The purpose of this note is to establish a strong convergence of a modified implicit iteration process to a common fixed point for a finite family of Z-operators.

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1. Introduction and preliminaries

We recall the following definitions in a metric space \((X,d)\). A mapping \(T : X \rightarrow X\) is called an \(a\)-contraction if

\[
d(Tx, Ty) \leq ad(x, y) \quad \forall x, y \in X,
\]

where \(a \in (0, 1)\).

The map \(T\) is called Kannan mapping [7] if there exists \(b \in (0, 1/2)\) such that

\[
d(Tx, Ty) \leq b [d(x, Tx) + d(y, Ty)] \quad \forall x, y \in X.
\]

A similar definition is due to Chatterjea [3]: there exists \(c \in (0, 1/2)\) such that

\[
d(Tx, Ty) \leq c [d(x, Ty) + d(y, Tx)] \quad \forall x, y \in X.
\]

Combining these three definitions, Zamfirescu [12] proved the following important result.

**Theorem 1.1.** Let \((X,d)\) be a complete metric space and \(T : X \rightarrow X\) a mapping for which there exists the real numbers \(a, b,\) and \(c\) satisfying \(a \in (0, 1), b, c \in (0, 1/2)\) such that for each pair \(x, y \in X,\) at least one of the following conditions holds:

\[
\begin{align*}
(z_1) & \quad d(Tx, Ty) \leq ad(x, y), \\
( z_2) & \quad d(Tx, Ty) \leq b [d(x, Tx) + d(y, Ty)], \\
( z_3) & \quad d(Tx, Ty) \leq c [d(x, Ty) + d(y, Tx)].
\end{align*}
\]
Then $T$ has a unique fixed point $p$ and the Picard iteration $\{x_n\}$ defined by
\[ x_{n+1} = Tx_n, \quad n \in \mathbb{N}, \tag{1.4} \]
converges to $p$ for any arbitrary but fixed $x_1 \in X$.

One of the most general contraction conditions, for which the unique fixed point can be approximated by means of Picard iteration, has been obtained by Ćirić [5]: there exists $0 < h < 1$ such that
\[ d(Tx, Ty) \leq h \max \{ d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx) \} \quad \forall x, y \in X. \tag{QC} \]

Remark 1.2. (1) A mapping satisfying (QC) is commonly called quasicontraction. It is obvious that each of the conditions $(1.1)$–$(1.3)$ and $(z_1)$–$(z_3)$ implies (QC).

(2) An operator $T$ satisfying the contractive conditions $(z_1)$–$(z_3)$ in the above theorem is called $Z$-operator.

Let $C$ be a nonempty closed convex subset of a normed space $E$.

Xu and Ori [11] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, \ldots, N\}$), with $\{\alpha_n\}$ a real sequence in $(0, 1)$, and an initial point $x_0 \in C$:
\begin{align*}
    x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1, \\
    x_2 &= \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2, \\
    & \vdots \\
    x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N, \\
    x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1 x_{N+1}, \\
    & \vdots \\
\end{align*}

which can be written in the following compact form:
\[ x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n \quad \forall n \geq 1, \tag{1.6} \]

where $T_n = T_n(\text{mod} N)$ (here the mod $N$ function takes values in $I$). Xu and Ori proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$.

In [13], Zhou and Chang studied the weak and strong convergences of this implicit process to a common fixed point for a finite family of nonexpansive mappings. More precisely, they proved the following result.

**Theorem 1.3** [13, Theorem 3]. Let $E$ be a uniformly convex Banach space and let $K$ be a nonempty closed convex subset of $E$. Let $\{T_i : i \in I\}$ be $N$ semicompact nonexpansive self-mappings of $K$ with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ (here $F(T_i)$ denotes the set of fixed points of $T_i$).
Suppose that \( x_0 \in K \) and \( \{ \alpha_n \} \subset (b, c) \) for some \( b, c \in (0, 1) \). Then the sequence \( \{ x_n \} \) defined by the implicit iteration process (1.6) converges strongly to a common fixed point in \( F \).

