We establish the characterizations of metric spaces under compact-covering (resp., pseudo-sequence-covering, sequence-covering) \( \pi \)-s-maps by means of cfp-covers (resp., sfp-covers, cs-covers) and \( \sigma \)-strong networks.

1. Introduction and definitions

In 1966, Michael [11] introduced the concept of compact-covering maps. Since many important kinds of maps are compact-covering, such as closed maps on paracompact spaces, much work has been done to seek the characterizations of metric spaces under various compact-covering maps, for example, compact-covering (open) \( s \)-maps, pseudo-sequence-covering (quotient) \( s \)-maps, sequence-covering (quotient) \( s \)-maps, and compact-covering (quotient) \( s \)-maps, see [3, 9, 12, 15, 16]. \( \pi \)-map is another important map which was introduced by Ponomarev [13] in 1960 and correspondingly, many spaces, including developable spaces, weak Cauchy spaces, \( g \)-developable spaces, and semimetrizable spaces, were characterized as the images of metric spaces under certain quotient \( \pi \)-maps, see [1, 4, 6, 7].

The purpose of this paper is to establish the characterizations of metric spaces under compact-covering (resp., pseudo-sequence-covering, sequence-covering) \( \pi \)-s-maps by means of cfp-covers (resp., sfp-covers, cs-covers) and \( \sigma \)-strong networks.

In this paper, all spaces are Hausdorff, and all maps are continuous and surjective. \( \mathbb{N} \) denotes the set of all natural numbers. \( \omega \) denotes \( \mathbb{N} \cup \{0\} \). \( \tau (X) \) denotes a topology on \( X \). For a collection \( \mathcal{P} \) of subsets of a space \( X \) and a map \( f : X \to Y \), denote \( \{ f(P) : P \in \mathcal{P} \} \) by \( f(\mathcal{P}) \). For the usual product space \( \prod_{i \in \mathbb{N}} X_i \), \( \pi_i \) denotes the projective \( \prod_{i \in \mathbb{N}} X_i \) onto \( X_i \). For a sequence \( \{ x_n \} \) in \( X \), denote \( \langle x_n \rangle = \{ x_n : n \in \mathbb{N} \} \).

**Definition 1.1.** Let \( f : X \to Y \) be a map.

1. \( f \) is called a compact-covering map [11] if each compact subset of \( Y \) is the image of some compact subset of \( X \).
2. \( f \) is called a sequence-covering map [14] if whenever \( \{ y_n \} \) is a convergent sequence in \( Y \), then there exists a convergent sequence \( \{ x_n \} \) in \( X \) such that each \( x_n \in f^{-1}(y_n) \).
(3) $f$ is called a pseudo-sequence-covering map [3] if each convergent sequence (including its limit point) of $Y$ is the image of some compact subset of $X$.

(4) $f$ is called an $s$-map, if $f^{-1}(y)$ is separable in $X$ for any $y \in Y$.

(5) $f$ is called a $\pi$-map [13], if $(X,d)$ is a metric space, and for each $y \in Y$ and its open neighborhood $V$ in $Y$, $d(f^{-1}(y), M \setminus f^{-1}(V)) > 0$.

(6) $f$ is called a $\pi$-s-map, if $f$ is both $\pi$-map and $s$-map.

It is easy to check that compact maps on metric spaces are $\pi$-s-maps.

**Definition 1.2.** Let $\{P_n\}$ be a sequence of covers of a space $X$ such that $P_{n+1}$ refines $P_n$ for each $n \in \mathbb{N}$.

1. $\bigcup\{P_n : n \in \mathbb{N}\}$ is called a $\sigma$-strong network [5] for $X$ if for each $x \in X$, $\{st(x, P_n)\}$ is a local network of $x$ in $X$. If every $P_n$ satisfies property $P$, then $\bigcup\{P_n : n \in \mathbb{N}\}$ is called a $\sigma$-strong network consisting of $P$-covers.

2. $\{P_n\}$ is called a weak development for $X$ if for each $x \in X$, $\{st(x, P_n)\}$ is a weak neighborhood base of $x$ in $X$.

