We obtain necessary and sufficient conditions for the asymptotic stability of the linear delay difference equation $x_{n+1} + p \sum_{j=1}^{N} x_{n-k+(j-1)l} = 0$, where $n = 0, 1, 2, \ldots$, $p$ is a real number, and $k$, $l$, and $N$ are positive integers such that $k > (N - 1)l$.

1. Introduction

In [4], the asymptotic stability condition of the linear delay difference equation

$$x_{n+1} - x_n + p \sum_{j=1}^{N} x_{n-k+(j-1)l} = 0,$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $p$ is a real number, and $k$, $l$, and $N$ are positive integers with $k > (N - 1)l$ is given as follows.

**Theorem 1.1.** Let $k$, $l$, and $N$ be positive integers with $k > (N - 1)l$. Then the zero solution of (1.1) is asymptotically stable if and only if

$$0 < p < \frac{2 \sin(\pi/2M) \sin(l\pi/2M) \sin(Nl\pi/2M)}{\sin(Nl\pi/2M)},$$

where $M = 2k + 1 - (N - 1)l$.

Theorem 1.1 generalizes asymptotic stability conditions given in [1, page 87], [2, 3, 5], and [6, page 65]. In this paper, we are interested in the situation when (1.1) does not depend on $x_n$, namely we are interested in the asymptotic stability of the linear delay difference equation of the form

$$x_{n+1} + p \sum_{j=1}^{N} x_{n-k+(j-1)l} = 0,$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $p$ is a real number, and $k$, $l$, and $N$ are positive integers with $k \geq (N - 1)l$. Our main theorem is the following.
Theorem 1.2. Let \( k, l, \) and \( N \) be positive integers with \( k \geq (N-1)l \). Then the zero solution of (1.3) is asymptotically stable if and only if

\[
-\frac{1}{N} < p < p_{\text{min}}, \tag{1.4}
\]

where \( p_{\text{min}} \) is the smallest positive real value of \( p \) for which the characteristic equation of (1.3) has a root on the unit circle.

2. Proof of theorem

The characteristic equation of (1.3) is given by

\[
F(z) = z^{k+1} + p(z^{(N-1)l} + \cdots + z^l + 1) = 0. \tag{2.1}
\]

For \( p = 0 \), \( F(z) \) has exactly one root at 0 of multiplicity \( k + 1 \). We first consider the location of the roots of (2.1) as \( p \) varies. Throughout the paper, we denote the unit circle by \( C \) and let \( M = 2k + 2 - (N-1)l \).

Proposition 2.1. Let \( z \) be a root of (2.1) which lies on \( C \). Then the roots \( z \) and \( p \) are of the form

\[
z = e^{w_m i}, \tag{2.2}
\]

\[
p = (-1)^{m+1} \frac{\sin(lw_m/2)}{\sin(Nlw_m/2)} \equiv p_m \tag{2.3}
\]

for some \( m = 0, 1, \ldots, M-1 \), where \( w_m = (2m/M)\pi \). Conversely, if \( p \) is given by (2.3), then \( z = e^{w_m i} \) is a root of (2.1).

Proof. Note that \( z = 1 \) is a root of (2.1) if and only if \( p = -1/N \), which agrees with (2.2) and (2.3) for \( w_m = 0 \). We now consider the roots of (2.1) which lie on \( C \) except the root \( z = 1 \). Suppose that the value \( z \) satisfies \( z^{Nl} = 1 \) and \( z^l \neq 1 \). Then \( z^{Nl} - 1 = (z^l - 1)(z^{(N-1)l} + \cdots + z^l + 1) = 0 \) which gives \( z^{(N-1)l} + \cdots + z^l + 1 = 0 \), and hence \( z \) is not a root of (2.1).

As a result, to determine the roots of (2.1) which lie on \( C \), it suffices to consider only the value \( z \) such that \( z^N \neq 1 \) or \( z^l = 1 \). For these values of \( z \), we may write (2.1) as

\[
p = -\frac{z^{k+1}}{z^{(N-1)l} + \cdots + z^l + 1}. \tag{2.4}
\]

Since \( p \) is real, we have

\[
p = -\frac{\overline{z}^{k+1}}{\overline{z}^{(N-1)l} + \cdots + \overline{z}^l + 1} = -\frac{z^{-k-1+(N-1)l}}{z^{(N-1)l} + \cdots + z^l + 1}, \tag{2.5}
\]

where \( \overline{z} \) denotes the conjugate of \( z \). It follows from (2.4) and (2.5) that

\[
z^{2k+2-(N-1)l} = 1 \tag{2.6}
\]

which implies that (2.2) is valid for \( m = 0, 1, \ldots, M-1 \) except for those integers \( m \) such that \( e^{Nlw_m} = 1 \) and \( e^{lw_m} \neq 1 \). We now show that \( p \) is of the form stated in (2.3). There are two cases to be considered as follows.
Case 1. \( z \) is of the form \( e^{w_m l} \) for some \( m = 1, 2, \ldots, M-1 \) and \( z^{N_l} \neq 1 \).

