We give de Leeuw-type transference theorems for bilinear multipliers. In particular, it is shown that bilinear multipliers arising from regulated functions $m(\xi, \eta)$ in $\mathbb{R} \times \mathbb{R}$ can be transferred to bilinear multipliers acting on $\mathbb{T} \times \mathbb{T}$ and $\mathbb{Z} \times \mathbb{Z}$. The results follow from the description of bilinear multipliers on the discrete real line acting on $L^p$-spaces.

1. Introduction

Let $(p_1, p_2, p_3)$ be such that $0 < p_1, p_2, p_3 \leq \infty$, $1/p_1 + 1/p_2 = 1/p_3$ and let $m(\xi, \eta)$ be a bounded measurable function on $\mathbb{R}^2$. It is said to be a bilinear $(p_1, p_2)$-multiplier on $\mathbb{R} \times \mathbb{R}$ if

$$\mathcal{C}_m(f, g)(x) = \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)m(\xi, \eta)e^{2\pi i x(\xi+\eta)}d\xi d\eta \quad (1.1)$$

(defined for Schwarz test functions $f$ and $g$ in $\mathcal{S}$) extends to a bounded bilinear operator from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ into $L^{p_3}(\mathbb{R})$.

The theory of these multipliers has been tremendously developed after the results proved by Lacey and Thiele (see [16, 18, 17]) which establish that $m(\xi, \nu) = \text{sign}(\xi + \alpha \nu)$ is a $(p_1, p_2)$-multiplier for each triple $(p_1, p_2, p_3)$ such that $1 < p_1, p_2 \leq \infty$, $p_3 > 2/3$, and each $\alpha \in \mathbb{R} \setminus \{0, 1\}$.

The study of such multipliers was started by Coifman and Meyer (see [3, 4, 19]) for smooth symbols and new results for nonsmooth symbols, extending the ones given by the bilinear Hilbert transform, have been achieved by Gilbert and Nahmod (see [8, 9, 10]) and also by Muscalu et al. (see [20]).

We refer the reader also to [7, 12, 11, 15] for new results on bilinear multipliers and related topics.

In a recent paper (see [7]), Fan and Sato have shown certain de Leeuw-type theorems for transferring multilinear operators on Lebesgue and Hardy spaces from $\mathbb{R}^n$ to $\mathbb{T}^n$. Here we will consider bilinear multipliers on Lebesgue spaces $L^p(\mathbb{R})$ and get a characterization which allows us to transfer not only to the bilinear multipliers on $\mathbb{T}$ but also on $\mathbb{Z}$. Our approach will follow closely the ideas in the original paper by de Leeuw (see [6]) and will
provide an alternative proof of some results in [7], whose proof follows, in the multilinear case, the approach used by Stein and Weiss (see [21, page 260]).

We start by setting up natural analogous versions of bilinear multipliers in the periodic and discrete cases. Let \( m = (m_{k,k'}) \) be a bounded sequence and let \( \hat{m} \) be a periodic function on \( \mathbb{T} \times \mathbb{T} \). Define for \( \theta \in [-1/2,1/2] \),

\[
\mathcal{P}_m(f,g)(\theta) = \sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} \hat{f}(k)\hat{g}(k')m_{k,k'}e^{2\pi i\theta(k+k')}
\]

(1.2)

for functions \( f, g \) defined on \( \mathbb{T} \), and for \( k \in \mathbb{Z} \),

\[
\mathcal{D}_{\hat{m}}(a,b)(k) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} P(t)Q(s)\hat{m}(t,s)e^{2\pi ik(t+s)}\,dt\,ds
\]

(1.3)

for sequences \( a = (a(n))_{n \in \mathbb{Z}} \) and \( b = (b(n))_{n \in \mathbb{Z}} \), where \( P(t) = \sum_{n \in \mathbb{Z}} a(n)e^{2\pi int} \) and \( Q(t) = \sum_{n \in \mathbb{Z}} b(n)e^{2\pi int} \).

Now we say that \( m \) (resp., \( \hat{m} \)) is a bilinear \((p_1, p_2)\)-multiplier on \( \mathbb{Z} \times \mathbb{Z} \) (resp., \( \mathbb{T} \times \mathbb{T} \)) if \( \mathcal{P}_m \) (resp., \( \mathcal{D}_{\hat{m}} \)) defines a bounded bilinear operator from \( L^{p_1}(\mathbb{T}) \times L^{p_2}(\mathbb{T}) \) into \( L^{p_3}(\mathbb{T}) \) (resp., \( \ell^{p_1}(\mathbb{Z}) \times \ell^{p_2}(\mathbb{Z}) \) into \( \ell^{p_3}(\mathbb{Z}) \)), where \( 1/p_1 + 1/p_2 = 1/p_3 \).

