \right\}
\]

\[
= 0.03592 \leq 0.25 \times 0.8 = \frac{\epsilon}{||((f'_{x_0})^{-1}||},
\]

(4.4)

and \(||(F'_{x_0})^{-1}(F'_x - F'_x)|| < \epsilon = 0.25\). Also \(c_0 = 0.335\), \(c = 0.66833\), \(\eta = 0.0100125\), \(0 \leq ||(f'_{x_0})^{-1}(f(x_0) + g(x_0))|| \leq 0.0100125\), and \((c_0/(1 - c) + 1) = 2.0097891587\). Since \((c_0/(1 - c) + 1)\eta < r\), all the conditions of Theorem 2.1 are verified. Thus \(f(x) + g(x) = 0\) has a unique solution in  \(\overline{U}(0,0.4)\). However, \(F'_x\) is not continuous at zero but piecewise hemicontinuous in \(\overline{U}(0,0.4)\). It may be noted that Corollary 2.2 (due to Argyros) cannot be applied to prove that this equation has a solution, whereas Theorem 2.1 insures this.

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