From (2.4), we have

\[
p = -\frac{z^{k+1}(z^l - 1)}{e^{N_l w_m l} - 1} = -\frac{e^{(k+1)w_m l}(e^{l w_m l} - 1)}{e^{N_l w_m l/2} - e^{-N_l w_m l/2}}\]

\[
= -e^{(k+1)-(N-1)(l/2)w_m l} \frac{\sin(l w_m/2)}{\sin(N_l w_m/2)}
\]

\[
= -e^{m \pi i} \frac{\sin(l w_m/2)}{\sin(N_l w_m/2)} = (-1)^{m+1} \frac{\sin(l w_m/2)}{\sin(N_l w_m/2)} \equiv p_m.
\]

(2.7)

Case 2. \( z \) is of the form \( e^{w_m l} \) for some \( m = 1, 2, \ldots, M-1 \) and \( z^l = 1 \).

In this case, we have \( l w_m = 2q\pi \) for some positive integer \( q \). Then taking the limit of \( p_m \) as \( l w_m \to 2q\pi \), we obtain

\[
p = -\frac{(-1)^{m+q(N-1)}}{N}.
\]

(2.8)

From these two cases, we conclude that \( p \) is of the form in (2.3) for \( m = 1, 2, \ldots, M-1 \) except for those \( m \) such that \( e^{N_l w_m l} = 1 \) and \( e^{l w_m l} \neq 1 \).

Conversely, if \( p \) is given by (2.3), then it is obvious that \( z = e^{w_m l} \) is a root of (2.1). This completes the proof of the proposition. \( \square \)

From Proposition 2.1, we may consider \( p \) as a holomorphic function of \( z \) in a neighborhood of each \( z_m \). In other words, in a neighborhood of each \( z_m \), we may consider \( p \) as a holomorphic function of \( z \) given by

\[
p(z) = -\frac{z^{k+1}}{z^{(N-1)l} + \cdots + z^l + 1}.
\]

(2.9)

Then we have

\[
\frac{dp(z)}{dz} = -\frac{(k+1)z^k}{z^{(N-1)l} + \cdots + z^l + 1} + \frac{z^k\{(N-1)lz^{(N-1)l} + \cdots +lz^l\}}{(z^{(N-1)l} + \cdots + z^l + 1)^2}.
\]

(2.10)

From this, we have the following lemma.

**Lemma 2.2.** \( dp/dz \big|_{z=e^{w_m l}} \neq 0 \). In particular, the roots of (2.1) which lie on \( C \) are simple.

**Proof.** Suppose on the contrary that \( dp/dz \big|_{z=e^{w_m l}} = 0 \). We divide (2.10) by \( p(z)/z \) to obtain

\[
k + 1 - \frac{l\{(N-1)z^{(N-1)l} + \cdots + z^l\}}{z^{(N-1)l} + \cdots + z^l + 1} = 0.
\]

(2.11)

Substituting \( z \) by \( 1/z \) in (2.10), we obtain

\[
k + 1 - \frac{l\{(N-1)+(N-2)z^l + \cdots + z^{(N-2)l}\}}{z^{(N-1)l} + \cdots + z^l + 1} = 0.
\]

(2.12)
By adding (2.11) and (2.12), we obtain
\[ 2k + 2 - (N - 1)l = 0 \]  
(2.13)
which contradicts \( k \geq (N - 1)l \). This completes the proof. □

From Lemma 2.2, there exists a neighborhood of \( z = e^{wmi} \) such that the mapping \( p(z) \) is one to one and the inverse of \( p(z) \) exists locally. Now, let \( z \) be expressed as \( z = re^{it} \). Then we have
\[ \frac{dz}{dp} = \frac{r}{z} \left\{ \frac{dr}{dp} + ir \frac{d\theta}{dp} \right\} \]  
(2.14)
which implies that
\[ \frac{dr}{dp} = \text{Re} \left\{ \frac{r}{z} \frac{dz}{dp} \right\} \]  
(2.15)
as \( p \) varies and remains real. The following result describes the behavior of the roots of (2.1) as \( p \) varies.

