This paper deals with the existence and uniqueness of solutions for a class of infinite-horizon systems derived from optimal control. An existence and uniqueness theorem is proved for such Hamiltonian systems under some natural assumptions.

1. Introduction

We begin with a simple example to introduce the background of the considered problem. Let $U$ be a bounded closed subset of $\mathbb{R}^m$ and let functions $f : \mathbb{R}^n \times \mathbb{R}^m \times [a, \infty) \to \mathbb{R}^n$, $L : \mathbb{R}^n \times \mathbb{R}^m \times [a, \infty) \to \mathbb{R}$ be differentiable with respect to the first variable. Consider an optimal control system of the form

$$\text{Minimize } J[u(\cdot)] = \int_a^\infty L(x(t), u(t), t) dt \quad (1.1)$$

over all admissible controls $u(\cdot) \in L^2([a, \infty); U)$, where the trajectories $x : [a, \infty) \to \mathbb{R}^n$ are differentiable on $[a, \infty)$ and satisfy the dynamic system

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(a) = x_0. \quad (1.2)$$

From control theory, the well-known Pontryagin maximum principle, an important necessary optimality condition, is usually applied to get optimal controls for this system. By doing this, the following infinite-horizon Hamiltonian system is derived:

$$\dot{x}(t) = \frac{\partial H(x(t), p(t), t)}{\partial p}, \quad x(a) = x_0,$n

$$\dot{p}(t) = -\frac{\partial H(x(t), p(t), t)}{\partial x}, \quad x(\cdot) \in L^2([a, \infty); \mathbb{R}^n), \quad p(\cdot) \in L^2([a, \infty); \mathbb{R}^n). \quad (1.3)$$
Here, $H(x, p, t) = \lambda L(x, \tilde{u}, t) + \langle p, f(x, \tilde{u}, t) \rangle$ is the Hamiltonian function for (1.1)-(1.2), $\langle \cdot, \cdot \rangle$ stands for inner product in $\mathbb{R}^n$, $\tilde{u}$ is an optimal control, and $x(t)$ is the optimal trajectory corresponding to the optimal control $\tilde{u}$.

The existence and uniqueness of solutions for system (1.3) is a very interesting question; if solutions to (1.3) are unique, then the optimal control for system (1.1)-(1.2) can be solved analytically or numerically through (1.3). When we consider the generalization of (1.3) in infinite-dimensional spaces, the following Hamiltonian system is obtained:

$$
\begin{align*}
\dot{x}(t) &= A(t)x(t) + F(x(t), p(t), t), \\
x(a) &= x_0, \\
\dot{p}(t) &= -A^*(t)p(t) + G(x(t), p(t), t), \\
x(\cdot) &\in L^2([a, \infty); X), \quad p(\cdot) \in L^2([a, \infty); X),
\end{align*}
$$

(1.4)

where both $x(t)$ and $p(t)$ take values in a Hilbert space $X$ for $a \leq t < \infty$. It is always assumed that $F, G : X \times X \times [a, \infty) \rightarrow X$ are nonlinear operators, that $A(t)$ is a closed operator for each $t \in [a, \infty)$, and that $A^*(t)$ is the adjoint operator of $A(t)$.

The following system is called a linear Hamiltonian system, which is a special case of (1.4),

$$
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)p(t) + \varphi(t), \\
x(a) &= x_0, \\
\dot{p}(t) &= -A^*(t)p(t) + C(t)x(t) + \psi(t), \\
x(\cdot) &\in L^2([a, \infty); X), \quad p(\cdot) \in L^2([a, \infty); X),
\end{align*}
$$

(1.5)

where $\varphi(\cdot), \psi(\cdot) \in L^2([a, \infty); X)$, and $B(t), C(t)$ are selfadjoint linear operators from $X$ to $X$ for all $t \in [a, \infty)$.

In [2], Lions has discussed the existence and uniqueness of solutions for system (1.5) and gave an existence and uniqueness result. In [1], Hu and Peng considered the existence and uniqueness of solutions for a class of nonlinear forward-backward stochastic differential equations similar to (1.3) but on finite horizon, they provided an existence and uniqueness theorem for (1.3). Peng and Shi in [3] dealt with the existence and uniqueness of solutions for (1.3) using the techniques developed in [1]. In this paper, we consider the existence and uniqueness of solutions for infinite-dimensional system (1.4).

Throughout the paper, the following basic assumptions hold.