Of course we can see these three cases as instances of the general bilinear multiplier acting on different groups. Let \( G \) be a locally compact abelian group and \( \hat{G} \) its dual group with Haar measure \( \mu \). Let \( 1 \leq p_1, p_2 \leq \infty \) and let \( m \) be a bounded measurable function on \( \hat{G} \times \hat{G} \). We say that \( m \) is a \((p_1, p_2)\)-multiplier on \( \hat{G} \times \hat{G} \) if the operator

\[
T_m(f,g)(x) = \int_{\hat{G}} \int_{\hat{G}} \overline{f}(\gamma_1)\overline{f}(\gamma_2)m(\gamma_1,\gamma_2)\gamma_1(-x)\gamma_2(-x)d\mu(\gamma_1)d\mu(\gamma_2)
\]

(1.4)

(defined for simple functions \( f \) and \( g \)) extends to a bounded bilinear operator from \( L^{p_1}(G) \times L^{p_2}(G) \) to \( L^{p_3}(G) \), where \( 1/p_1 + 1/p_2 = 1/p_3 \). The reader is referred to [14] for the general theory in the linear case.

The first transference results on linear multipliers were given by de Leeuw (see [6]). He showed, among other things, that if \( m \) is regulated (all its points are Lebesgue points) and \( m \) is a \( p \)-multiplier on \( \mathbb{R} \), then \( (m(ek))_k \) is a uniformly bounded \( p \)-multiplier for all \( \varepsilon > 0 \) on \( \mathbb{Z} \) (see [21, page 264]) for the converse of this result for continuous multipliers. Transference results of similar nature are presented in [1].

A general transference method was considered by [5] (see also the generalization given by [13]), but we will not consider these approaches in our bilinear generalization in the paper.

In [7], the multilinear version of the continuous result was shown, namely that for any continuous function \( m(\xi, \eta) \), one has that \( m \) is a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \) if and only if \( m(\varepsilon k, \varepsilon k')_{k,k'} \) is a uniformly bounded multiplier on \( \mathbb{Z} \times \mathbb{Z} \) for \( \varepsilon > 0 \). An extension of the result to Lorentz spaces was achieved in [2].
We will first characterize the boundedness of bilinear multipliers on \( \mathbb{R} \times \mathbb{R} \) by the existence of a constant \( K \) such that
\[
\left| \sum_{t \in \mathbb{R}} \sum_{s \in \mathbb{R}} m(t, s) \mu(\{t\}) \nu(\{s\}) \lambda(t + s) \right| \leq K \| \hat{\mu} \|_{B_{p_1}} \| \hat{\nu} \|_{B_{p_2}} \| \hat{\lambda} \|_{B_{p_3}} \tag{1.5}
\]
for all measures \( \mu, \nu, \lambda \) of finite supports.

This allows us to transfer from the continuous \( \mathcal{C}_m \) to the discrete case \( \mathcal{D}_m \) recovering some of the Fan-Sato results in [7].

We also obtain the transference from the continuous case \( \mathcal{C}_m \) to the periodic case \( \mathcal{D}_m \). Our main result establishes that \( m \) is a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \) \( \Longleftrightarrow \) if and only if \( D_{\varepsilon} m = m_{\varepsilon, \varepsilon} \chi_{[-1/2, 1/2] \times [-1/2, 1/2]} \) (extended by periodicity) are uniformly bounded \((p_1, p_2)\)-multipliers on \( \mathbb{T} \times \mathbb{T} \).

The reader should be aware that the results of the paper can be stated for multilinear notions, for instance, for \( m(\xi_1, \ldots, \xi_n) \), one has
\[
\mathcal{C}_m(f_1, \ldots, f_n)(x) = \int_{\mathbb{R}^n} \hat{f}_1(\xi) \cdots \hat{f}_n(\xi_n) m(\xi_1, \ldots, \xi_n) e^{2\pi i x_1 \xi_1 + \cdots + \xi_n} d\xi_1 \cdots d\xi_n, \tag{1.6}
\]
and similar modifications for \( \mathcal{D}_m \) and \( \mathcal{D}_m \). We simply do the bilinear case for the sake of simplicity.

Throughout the paper, \( 1 \leq p_1, p_2, p_3 \leq \infty \) and \( 1/p_3 = 1/p_1 + 1/p_2 \). For a given finite Borel measure \( \mu \) on \( \mathbb{R} \), we write \( \hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-2\pi i t \xi} d\mu(t) \) and, for an almost periodic function \( g \), we denote \( \|g\|_{B_p} = \lim_{T \to \infty} (\|1/2T\| \int_{-T}^{T} |g(t)| \, |dt|)^{1/p} \). We will use the notations \( D_{\varepsilon} m(x, y) = m(\varepsilon x, \varepsilon y) \) and \( \Phi(x) = (1/\varepsilon) \Phi(x/\varepsilon) \).

2. Bilinear multipliers on \( \mathbb{R} \times \mathbb{R} \)

We start by reformulating the condition of \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \) using duality. The proof is straightforward and is left to the reader.