**Definition 1.3** [2]. Let $X$ be a space.

1. Let $\{x_n\}$ be a convergent sequence in $X$, and $P \subset X$. $\{x_n\}$ is eventually in $P$ if whenever $\{x_n\}$ converges to $x$, then $\{x\} \cup \{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$.

2. Let $x \in P \subset X$. $P$ is called a sequential neighborhood of $x$ in $X$ if whenever a sequence $\{x_n\}$ in $X$ converges to $x$, then $\{x_n\}$ is eventually in $P$.

3. Let $P \subset X$. $P$ is called a sequentially open subset in $X$ if $P$ is a sequential neighborhood of $x$ in $X$ for any $x \in P$.

4. $X$ is called a sequential space if each sequentially open subset in $X$ is open.

**Definition 1.4** [10]. Let $\mathcal{P}$ be a collection of subsets of a space $X$.

1. $\mathcal{P}$ is called a cfp-cover (i.e., compact-finite-partition cover) of compact subset $K$ in $X$ if there are a finite collection $\{K_\alpha : \alpha \in J\}$ of closed subsets of $K$ and $\{P_\alpha : \alpha \in J\} \subset \mathcal{P}$ such that $K = \bigcup\{K_\alpha : \alpha \in J\}$ and each $K_\alpha \subset P_\alpha$.

2. $\mathcal{P}$ is called a cfp-cover for $X$ if for any compact subset $K$ of $X$, there exists a finite subcollection $\mathcal{P}^* \subset \mathcal{P}$ such that $\mathcal{P}^*$ is a cfp-cover of $K$ in $X$.

3. $\mathcal{P}$ is called an sfp-cover (i.e., sequence-finite-partition cover) for $X$ if for any convergent sequence (including its limit point) $K$ in $X$, there exists a finite subcollection $\mathcal{P}^* \subset \mathcal{P}$ such that $\mathcal{P}^*$ is a cfp-cover of $K$ in $X$.

4. $\mathcal{P}$ is called a cs-cover for $X$, if every convergent sequence in $X$ is eventually in some element of $\mathcal{P}$.

2. Results

**Theorem 2.1.** A space $X$ is the compact-covering $\pi$-s-image of a metric spaces if and only if $X$ has a $\sigma$-strong network consisting of point-countable cfp-covers.

**Proof.** To prove the only if part, suppose $f : (M,d) \rightarrow X$ is a compact-covering $\pi$-s-map, where $(M,d)$ is a metric space. For each $n \in \mathbb{N}$, put $\mathcal{F}_n = \{f(B(z,1/n)) : z \in M\}$, where $B(z,1/n) = \{y \in M : d(z,y) < 1/n\}$. Obviously, $\mathcal{F}_n$ refines $\mathcal{F}_n$. We claim that $\bigcup\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a $\sigma$-strong network for $X$. In fact, for each $x \in X$, and its open neighborhood $U$, since $f$ is a $\pi$-map, then there exists $n \in \mathbb{N}$ such that $d(f^{-1}(x), M \setminus f^{-1}(U)) > 1/n$. 


We can pick $m \in \mathbb{N}$ such that $m \geq 2n$. If $z \in M$ with $x \in f(B(z, 1/m))$, then
\[ f^{-1}(x) \cap B(z, 1/m) \neq \emptyset. \] (2.1)

If $B(z, 1/m) \notin f^{-1}(U)$, then
\[ d(f^{-1}(x), M \setminus f^{-1}(U)) \leq \frac{2}{m} \leq \frac{1}{n}, \] (2.2)

which is a contradiction. Thus $B(z, 1/m) \subset f^{-1}(U)$, so $f(B(z, 1/m)) \subset U$. Hence $\text{st}(x, \mathcal{F}_m) \subset U$. Therefore $\bigcup \{\mathcal{F}_n : n \in \mathbb{N}\}$ is a $\sigma$-strong network for $X$.