(1) There exists a real number $L > 0$ such that

$$
\begin{align*}
\|F(x_1, p_1, t) - F(x_2, p_2, t)\| &\leq L(\|x_1 - x_2\| + \|p_1 - p_2\|), \\
\|G(x_1, p_1, t) - G(x_2, p_2, t)\| &\leq L(\|x_1 - x_2\| + \|p_1 - p_2\|)
\end{align*}
$$

(1.6)

for all $x_1, p_1, x_2, p_2 \in X$ and $t \in [a, \infty)$. 

(II) There exists a real number $\alpha > 0$ such that
\[
\langle F(x_1, p_1, t) - F(x_2, p_2, t), p_1 - p_2 \rangle + \langle G(x_1, p_1, t) - G(x_2, p_2, t), x_1 - x_2 \rangle \\
\leq -\alpha(\|x_1 - x_2\| + \|p_1 - p_2\|)
\] (1.7)
for all $x_1, p_1, x_2, p_2 \in X$ and $t \in [a, \infty)$. 

2. Lemmas
Two lemmas are given in this section. They are essential to prove the main theorem.

Lemma 2.1. Consider the Hamiltonian system
\[
\dot{x}(t) = A(t)x(t) + F_\beta(x, p, t) + \varphi(t), \\
x(a) = x_0,
\]
\[
\dot{p}(t) = -A^*(t)p(t) + G_\beta(x, p, t) + \psi(t),
\]
\[
x(\cdot) \in L^2([a, \infty); X), \quad p(\cdot) \in L^2([a, \infty); X),
\] (2.1)
where $\varphi(\cdot), \psi(\cdot) \in L^2([a, \infty); X)$. The functions $F_\beta$ and $G_\beta$ are defined as
\[
F_\beta(x, p, t) := -(1 - \beta)\alpha p + \beta F(x, p, t),
\]
\[
G_\beta(x, p, t) := -(1 - \beta)ax + \beta G(x, p, t).
\] (2.2)

Assume that (2.1) has a unique solution for some real number $\beta = \beta_0 \geq 0$ and any $\varphi(t), \psi(t)$. There exists a real number $\delta > 0$, which is independent of $\beta_0$, such that (2.1) has a unique solution for any $\varphi(t), \psi(t)$, and $\beta \in [\beta_0, \beta_0 + \delta]$.

Proof. For any given $\varphi(\cdot), \psi(\cdot), x(\cdot), p(\cdot) \in L^2([a, \infty); X)$ and $\delta > 0$, construct the following Hamiltonian system:
\[
\dot{X}(t) = A(t)X(t) + F_{\beta_0}(X, P, t) + F_{\beta_0 + \delta}(x, p, t) - F_{\beta_0}(x, p, t) + \varphi(t),
\]
\[
X(a) = x_0,
\]
\[
\dot{P}(t) = -A^*(t)P(t) + G_{\beta_0}(X, P, t) + G_{\beta_0 + \delta}(x, p, t) - G_{\beta_0}(x, p, t) + \psi(t),
\]
\[
X(\cdot) \in L^2([a, \infty); X), \quad P(\cdot) \in L^2([a, \infty); X).
\] (2.3)

Note that
\[
F_{\beta_0 + \delta}(x, p, t) - F_{\beta_0}(x, p, t) \\
= -(1 - \beta_0 - \delta)\alpha p + (\beta_0 + \delta)F(x, p, t) + (1 - \beta_0)\alpha p - \beta_0 F(x, p, t) \\
= \alpha \delta p + \delta F(x, p, t),
\] (2.4)
\[
G_{\beta_0 + \delta}(x, p, t) - G_{\beta_0}(x, p, t) \\
= -(1 - \beta_0 - \delta)ax + (\beta_0 + \delta)G(x, p, t) + (1 - \beta_0)ax - \beta_0 G(x, p, t) \\
= \alpha \delta x + \delta G(x, p, t).
The assumption of Lemma 2.1 implies that (2.3) has a unique solution for each pair \((x(\cdot), p(\cdot)) \in L^2([a, \infty); X) \times L^2([a, \infty); X)\). Therefore, the mapping \(J\),

\[
L^2([a, \infty); X) \times L^2([a, \infty); X) \to L^2([a, \infty); X) \times L^2([a, \infty); X),
\]

given by

\[
J(x(\cdot), p(\cdot)) := (X(\cdot), P(\cdot))
\]

is well defined.