**Proposition 2.3.** The moduli of the roots of (2.1) at \( z = e^{wmi} \) increase as \( |p| \) increases.

**Proof.** Let \( r \) be the modulus of \( z \). Let \( z = e^{wmi} \) be a root of (2.1) on \( C \). To prove this proposition, it suffices to show that
\[ \left. \left( \frac{dr}{dp} \cdot p \right) \right|_{z = e^{wmi}} > 0. \]  
(2.16)
There are two cases to be considered.

**Case 1** \( (z^N \neq 1) \). In this case, we have
\[ p(z) = -\frac{z^{k+1}(z^l - 1)}{z^N - 1} = -\frac{z^k f(z)}{z^{Nl} - 1}, \]  
(2.17)
where \( f(z) = z(z^l - 1) \). Then
\[ \frac{dp}{dz} = \frac{-z^{k-1}g(z)}{(z^{Nl} - 1)^2}, \]  
(2.18)
where \( g(z) = (kf(z) + zf'(z))(z^{Nl} - 1) - Nlz^{Nl}f(z) \). Letting \( w(z) = -(z^{Nl} - 1)^2/(z^k g(z)) \), we obtain
\[ \frac{dr}{dp} = \text{Re} \left( \frac{r}{z} \frac{dz}{dp} \right) = r \text{Re}(w). \]  
(2.19)
We now compute \( \text{Re}(w) \). We note that
\[ f(\bar{z}) = -\frac{f(z)}{z^l}, \quad f'(\bar{z}) = \frac{h(z)}{z^l}, \]  
(2.20)
where \( h(z) = l + 1 - z^l \). From the above equalities and as \( z^M = 1 \), we have

\[
 z^k g(z) = \frac{1}{z^k} \left\{ \left( k f(z) + \frac{1}{z} f'(z) \right) \left( \frac{1}{z^{Nl}} - 1 \right) - \frac{NL}{z^{Nl}} f(z) \right\}
\]

\[
 = \frac{-k f(z) + zh(z)(1 - z^{Nl}) + Nl f(z)}{z^{Nl + l + 2 + k}}
\]

\[
 = \frac{-k f(z) + zh(z)(1 - z^{Nl}) + Nl f(z)}{z^{2Nl - k}}.
\]

It follows that

\[
 \text{Re}(w) = \frac{w + \overline{w}}{2}
\]

\[
 = -\frac{1}{2} \left\{ \frac{(z^{Nl} - 1)^2}{z^k g(z)} + \frac{(z^{Nl} - 1)^2}{z^k g(z)} \right\}
\]

\[
 = -\frac{1}{2} \left\{ \frac{z^k g(z)(z^{Nl} - 1)^2 + z^k g(z)(\overline{z}^{Nl} - 1)^2}{|g(z)|^2} \right\}
\]

\[
 = -\frac{1}{2} \left\{ \left( -k f(z) + zh(z) \right)(1 - z^{Nl}) + Nl f(z) \right\} \cdot \left( \frac{1}{z^{Nl} - 1} \right)^2
\]

\[
+ z^k \left( (k f(z) + zf'(z))(z^{Nl} - 1) - Nlz^{NL} f(z) \right) \left( \frac{1}{z^{Nl} - 1} \right)^2
\]

\[
= -\frac{(z^{Nl} - 1)^2 z^k}{2z^{2NL} |g(z)|^2} \left\{ (k f(z) - zh(z))(z^{Nl} - 1) + Nl f(z)
\right.
\]

\[
+ ((k f(z) + zf'(z))(z^{Nl} - 1)) - Nlz^{NL} f(z) \right\}
\]

\[
= -\frac{(z^{Nl} - 1)^2 z^k}{2z^{2NL} |g(z)|^2} \left\{ 2k f(z) + z(f'(z) - h(z)) - Nl f(z) \right\}.
\]

(2.21)

Since

\[
2k f(z) + z(f'(z) - h(z)) - Nl f(z) = M f(z),
\]

we obtain

\[
\text{Re}(w) = \frac{(z^{Nl} - 1)^4 M}{2z^{2NL} |g(z)|^2} \cdot \frac{-z^k f(z)}{z^{Nl} - 1} = \frac{(z^{Nl} - 1)^4 M p}{2z^{2NL} |g(z)|^2}.
\]

(2.22)

The value of \( \text{Re}(w) \) at \( z = e^{\omega t} \) is

\[
\text{Re}(w) = \frac{(z^{Nl} - 1)^4}{z^{2NL}} \cdot \frac{M p}{2 |g(z)|^2} = \left( \frac{2 \cos Nlw_m - 2}{2} \right)^2 \cdot \frac{M p}{2 |g(z)|^2} > 0.
\]

(2.25)
A note on asymptotic stability conditions

Therefore,
\[ \frac{dr}{dp} = \frac{2r(\cos Nlwm - 1)^2 Mp}{|g(z)|^2} \tag{2.26} \]
and it follows that (2.16) holds at \( z = e^{w_m i} \).