For each $n \in \mathbb{N}$, let $\mathcal{B}_n$ be a locally finite open refinement of $\{B(z, 1/n) : z \in M\}$. Since locally finite collections are closed under finite intersections, we can assume that $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$ for each $n \in \mathbb{N}$. Put $\mathcal{P}_n = f(\mathcal{B}_n)$. Obviously, $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$. Since $f$ is an $s$-map, each $\mathcal{P}_n$ is point-countable in $X$. Because $\mathcal{P}_n$ refines $\mathcal{F}_n$ for each $n \in \mathbb{N}$, then $\bigcup \{\mathcal{P}_n : n \in \mathbb{N}\}$ is also a $\sigma$-strong network for $X$.

We now show that each $\mathcal{P}_n$ is a cfp-cover for $X$. Suppose $K$ is compact in $X$, since $f$ is compact-covering, then $f(L) = K$ for some compact subset $L$ of $M$. Since $\mathcal{B}_n$ is an open cover of $L$ in $M$, $\mathcal{B}_n$ have a finite subcover $\mathcal{B}_{n,L}^n$. Thus $\mathcal{B}_{n,L}^n$ can be precisely refined by some finite cover of $L$ consisting of closed subsets of $L$, denoted by $\{L_n : \alpha \in \mathcal{J}_n\}$. Put $\mathcal{P}_n^K = f(\mathcal{B}_{n,L}^n)$, since $\mathcal{P}_n^K$ is precisely refined by closed cover $\{f(L_n) : \alpha \in \mathcal{J}_n\}$ of $K$, then $\mathcal{P}_n^K$ is a cfp-cover of $K$ in $X$. Hence each $\mathcal{P}_n$ is a cfp-cover for $X$.

To prove the if part, suppose $\bigcup \{\mathcal{P}_i : i \in \mathbb{N}\}$ is a $\sigma$-strong network for $X$ consisting of point-countable cfp-covers. For each $i \in \mathbb{N}$, $\mathcal{P}_i$ is a point-countable cfp-cover for $X$. Let $\mathcal{P}_i = \{P_\alpha : \alpha \in \Lambda_i\}$, endow $\Lambda_i$ with the discrete topology, then $\Lambda_i$ is a metric space. Put
\[ M = \left\{ \alpha = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Lambda_i : \langle P_\alpha \rangle \text{ forms a local network at some point } x_\alpha \text{ in } X \right\}, \] (2.3)

and endow $M$ with the subspace topology induced from the usual product topology of the collection $\{\Lambda_i : i \in \mathbb{N}\}$ of metric spaces, then $M$ is a metric space. Since $X$ is Hausdorff, $x_\alpha$ is unique in $X$. For each $\alpha \in M$, we define $f : M \to X$ by $f(\alpha) = x_\alpha$. For each $x \in X$ and $i \in \mathbb{N}$, there exists $\alpha_i \in \Lambda_i$ such that $x \in P_{\alpha_i}$. Since $\bigcup \{\mathcal{P}_i : i \in \mathbb{N}\}$ is a $\sigma$-strong network for $X$, then $\{P_{\alpha_i} : i \in \mathbb{N}\}$ is a local network of $x$ in $X$. Put $\alpha = (\alpha_i)$, then $\alpha \in M$ and $f(\alpha) = x$. Thus $f$ is surjective. Suppose $\alpha = (\alpha_i) \in M$ and $f(\alpha) = x \in U \in \tau(X)$, then there exists $n \in \mathbb{N}$ such that $P_{\alpha_n} \subset U$. Put
\[ V = \{\beta \in M : \text{the } n\text{th coordinate of } \beta \text{ is } x_{\alpha_n}\}, \] (2.4)

then $V$ is an open neighborhood of $\alpha$ in $M$, and $f(V) \subset P_{\alpha_n} \subset U$. Hence $f$ is continuous. For each $\alpha, \beta \in M$, we define
\[ d(\alpha, \beta) = \begin{cases} 0, & \alpha = \beta, \\ \max \{1/k : \pi_k(\alpha) \neq \pi_k(\beta)\}, & \alpha \neq \beta, \end{cases} \] (2.5)