In [4], Chidume and Shahzad studied the strong convergence of the implicit process (1.6) to a common fixed point for a finite family of nonexpansive mappings. They proved the following results.

**Theorem 1.4 [4, Theorem 3.3].** Let \( E \) be a uniformly convex Banach space and let \( K \) be a nonempty closed convex subset of \( E \). Let \( \{ T_i : i \in I \} \) be \( N \) nonexpansive self-mappings of \( K \) with \( F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \). Suppose that one of the mappings in \( \{ T_i : i \in I \} \) is semi-compact. Let \( \{ \alpha_n \}_{n \geq 1} \subset [\delta, 1 - \delta] \) for some \( \delta \in (0, 1) \). From arbitrary \( x_0 \in K \), define the sequence \( \{ x_n \} \) by the implicit iteration process (1.6). Then \( \{ x_n \} \) converges strongly to a common fixed point of the mappings \( \{ T_i : i \in I \} \).

**Remark 1.5.** It is worth mentioning here that [13, Theorem 1] by Zhou and Chang is “for convergence of modified implicit iteration process for a finite family of asymptotically nonexpansive mappings in uniformly convex Banach spaces.”

Let \( C \) be a nonempty closed convex subset of a normed space \( E \). Inspired and motivated by the above said facts, we suggest the following implicit iteration process with errors and define the sequence \( \{ x_n \} \) as follows:

\[
x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n + u_n \quad \forall n \geq 1,
\]

where \( T_n = T_n(\text{mod} N), \{ \alpha_n \} \) is a sequence in \( (0, 1) \), and \( \{ u_n \} \) is a summable sequence in \( C \).

Clearly, this iteration process contains the process (1.6) as its special case.

The purpose of this note is to study the strong convergence of implicit iteration process (1.7) to a common fixed point for a finite family of \( Z \)-operators in normed spaces.

The following lemma is proved in [2].

**Lemma 1.6.** Let \( \{ r_n \}, \{ s_n \}, \text{and} \{ t_n \} \) be sequences of nonnegative numbers satisfying

\[
r_{n+1} \leq (1 - s_n) r_n + s_n t_n \quad \forall n \geq 1.
\]

If \( \sum_{n=1}^{\infty} s_n = \infty \) and \( \lim_{n \to \infty} t_n = 0 \), then \( \lim_{n \to \infty} r_n = 0 \).

**2. Main results**

**Theorem 2.1.** Let \( C \) be a nonempty closed convex subset of a normed space \( E \). Let \( \{ T_1, T_2, \ldots, T_N \} : C \to C \) be \( N \) \( Z \)-operators with \( F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \). From arbitrary \( x_0 \in C \), define the sequence \( \{ x_n \} \) by the implicit iteration process (1.7) satisfying \( \sum_{n=1}^{\infty} (1 - \alpha_n) = \infty \) and \( \| u_n \| = 0(1 - \alpha_n) \). Then \( \{ x_n \} \) converges strongly to a common fixed point of \( \{ T_1, T_2, \ldots, T_N \} \).

**Proof.** It follows from \( F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \) that the operators \( \{ T_1, T_2, \ldots, T_N \} \) have a common fixed point in \( C \), say \( w \). Consider \( x, y \in C \). Since each \( T_i : i \in I \) is a \( Z \)-operator, at least one of the conditions (\( z_1 \)), (\( z_2 \)), and (\( z_3 \)) is satisfied. If (\( z_2 \)) holds, then

\[
\| T_i x - T_i y \| \leq b[\| x - T_i x \| + \| y - T_i y \|] \leq b[\| x - T_i x \| + \| y - x \| + \| x - T_i x \| + \| T_i x - T_i y \|]
\]

(2.1)
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implies

\[(1 - b)\|T_ix - T_iy\| \leq b\|x - y\| + 2b\|x - T_ix\|,\] \quad (2.2)

which yields (using the fact that \(0 \leq b < 1\))

\[\|T_ix - T_iy\| \leq \frac{b}{1 - b}\|x - y\| + \frac{2b}{1 - b}\|x - T_ix\|.\] \quad (2.3)

