ON HORIZONTAL AND COMPLETE LIFTS FROM A MANIFOLD WITH $f_{\lambda}(7,1)$-STRUCTURE TO ITS COTANGENT BUNDLE

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The horizontal and complete lifts from a manifold $M^n$ to its cotangent bundles $T^*(M^n)$ were studied by Yano and Ishihara, Yano and Patterson, Nivas and Gupta, Dambrowski, and many others. The purpose of this paper is to use certain methods by which $f_{\lambda}(7,1)$-structure in $M^n$ can be extended to $T^*(M^n)$. In particular, we have studied horizontal and complete lifts of $f_{\lambda}(7,1)$-structure from a manifold to its cotangent bundle.

1. Introduction

Let $M$ be a differentiable manifold of class $c^\infty$ and of dimension $n$ and let $C_{TM}$ denote the cotangent bundle of $M$. Then $C_{TM}$ is also a differentiable manifold of class $c^\infty$ and dimension $2n$.

The following are notations and conventions that will be used in this paper.

(1) $\mathcal{S}^r_\chi(M)$ denotes the set of tensor fields of class $c^\infty$ and of type $(r,s)$ on $M$. Similarly, $\mathcal{S}^r_\chi(C_{TM})$ denotes the set of such tensor fields in $C_{TM}$.

(2) The map $\Pi$ is the projection map of $C_{TM}$ onto $M$.

(3) Vector fields in $M$ are denoted by $X, Y, Z, \ldots$ and Lie differentiation by $L_X$. The Lie product of vector fields $X$ and $Y$ is denoted by $[X, Y]$.

(4) Suffixes $a, b, c, \ldots, h, i, j, \ldots$ take the values 1 to $n$ and $\tilde{a} = i + n$. Suffixes $A, B, C, \ldots$ take the values 1 to $2n$.

If $A$ is a point in $M$, then $\Pi^{-1}(A)$ is fiber over $A$. Any point $p \in \Pi^{-1}(A)$ is denoted by the ordered pair $(A, p_A)$, where $p$ is 1-form in $M$ and $p_A$ is the value of $p$ at $A$. Let $U$ be a coordinate neighborhood in $M$ such that $A \in U$. Then $U$ induces a coordinate neighborhood $\Pi^{-1}(U)$ in $C_{TM}$ and $p \in \Pi^{-1}(U)$.

2. Complete lift of $f_{\lambda}(7,1)$-structure

Let $f(\neq 0)$ be a tensor field of type $(1,1)$ and class $c^\infty$ on $M$ such that

$$f^2 + \lambda^2 f = 0,$$  \hspace{1cm} (2.1)
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where \( \lambda \) is any complex number not equal to zero. We call the manifold \( M \) satisfying (2.1) as \( f_{\lambda}(7,1) \)-structure manifold. Let \( f_i^h \) be components of \( f \) at \( A \) in the coordinate neighborhood \( U \) of \( M \). Then the complete lift \( f^c \) of \( f \) is also a tensor field of type \((1,1)\) in \( C^{\text{TM}} \) whose components \( \tilde{f}_A^B \) in \( \Pi^{-1}(U) \) are given by [2]

\[
\begin{align*}
\tilde{f}_i^h &= f_i^h; & f_{ij} \equiv 0, \quad (2.2) \\
\tilde{f}_i^h &= \rho_d \left( \frac{\partial f_a^i}{\partial x^i} \frac{\partial f_a^h}{\partial x^h} \right); & \tilde{f}_i^h = f_i^h, \quad (2.3)
\end{align*}
\]

where \((x^1,x^2,\ldots,x^n)\) are coordinates of \( A \) relative to \( U \) and \( p_A \) has a component \((p_1,p_2,\ldots,p_n)\).

Thus we can write

\[
f^C = (\tilde{f}_B^A) = \begin{pmatrix} f_i^h & 0 \\ \rho_d(i) f_a^i - \rho_d(j) f_a^j & f_i^j \end{pmatrix}, \quad (2.4)
\]

where \( \partial_i = \partial / \partial x^i \).

If we put

\[
\partial_i f_a^i - \partial_h f_a^h = 2\partial[i f_a^a], \quad (2.5)
\]

then we can write (2.4) in the form

\[
f^C = (\bar{f}_B^A) = \begin{pmatrix} f_i^h & 0 \\ 2\rho_d(i) f_a^i & f_i^j \end{pmatrix}, \quad (2.6)
\]

Thus we have

\[
\begin{pmatrix} f_i^h & 0 \end{pmatrix} \begin{pmatrix} f_j^i & 0 \end{pmatrix}, \quad (2.7)
\]

or

\[
\begin{pmatrix} f_i^h f_j^i & 0 \end{pmatrix} \begin{pmatrix} 2\rho_d(i) f_a^i & f_i^j \end{pmatrix}, \quad (2.8)
\]