then $d$ is a distance on $M$. Because the topology of $M$ is the subspace topology induced from the usual product topology of the collection $\{\Lambda_i : i \in \mathbb{N}\}$ of discrete spaces, thus $d$
is a metric on $M$. For each $x \in U \in \tau(X)$, there exists $n \in \mathbb{N}$ such that $st(x, \mathcal{P}_n) \subset U$. For $\alpha \in f^{-1}(x)$, $\beta \in M$, if $d(\alpha, \beta) < 1/n$, then $\pi_i(\alpha) = \pi_i(\beta)$ whenever $i \leq n$. So $x \in P_{\pi_i(\alpha)} = P_{\pi_i(\beta)}$. Thus,

$$f(\beta) \in \bigcap_{i \in \mathbb{N}} P_{\pi_i(\beta)} \subset P_{\pi(\beta)} \subset U. \quad (2.6)$$

Hence

$$d(f^{-1}(x), M \setminus f^{-1}(U)) \geq \frac{1}{n}. \quad (2.7)$$

Therefore $f$ is a $\pi$-map.

For each $x \in X$, it follows from the point-countable property of $\mathcal{P}_n$ that $\{\alpha \in \Lambda_i : x \in P_{\alpha} \}$ is countable. Put

$$L = \left( \bigcap_{i \in \mathbb{N}} \{ \alpha \in \Lambda_i : x \in P_{\alpha} \} \right) \bigcap M, \quad (2.8)$$

then $L$ is a hereditarily separable subspace of $M$, and $f^{-1}(x) \subset L$. Thus $f^{-1}(x)$ is separable in $M$, that is, $f$ is an $s$-map.

We will prove that $f$ is compact-covering. Suppose $K$ is compact in $X$. Since each $\mathcal{P}_n$ is a cfp-cover for $X$, there exists finite subcollection $\mathcal{P}_n^K$ such that it is a cfp-cover of $K$ in $X$. Thus there are a finite collection $\{K_\alpha : \alpha \in J_n\}$ of closed subsets of $K$ and $\{P_\alpha : \alpha \in J_n\} \subset \mathcal{P}_n^K$ such that $K = \bigcup \{K_\alpha : \alpha \in J_n\}$ and each $K_\alpha \subset P_\alpha$. Obviously, each $K_\alpha$ is compact in $X$. Put

$$L = \left\{ (\alpha_i) : \alpha_i \in J_i, \bigcap_{i \in \mathbb{N}} K_{\alpha_i} \neq \emptyset \right\}, \quad (2.9)$$

then

(i) $L$ is compact in $M$.  
In fact, for all $(\alpha_i) \notin L$, $\bigcap_{i \in \mathbb{N}} K_{\alpha_i} = \emptyset$. From $\bigcap_{i \in \mathbb{N}} K_{\alpha_i} = \emptyset$, there exists $n_0 \in \mathbb{N}$ such that $\bigcap_{i = 1}^{n_0} K_{\alpha_i} = \emptyset$. Put

$$W = \{ (\beta_i) : \beta_i \in J_i, \beta_i = \alpha_i, 1 \leq i \leq n_0 \}, \quad (2.10)$$

then $W$ is an open neighborhood of $(\alpha_i)$ in $\prod_{i \in \mathbb{N}} J_i$, and $W \bigcap L = \emptyset$. Thus $L$ is closed in $\prod_{i \in \mathbb{N}} J_i$. Since $\prod_{i \in \mathbb{N}} J_i$ is compact in $\prod_{i \in \mathbb{N}} \Lambda_i$, $L$ is compact in $M$.

(ii) $L \subset M$, $f(L) = K$.  
In fact, for all $(\alpha_i) \in L$, $\bigcap_{i \in \mathbb{N}} K_{\alpha_i} \neq \emptyset$. Pick $x \in \bigcap_{i \in \mathbb{N}} K_{\alpha_i}$, then $(P_{\alpha_i})$ is a local network of $x$ in $X$, so $(\alpha_i) \in M$. This implies $L \subset M$.