Let \(J(x_1(\cdot), p_1(\cdot)) = (X_1(\cdot), P_1(\cdot))\) and \(J(x_2(\cdot), p_2(\cdot)) = (X_2(\cdot), P_2(\cdot))\). Since \(X_1(\cdot) - X_2(\cdot) \in L^2([a, \infty); X)\) and \(P_1(\cdot) - P_2(\cdot) \in L^2([a, \infty); X)\), there exists a sequence of real numbers \(a < t_1 < t_2 < \cdots < t_k < \cdots\) such that \(t_k \to \infty\) as \(k \to \infty\) and

\[
X_1(t_k) - X_2(t_k) \to 0, \quad P_1(t_k) - P_2(t_k) \to 0, \quad \text{as} \quad k \to \infty.
\]

Note that

\[
\frac{d}{dt} \langle X_1(t) - X_2(t), P_1(t) - P_2(t) \rangle = \langle F_{\beta_0}(X_1, P_1, t) - F_{\beta_0}(X_2, P_2, t), P_1 - P_2 \rangle + \alpha \delta(x_1 - x_2) + \delta(F(x_1, p_1, t) - F(x_2, p_2, t))\]

implies that

\[
I_1 = -\alpha(1 - \beta_0) \|P_1 - P_2\|^2 + \beta_0 \langle F(X_1, P_1, t) - F(X_2, P_2, t), P_1 - P_2 \rangle
\]

\[
+ \alpha \delta(x_1 - x_2) + \delta(F(x_1, p_1, t) - F(x_2, p_2, t), P_1 - P_2),
\]

similarly,

\[
G_{\beta_0}(X_1, P_1, t) - G_{\beta_0}(X_2, P_2, t) = -\alpha(1 - \beta_0) (X_1 - X_2) + \beta_0 (G(X_1, P_1, t) - G(X_2, P_2, t))
\]

implies that

\[
I_2 = -\alpha(1 - \beta_0) \|X_1 - X_2\|^2 + \beta_0 \langle G(X_1, P_1, t) - G(X_2, P_2, t), X_1 - X_2 \rangle
\]

\[
+ \alpha \delta(x_1 - x_2, X_1 - X_2) + \delta(G(x_1, p_1, t) - G(x_2, p_2, t), X_1 - X_2).
\]
It follows from the estimates for $I_1, I_2$, and the assumption (I) that

$$I_1 + I_2 \leq -\alpha (||X_1 - X_2||^2 + ||P_1 - P_2||^2)$$

$$+ \alpha \delta (||P_1 - P_2|| ||P_1 - P_2|| + ||x_1 - x_2|| ||X_1 - X_2||)$$

$$+ \delta \| F(x_1, p_1, t) - F(x_2, p_2, t) || P_1 - P_2 \|$$

$$+ \delta \| G(x_1, p_1, t) - G(x_2, p_2, t) || ||X_1 - X_2||$$

$$\leq -\alpha (||X_1 - X_2||^2 + ||P_1 - P_2||^2)$$

$$+ \delta (2L + \alpha) (||X_1 - X_2||^2 + ||P_1 - P_2||^2 + ||x_1 - x_2||^2 + ||P_1 - P_2||^2).$$

Therefore,

$$\frac{d}{dt} \langle X_1(t) - X_2(t), P_1(t) - P_2(t) \rangle$$

$$\leq -\alpha (||X_1 - X_2||^2 + ||P_1 - P_2||^2)$$

$$+ \delta (2L + \alpha) (||X_1 - X_2||^2 + ||P_1 - P_2||^2 + ||x_1 - x_2||^2 + ||P_1 - P_2||^2).$$

Integrating between $a$ and $t$, we have

$$\langle X_1(t_k) - X_2(t_k), P_1(t_k) - P_2(t_k) \rangle - \langle X_1(a) - X_2(a), P_1(a) - P_2(a) \rangle$$

$$\leq -\alpha \int_a^{t_k} (||X_1 - X_2||^2 + ||P_1 - P_2||^2) dt + \delta (2L + \alpha)$$

$$\times \int_a^{t_k} (||X_1 - X_2||^2 + ||P_1 - P_2||^2 + ||x_1 - x_2||^2 + ||P_1 - P_2||^2) dt.$$

Letting $k \to \infty$ and noting that (2.7), we obtain

$$\int_a^{\infty} (||X_1 - X_2||^2 + ||P_1 - P_2||^2) dt \leq \frac{2\delta L + \delta \alpha}{\alpha - 2\delta L - \delta \alpha} \int_a^{\infty} (||X_1 - X_2||^2 + ||P_1 - P_2||^2) dt.$$