If we put

\[
2\rho_d(i) f_a^i \partial[i f_a^a] + 2\rho_d(j) f_a^j \partial[j f_a^a] = L_{ij}, \quad (2.9)
\]

then (2.8) takes the form

\[
\begin{pmatrix} f_i^h f_j^i & 0 \end{pmatrix} \begin{pmatrix} L_{ij} & f_i^j f_i^j \end{pmatrix}, \quad (2.10)
\]
Thus we have

$$(f^C)^4 = \begin{pmatrix} f^h_i f^j_i & 0 \\ L_{hj} & f^j_i f^k_i \end{pmatrix} \begin{pmatrix} f^k_i f^l_i & 0 \\ L_{jl} & f^l_i f^m_i \end{pmatrix},$$  \hspace{1cm} (2.11)

or

$$(f^C)^4 = \begin{pmatrix} f^h_i f^j_i f^k_i & 0 \\ f^j_i f^k_i f^l_i L_{hj} + f^j_i f^h_i L_{jl} & f^j_i f^k_i f^l_i f^m_i \end{pmatrix}.$$  \hspace{1cm} (2.12)

Putting again

$$f^j_i f^k_i L_{hj} + f^j_i f^h_i L_{jl} = P_{hl},$$  \hspace{1cm} (2.13)

then we can put (2.12) in the form

$$(f^C)^4 = \begin{pmatrix} f^h_i f^j_i f^k_i & 0 \\ P_{hl} & f^j_i f^k_i f^l_i f^m_i \end{pmatrix}.$$  \hspace{1cm} (2.14)

Thus,

$$(f^C)^6 = \begin{pmatrix} f^h_i f^j_i f^k_i f^l_i f^m_i & 0 \\ P_{hl} & f^j_i f^k_i f^l_i f^m_i \end{pmatrix} \begin{pmatrix} f^m_i f^m_i & 0 \\ f^l_i f^m_i f^m_i & L_{ln} \end{pmatrix},$$  \hspace{1cm} (2.15)

$$(f^C)^6 = \begin{pmatrix} f^h_i f^j_i f^k_i f^l_i f^m_i f^m_i & 0 \\ P_{hl} f^j_i f^m_i f^m_i + L_{ln} f^j_i f^k_i f^l_i f^m_i & f^m_i f^m_i f^m_i f^m_i f^m_i f^m_i \end{pmatrix}.$$  \hspace{1cm} (2.16)

Putting again

$$P_{hl} f^j_i f^m_i f^m_i + L_{ln} f^j_i f^k_i f^l_i f^m_i f^m_i = Q_{hn},$$  \hspace{1cm} (2.17)

then (2.16) takes the form

$$(f^C)^6 = \begin{pmatrix} f^h_i f^j_i f^k_i f^l_i f^m_i f^m_i & 0 \\ Q_{hn} & f^m_i f^m_i f^m_i f^m_i f^m_i f^m_i \end{pmatrix}.$$  \hspace{1cm} (2.18)
Thus,\[ (f^C)^7 = \left( \begin{array}{ccc} f^h f_j f_k f^i f_m f_n & 0 \\ Q_{hn} f^m f^i f_k f_j f_i f_h & 0 \end{array} \right) \left( \begin{array}{c} f^p_n \\ 2 p_r \partial [ p f^r_n ] \\ f^p_r \end{array} \right), \] (2.19)

\[ (f^C)^7 = \left( \begin{array}{ccc} f^h f_j f_k f^i f_m f_n f^p & 0 \\ Q_{hn} f^p_n + 2 p_r \partial [ p f^r_n ] f^m f^i f_k f_j f_i f_h & f^p_n f^m f^i f_k f_j f_i f_h \end{array} \right). \] (2.20)

In view of (2.1), we have\[ f_i^h f_j f_k f^i f_m f_n f^p = - \lambda^2 f^h_p, \] (2.21)

and also putting
\[ Q_{hn} f^p_n + 2 p_r \partial [ p f^r_n ] f^m f^i f_k f_j f_i f_h = - \lambda^2 p_s \partial [ p f^s_h ], \] (2.22)

then (2.20) can be given by
\[ (f^C)^7 = \left( \begin{array}{ccc} - \lambda^2 f^p_n \\ - \lambda^2 p_r \partial [ p f^r_n ] & - \lambda^2 f^p_r \end{array} \right), \] (2.23)

In view of (2.6) and (2.23), it follows that
\[ (f^C)^7 + \lambda^2 (f^C) = 0. \] (2.24)

Hence the complete lift $f^C$ of $f$ admits an $f^\lambda(7,1)$-structure in the cotangent bundle $C_{TM}$.

Thus we have the following theorem.