For all $x \in K$, for each $i \in \mathbb{N}$, pick $\alpha_i \in J_i$ such that $x \in K_{\alpha_i}$. Thus $f((\alpha_i)) = x$, so $K \subset f(L)$. Obviously, $f(L) \subset K$. Hence $f(L) = K$.

In a word, $f$ is compact-covering. \qed

**Corollary 2.2.** A space $X$ is the compact-covering, quotient, and $\pi$-s-image of a metric space if and only if $X$ has a weak-development consisting of point-countable cfp-covers.
Proof. To prove the only if part, suppose \( X \) is the compact-covering, quotient, and \( \pi \)-image of a metric space \( M \). From Theorem 2.1, \( X \) has a \( \sigma \)-strong network consisting of point-countable \( \text{cfp} \)-covers \( \bigcup \{ \mathcal{P}_n : n \in \mathbb{N} \} \). For each \( x \in X \), \( \text{st}(x, \mathcal{P}_n) \) is a sequential neighborhood of \( x \) in \( X \). Obviously, \( X \) is a sequential space. Thus \( \text{st}(x, \mathcal{P}_n) \) is a weak neighborhood base of \( x \) in \( X \). Hence \( \{ \mathcal{P}_n \} \) is a weak-development for \( X \).

To prove the if part, suppose \( X \) has a weak development consisting of point-countable \( \text{cfp} \)-covers. From Theorem 2.1, \( X \) is the image of a metric space under a compact-covering \( \pi \)-map \( f \). Obviously, \( X \) is sequential. By [8, Proposition 2.1.16], \( f \) is quotient.

Similar to the proofs of Theorem 2.1 and Corollary 2.2, we have the following theorem.

Theorem 2.3. A space \( X \) is the pseudo-sequence-covering \( \pi \)-image of a metric space if and only if \( X \) has a \( \sigma \)-strong network consisting of point-countable \( \text{sfp} \)-covers.

Corollary 2.4. A space \( X \) is the pseudo-sequence-covering, quotient, and \( \pi \)-image of a metric space if and only if \( X \) has a weak-development consisting of point-countable \( \text{sfp} \)-covers.

Theorem 2.5. A space \( X \) is the sequence-covering \( \pi \)-image of a metric space if and only if \( X \) has a \( \sigma \)-strong network consisting of point-countable \( \text{cs} \)-covers.

Proof. To prove the only if part, suppose \( f : (M, d) \rightarrow X \) is a sequence-covering \( \pi \)-map, where \((M, d)\) is a metric space. Similar to the proof of Theorem 2.1, we can show that \( \bigcup \{ \mathcal{P}_n : n \in \mathbb{N} \} \) is a \( \sigma \)-strong network consisting of point-countable covers. It suffices to show that each \( \mathcal{P}_n \) is a \( \text{cs} \)-cover for \( X \). Suppose \( \{ x_n \} \) converges to \( x \in X \). Since \( f \) is sequence-covering, then there exists a convergent sequence \( \{ z_i \} \) such that each \( z_i \in f^{-1}(x_i) \). Suppose \( \{ z_i \} \rightarrow z \), then \( z \in f^{-1}(x) \) and \( z \in B \) for some \( B \in \mathcal{B}_n \). Thus \( \{ z_i \} \) is eventually in \( B \), so \( \{ x_i \} \) is eventually in \( f(B) \in \mathcal{P}_n \). Hence each \( \mathcal{P}_n \) is a \( \text{cs} \)-cover for \( X \).