Choose a small $\delta$ (independent of $\beta_0$) such that

$$\frac{2\delta L + \delta \alpha}{\alpha - 2\delta L - \delta \alpha} \leq \frac{1}{2},$$

then $J$ is a contractive mapping and hence has a unique fixed point. Thus, (2.3) becomes

$$\dot{x}(t) = A(t)x(t) + F_{\beta_0 + \delta}(x, p, t) + \varphi(t),$$

$$x(a) = x_0,$$

$$\dot{p}(t) = -A^*(t)p(t) + G_{\beta_0 + \delta}(x, p, t) + \psi(t),$$

$$x(\cdot) \in L^2([a, \infty); X), \quad p(\cdot) \in L^2([a, \infty); X).$$

This shows that system (2.1) has a unique solution on $[a, \infty)$ for $\beta \in [\beta_0, \beta_0 + \delta]$. The proof is complete. \qed
Lemma 2.2. System (2.1) has a unique solution on \([a, \infty)\) for \(\beta = 0\), that is, the system

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) - \alpha p(t) + \varphi(t), \\
x(0) &= x_0, \\
\dot{p}(t) &= -A^*(t)p(t) - \alpha x(t) + \psi(t), \\
x(\cdot) &\in L^2([a, \infty); X), \\
p(\cdot) &\in L^2([a, \infty); X),
\end{align*}
\]

(2.19)

has a unique solution on \([a, \infty)\).

For the proof, see [2, Section 6.2, Chapter III].

3. Main theorem

Theorem 3.1. System (1.4) has a unique solution under assumptions (I) and (II).

Proof. By Lemma 2.2, system (2.1) has a unique solution on \([a, \infty)\) in the case \(\beta_0 = 0\). It follows from Lemma 2.1 that there exists a real number \(\delta > 0\) such that (2.1) has a unique solution on \([a, \infty)\) for any \(\beta \in [0, \delta]\) and \(\varphi, \psi \in L^2([a, \infty); X)\). Let \(\beta_0 = \delta\) in Lemma 2.1. Repeating this procedure implies that (2.1) has a unique solution on \([a, \infty)\) for any \(\beta \in [\delta, 2\delta]\) and \(\varphi, \psi \in L^2([a, \infty); X)\). After finitely many steps, one can show that system (2.1) has a unique solution for \(\beta = 1\). Therefore, it is proved that system (1.4) has a unique solution on \([a, \infty)\) by letting \(\beta = 1\), \(\varphi(t) \equiv 0\), and \(\psi(t) \equiv 0\). \(\square\)

Remark 3.2. Consider system (1.5). Note that

\[
\begin{align*}
\langle F(x_1, p_1, t) - F(x_2, p_2, t), p_1 - p_2 \rangle + \langle G(x_1, p_1, t) - G(x_2, p_2, t), x_1 - x_2 \rangle &= \langle B(t)(p_1 - p_2), p_1 - p_2 \rangle + \langle C(t)(x_1 - x_2), x_1 - x_2 \rangle.
\end{align*}
\]

(3.1)

By Theorem 3.1, system (1.5) has a unique solution if it is assumed that both \(B(t)\) and \(C(t)\) are uniformly negative definite on \([a, \infty)\), that is, there exists a real number \(\gamma > 0\) such that \((B(t)x, x) \leq -\gamma \|x\|^2\) and \((C(t)x, x) \leq -\gamma \|x\|^2\) for all \(x \in X, x \neq 0\), and \(t \in [a, \infty)\).

Remark 3.3. Consider the control system

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + Bu(t), \\
x(a) &= x_0,
\end{align*}
\]

(3.2)

with a quadratic cost functional

\[
J[u(\cdot)] = \int_a^\infty \left[ \langle Qx(t), x(t) \rangle + \langle Ru(t), u(t) \rangle \right] dt,
\]

(3.3)

where \(u(t)\) and \(x(t)\) take values in Hilbert spaces \(U\) and \(X\), where \(B \in \mathcal{L}[U, X]\), and where \(Q \in \mathcal{L}[X, X]\) and \(R \in \mathcal{L}[U, U]\) are selfadjoint operators.
From optimal control theory, the following Hamiltonian system is derived:

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) - BR^{-1}Bp(t), \\
x(a) &= x_0, \\
\dot{p}(t) &= -A^*(t)p(t) - Qx(t), \\
x(\cdot) &\in L^2([a, \infty); X), \\
p(\cdot) &\in L^2([a, \infty); X).
\end{align*}
\]

(3.4)

This is a special case of system (1.5). Therefore, system (3.4) has a unique solution if both $BR^{-1}B$ and $Q$ are positive definite.

References


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