**Case 2** (\( z^l = 1 \)). With an argument similar to Case 1, we obtain
\[ \frac{dr}{dp} = \frac{2rN^2 Mp}{|(M + 1)z - M + 1|^2} \tag{2.27} \]
which implies that (2.16) is valid for \( z = e^{w_m i} \).

This completes the proof. \( \square \)

We now determine the minimum of the absolute values of \( p_m \) given by (2.3). We have the following result.

**Proposition 2.4.** \( |p_0| = \min\{|p_m| : m = 0, 1, \ldots, M - 1\} \).

To prove Proposition 2.4, we need the following lemma, which was proved in [4].

**Lemma 2.5.** Let \( N \) be a positive integer, then
\[ \left| \frac{\sin Nt}{\sin t} \right| \leq N \tag{2.28} \]
holds for all \( t \in \mathbb{R} \).

**Proof of Proposition 2.4.** From (2.3), \( p_m = (-1)^{m+1}(\sin(lw_m/2)/\sin(Nlw_m/2)) \). For \( m = 0 \), it follows from L'Hospital's rule that \( p_0 = -1/N \). For \( m = 1, 2, \ldots, M - 1 \), we have
\[ |p_m| = \left| (-1)^{m+1} \frac{\sin(lw_m/2)}{\sin(Nlw_m/2)} \right| \geq \frac{1}{N} \tag{2.29} \]
by Lemma 2.5. This completes the proof. \( \square \)

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Note that \( F(1) = 1 + Np \leq 0 \) if and only if \( p \leq -1/N \). Since \( \lim_{z \to +\infty} F(z) = +\infty \), it follows that (2.1) has a positive root \( \alpha \) such that \( \alpha > 1 \) when \( p \leq -1/N \). We claim that if \( |p| \) is sufficiently small, then all the roots of (2.1) are inside the unit disk. To this end, we note that when \( p = 0 \), (2.1) has exactly one root at 0 of multiplicity \( k + 1 \). By the continuity of the roots with respect to \( p \), this implies that our claim is true. By Proposition 2.4, \( p_0 = -1/N \) and \( |p_m| \geq 1/N \) which implies that \( |p_0| = 1/N \) is the smallest positive value of \( p \) such that a root of (2.1) intersects the unit circle as \( |p| \) increases. Moreover, Proposition 2.3 implies that if \( p > p_{\min} \), then there exists a root \( \alpha \) of (2.1) such that \( |\alpha| \geq 1 \), where \( p_{\min} \) is the smallest positive real value of \( p \) for which (2.1) has a root on \( C \). We conclude that all the roots of (2.1) are inside the unit disk if and only if \( -1/N < p < p_{\min} \). In other words, the zero solution of (1.3) is asymptotically stable if and only if condition (1.4) holds. This completes the proof. \( \square \)
3. Examples

**Example 3.1.** In (1.3), let \( l \) and \( k \) be even positive integers, then we have

\[
F(-1) = -1 + pN. \tag{3.1}
\]

Thus if \( p = 1/N \), then \( F(-1) = 0 \) and we conclude that (1.3) is asymptotically stable if and only if \(-1/N < p < 1/N\).

**Example 3.2.** In (1.3), let \( N = 3, l = 3, \) and \( k = 6 \). Then \( M = 8 \) and we obtain \( p_0 = -1/3, p_1 = \sin(3/8)\pi/\sin(9/8)\pi, \) \( p_2 = -\sin(3/4)\pi/\sin(9/4)\pi, \) \( p_3 = \sin(9/8)\pi/\sin(27/8)\pi, \) \( p_4 = -\sin(3/2)\pi/\sin(9/2)\pi, \) \( p_5 = \sin(15/8)\pi/\sin(45/8)\pi, \) \( p_6 = -\sin(9/4)\pi/\sin(27/4)\pi, \) and \( p_7 = \sin(21/8)\pi/\sin(63/8)\pi \). Thus, \( p_3 = p_5 = \sin(\pi/8)/\sin(3\pi/8) \) is the smallest positive real value of \( p \) such that (2.1) has a root on \( C \). We conclude that (1.3) is asymptotically stable if and only if \(-1/3 < p < \sin(\pi/8)/\sin(3\pi/8)\).

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References