If (z3) holds, then similarly we obtain

\[\|T_ix - T_iy\| \leq \frac{c}{1 - c}\|x - y\| + \frac{2c}{1 - c}\|x - T_ix\|.\] \quad (2.4)

Denote

\[\delta = \max \left\{ a, \frac{b}{1 - b}, \frac{c}{1 - c} \right\}.\] \quad (2.5)

Then we have \(0 \leq \delta < 1\) and in view of (z1), (2.3)–(2.5) it results that the inequality

\[\|T_ix - T_iy\| \leq \delta\|x - y\| + 2\delta\|x - T_ix\|\] \quad (AR)

holds for all \(x, y \in C\).

Using (1.6), we have

\[\|x_n - w\| = \|\alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n + u_n - w\|
\]

\[= \|\alpha_n (x_{n-1} - w) + (1 - \alpha_n) (T_n x_n - w) + u_n\|
\]

\[\leq \alpha_n \|x_{n-1} - w\| + (1 - \alpha_n) \|T_n x_n - w\| + \|u_n\|.\] \quad (2.6)

Now for \(y = x_n\) and \(x = w\), (AR) gives

\[\|T x_n - w\| \leq \delta\|x_n - w\|,\] \quad (2.7)

and hence, by (2.6), (2.7) we obtain

\[\|x_n - w\| \leq \frac{\alpha_n}{1 - \delta(1 - \alpha_n)} \|x_{n-1} - w\| + \frac{1}{1 - \delta(1 - \alpha_n)} \|u_n\|.\] \quad (2.8)

Let

\[A_n = \alpha_n,\]

\[B_n = 1 - \delta(1 - \alpha_n),\] \quad (2.9)

and consider

\[\beta_n = 1 - \frac{A_n}{B_n} = 1 - \frac{\alpha_n}{1 - \delta(1 - \alpha_n)}\]

\[= \frac{(1 - \delta)(1 - \alpha_n)}{1 - \delta(1 - \alpha_n)} \geq (1 - \delta)(1 - \alpha_n).\] \quad (2.10)
Indeed

\[ 1 - \delta \leq 1 - \delta (1 - \alpha_n) \leq 1 \]  

(2.11)

implies

\[ \frac{A_n}{B_n} \leq 1 - (1 - \delta) (1 - \alpha_n). \]  

(2.12)

Thus from (2.8), we get

\[ \| x_n - w \| \leq \left[ 1 - (1 - \delta) (1 - \alpha_n) \right] \| x_{n-1} - w \| + \frac{1}{1 - \delta} \| u_n \|. \]  

(2.13)

With the help of Lemma 1.6 and using the fact that \( 0 \leq \delta < 1, 0 < \alpha_n < 1, \sum_{n=1}^{\infty} (1 - \alpha_n) = \infty, \) and \( \| u_n \| = 0(1 - \alpha_n), \) it results that

\[ \lim_{n \to \infty} \| x_n - w \| = 0. \]  

(2.14)

Consequently \( x_n \to w \in F \) and this completes the proof.

**Corollary 2.2.** Let \( C \) be a nonempty closed convex subset of a normed space \( E. \) Let \( \{ T_1, T_2, \ldots, T_N \} : C \to C \) be \( N \) operators satisfying condition \( Z \) with \( F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset. \) From arbitrary \( x_0 \in C, \) define the sequence \( \{ x_n \} \) by the implicit iteration process (1.6) satisfying \( \sum_{n=1}^{\infty} (1 - \alpha_n) = \infty. \) Then \( \{ x_n \} \) converges strongly to a common fixed point of \( \{ T_1, T_2, \ldots, T_N \}. \)

**Remark 2.3.** (1) Chatterjea’s and Kannan’s contractive conditions (1.3) and (1.2) are both included in the class of Zamfirescu operators.

(2) Recently the convergence problems of an implicit (or nonimplicit) iterative process to a common fixed point of finite family of nonexpansive mappings in Hilbert spaces have been considered by several authors (see, e.g., [1, 6, 8–11, 13]).

**References**


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