To prove the if part, suppose \( \bigcup \{ \mathcal{P}_i : i \in \mathbb{N} \} \) is a \( \sigma \)-strong network consisting of point-countable \( \text{cs} \)-covers for \( X \). For each \( i \in \mathbb{N} \), \( \mathcal{P}_i \) is a point-countable \( \text{cs} \)-cover for \( X \). Let \( \mathcal{P}_i = \{ \alpha_\alpha : \alpha \in \Lambda_i \} \). Similar to the proof of Theorem 2.1, we can show that \( f \) is a \( \pi \)-map. It suffices to show that \( f \) is sequence-covering. Suppose \( \{ x_n \} \) converges to \( x \) in \( X \). For each \( i \in \mathbb{N} \), since \( \mathcal{P}_i \) is a \( \text{cs} \)-cover for \( X \), then there exists \( P_{\alpha_i} \in \mathcal{P}_i \) such that \( \{ x_n \} \) is eventually in \( P_{\alpha_i} \). For each \( n \in \mathbb{N} \), if \( x_n \in P_{\alpha_i} \), let \( \alpha_{i_n} = \alpha_i \); if \( x_n \notin P_{\alpha_i} \), pick \( \alpha_{i_n} \in \Lambda_i \) such that \( x_n \in P_{\alpha_{i_n}} \). Thus there exists \( n_i \in \mathbb{N} \) such that \( \alpha_{i_n} = \alpha_i \) for all \( n > n_i \). So \( \{ \alpha_{i_n} \} \) converges to \( \alpha_i \). For each \( n \in \mathbb{N} \), put

\[
\beta_n = (\alpha_{i_n}) \in \prod_{i \in \mathbb{N}} \Lambda_i,
\] (2.11)

then \( (\beta_n) \in f^{-1}(x_n) \) and \( \{ \beta_n \} \) converges to \( x \). Thus \( f \) is sequence-covering.

Similar to the proof of Corollary 2.2, we have the following corollary.

Corollary 2.6. A space \( X \) is the sequence-covering, quotient, and \( \pi \)-image of a metric space if and only if \( X \) has a weak-development consisting of point-countable \( \text{cs} \)-covers.

We give examples to illustrate the theorems of this paper.
Example 2.7. Let $Z$ be the topological sum of the unit interval $[0,1]$, and the collection 
$\{S(x) : x \in [0,1]\}$ of $2^\omega$ convergent sequence $S(x)$. Let $X$ be the space obtained from $Z$ 
by identifying the limit point of $S(x)$ with $x \in [0,1]$, for each $x \in [0,1]$. Then, from [8, 
Example 2.9.27], or see [3, Example 9.8], we have the following facts.

(1) $X$ is the compact-covering, quotient compact image of a locally compact metric 
space.

(2) $X$ has no point-countable cs-network.

The above facts together with [9, Theorem 1] yield the following conclusion: compact-
covering (quotient) $\pi$-s-images of metric spaces are not sequence-covering (quotient) 
$\pi$-s-images of metric spaces.

Example 2.8. Let $X$ be a sequential fan $S_\omega$ (see [8, Example 1.8.7]), then $X$ is a Fréchet 
and $\aleph_0$-space. So $X$ is the sequence-covering $s$-image of a metric space. Because $X$ is 
not $g$-first countable, thus $X$ is not the pseudo-sequence-covering $\pi$-image of a metric 
space. Hence the following holds: sequence-covering (resp., pseudo-sequence-covering) 
$s$-images of metric spaces are not sequence-covering (resp., pseudo-sequence-covering) 
$\pi$-s-images of metric spaces.

Example 2.9. Let $X$ be a Gillman-Jerison space $\psi(\mathbb{N})$ (see [8, Example 1.8.4]). Since $X$ is 
developable, then $X$ is the sequence-covering, quotient $\pi$-image of a metric space by [10, 
Corollary 3.1.12]. But $X$ has no point-countable cs*-networks. Then, it follows from [8, 
Theorem 2.7.5] that $X$ is not the pseudo-sequence-covering $s$-image of a metric space. Thus,

(1) sequence-covering (quotient) $\pi$-images of metric spaces are not sequence-
covering (quotient) $\pi$-s-images of metric spaces,

(2) pseudo-sequence-covering (quotient) $\pi$-images of metric spaces are not pseudo-
sequence-covering (quotient) $\pi$-s-images of metric spaces.

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