**Theorem 2.1.** In order that the complete lift of $f^C$ of a $(1,1)$ tensor field $f$ admitting $f^\lambda(7,1)$-structure in $M$ may have the similar structure in the cotangent bundle $C_{TM}$, it is necessary and sufficient that
\[ Q_{hn} f^p_n + 2 p_r \partial [ p f^r_n ] f^m f^i f_k f_j f_i f_h = - \lambda^2 p_s \partial [ p f^s_h ]. \] (2.25)

**3. Horizontal lift of $f^\lambda(7,1)$-structure**

Let $f, g$ be two tensor fields of type $(1,1)$ on the manifold $M$. If $f^H$ denotes the horizontal lift of $f$, we have
\[ f^H g^H + g^H f^H = (f g + g f)^H. \] (3.1)

Taking $f$ and $g$ identical, we get
\[ (f^H)^2 = (f^2)^H. \] (3.2)
Multiplying both sides by \( f^H \) and making use of the same (3.2), we get

\[
(f^H)^3 = (f^3)^H
\]  
(3.3)

and so on. Thus it follows that

\[
(f^H)^4 = (f^4)^H, \quad (f^H)^5 = (f^5)^H,
\]  
(3.4)

and so on. Thus,

\[
(f^H)^7 = (f^7)^H.
\]  
(3.5)

Since \( f \) gives on \( M \) the \( f_\lambda(7,1) \)-structure, we have

\[
f^7 + \lambda^2 f = 0.
\]  
(3.6)

Taking horizontal lift, we obtain

\[
(f^7)^H + \lambda^2 (f^H) = 0.
\]  
(3.7)

In view of (3.5) and (3.7), we can write

\[
(f^H)^7 + \lambda^2 (f^H) = 0.
\]  
(3.8)

Thus the horizontal lift \( f^H \) of \( f \) also admits a \( f_\lambda(7,1) \)-structure. Hence we have the following theorem.

**Theorem 3.1.** Let \( f \) be a tensor field of type \((1,1)\) admitting \( f_\lambda(7,1) \)-structure in \( M \). Then the horizontal lift \( f^H \) of \( f \) also admits the similar structure in the cotangent bundle \( C_{TM} \).

4. Nijenhuis tensor of complete lift of \( f^7 \)

The Nijenhuis tensor of a \((1,1)\) tensor field \( f \) on \( M \) is given by

\[
N_{f,f}(X,Y) = [fX, fY] - f[fX,Y] - f[X,fY] + f^2[X,Y].
\]  
(4.1)

Also for the complete lift of \( f^7 \), we have

\[
N(f^7)^C, (f^7)^C(X^C,Y^C) = \left( (f^7)^C X^C, (f^7)^C Y^C \right) - (f^7)^C \left[ (f^7)^C X^C, Y^C \right] - (f^7)^C \left[ X^C, (f^7)^C Y^C \right] + (f^7)^C (f^7)^C \left[ X^C, Y^C \right].
\]  
(4.2)

In view of (2.1), the above (4.2) takes the form

\[
N(f^7)^C, (f^7)^C(X^C,Y^C)
\]

\[
= \left[ (-\lambda^2 f)^C X^C, (-\lambda^2 f)^C Y^C \right] - (-\lambda^2 f)^C \left[ (-\lambda^2 f)^C X^C, Y^C \right] - (-\lambda^2 f)^C \left[ X^C, (-\lambda^2 f)^C Y^C \right] + (-\lambda^2 f)^C (-\lambda^2 f)^C \left[ X^C, Y^C \right],
\]  
(4.3)
or
\[
N(f^7)^C, (f^7)^C(X^C, Y^C) = \lambda^4 \left\{ \left[ (f)^C X^C, (f)^C Y^C \right] - (f)^C \left[ (f)^C X^C, Y^C \right] \right\}.
\]

(4.4)

We also know that
\[
(f)^C X^C = (fX)^C + \nu(\mathcal{L}_X f),
\]

(4.5)

where \( \nu f \) has components
\[
\nu f = \begin{pmatrix} O^a \\ p_a f_i \end{pmatrix}.
\]

(4.6)

In view of (4.5), (4.4) takes the form
\[
N(f^7)^C, (f^7)^C(X^C, Y^C)
\]

= \lambda^4 \left\{ \left[ (fX)^C, (fY)^C \right] + \nu(\mathcal{L}_X f), (fY)^C \right\] - \left[ (fX)^C, Y^C \right] \right\}.

(4.7)

We now suppose that
\[
\mathcal{L}_X f = \mathcal{L}_Y f = 0.
\]

(4.8)

Then from (4.7), we have
\[
N(f^7)^C, (f^7)^C(X^C, Y^C) = \lambda^4 \left\{ \left[ (fX)^C, (fY)^C \right] - (f)^C \left[ (fX)^C, Y^C \right] \right\}.
\]

(4.9)

Further, if \( f \) acts as an identity operator on \( M \), that is,
\[
fX = X \quad \forall X \in \mathcal{S}_0(M),
\]

(4.10)

then we have from (4.9)
\[
N(f^7)^C, (f^7)^C(X^C, Y^C) = \lambda^8 \left\{ [X^C, Y^C] - [X^C, Y^C] - [X^C, Y^C] + [X^C, Y^C] \right\} = 0.
\]

(4.11)

Hence we have the following theorem.

**Theorem 4.1.** The Nijenhuis tensor of the complete lift of \( f^7 \) vanishes if the Lie derivatives of the tensor field \( f \) with respect to \( X \) and \( Y \) are both zero and \( f \) acts as an identity operator on \( M \).
References


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