**Lemma 2.1.** Let \( m(\xi, \eta) \) be a bounded measurable function on \( \mathbb{R} \times \mathbb{R} \). Then \( m \) is a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \) if and only if there exists a constant \( K \) so that
\[
\left| \int_{\mathbb{R}^2} \phi(\xi) \psi(\eta) \nu(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| \leq K \| \hat{\phi} \|_{p_1} \| \hat{\psi} \|_{p_2} \| \hat{\nu} \|_{p_3} \tag{2.1}
\]
for all \( \phi, \psi, \eta \in \mathcal{S} \).

Now we present some behavior of multipliers on \( \mathbb{R} \times \mathbb{R} \) with respect to convolution and dilation operators to be used later on.

**Lemma 2.2.** Let \( m(\xi, \eta) \) be a bounded measurable function on \( \mathbb{R} \times \mathbb{R} \). If \( \Phi \in L^1(\mathbb{R}^2) \) and \( m \) is a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \), then \( \Phi \ast m \) is a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \) and \( \|\Phi \ast m\| \leq \|\Phi\|_1 \|m\| \), where \( \|m\| \) stands for the norm of the corresponding bilinear map from \( L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \) into \( L^{p_3}(\mathbb{R}) \).
Proof. Let \( f_s(x) = f(x + s) \) for any \( s \in \mathbb{R} \) and function \( f \). Then for any \( s, t \in \mathbb{R} \) and \( \phi, \psi, \nu \in \mathcal{S} \) with \( \| \hat{\phi} \|_{p_1} = \| \hat{\psi} \|_{p_2} = \| \hat{\nu} \|_{p_3} = 1 \), we have

\[
\left| \int_{\mathbb{R}^2} \phi_s(\xi) \psi_t(\eta) \nu_{t+s}(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| \leq K. \tag{2.2}
\]

Now

\[
\begin{align*}
\int_{\mathbb{R}^2} \phi(\xi) \psi(\eta) \nu(\xi + \eta) \Phi * m(\xi, \eta) d\xi d\eta &= \int_{\mathbb{R}^2} \phi(\xi) \psi(\eta) \nu(\xi + \eta) \left( \int_{\mathbb{R}^2} m(\xi - s, \eta - t) \Phi(s, t) d\xi d\eta \right) d\xi d\eta \tag{2.3} \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(\xi + s) \psi(\eta + t) \nu(\xi + \eta + s + t) m(\xi, \eta) \Phi(s, t) d\xi d\eta ds dt.
\end{align*}
\]

And the result follows by Lemma 2.1. \( \square \)

**Lemma 2.3.** Let \( \epsilon > 0 \) and \( m(\xi, \eta) \) be a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \). Then \( m(\epsilon \xi, \epsilon \eta) \) is also a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \) and \( \| c_{m(\epsilon, \epsilon \cdot)} \| \leq \| c_m \| \).

**Proof.** For \( \phi, \psi, \nu \in \mathcal{S} \) and \( \| \hat{\phi} \|_{p_1} = \| \hat{\psi} \|_{p_2} = \| \hat{\nu} \|_{p_3} = 1 \), we have

\[
\begin{align*}
\int_{\mathbb{R}^2} \phi(\xi) \psi(\eta) \nu(\xi + \eta) m(\epsilon \xi, \epsilon \eta) d\xi d\eta &= \int_{\mathbb{R}^2} \frac{1}{\epsilon^{1/p_1}} \phi\left(\frac{\xi}{\epsilon}\right) \frac{1}{\epsilon^{1/p_2}} \psi\left(\frac{\eta}{\epsilon}\right) \frac{1}{\epsilon^{1/p_3}} \nu\left(\frac{\xi + \eta}{\epsilon}\right) m(\xi, \eta) d\xi d\eta. \tag{2.4}
\end{align*}
\]

The proof is finished invoking Lemma 2.1 again. \( \square \)

**Theorem 2.4.** Let \( m(\xi, \eta) \) be a bounded continuous function on \( \mathbb{R} \times \mathbb{R} \). The following are equivalent:

(i) \( m \) is a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \);

(ii) there exists a constant \( K \) so that

\[
\left| \sum_{t \in \mathbb{R}} \sum_{s \in \mathbb{R}} m(t, s) \mu(\{t\}) \nu(\{s\}) \lambda(\{t + s\}) \right| \leq K \| \hat{\mu} \|_{B_{p_1}} \| \hat{\nu} \|_{B_{p_2}} \| \hat{\lambda} \|_{B_{p_3}} \tag{2.5}
\]

for all measures \( \mu, \nu, \lambda \) supported on a finite number of points.

**Proof.** (i) \( \Rightarrow \) (ii). Assume that \( m \) is a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \). Denote by \( \phi \) the Gaussian function \( \phi(x) = e^{-x^2/2} \). Then for any \( \alpha > 0 \) and \( a \in \mathbb{R} \),

\[
\left( \frac{1}{\epsilon} \right)^{\alpha} \phi^a\left(\frac{\xi - a}{\epsilon}\right) = \delta_a \ast (\phi_\epsilon)^a(\xi). \tag{2.6}
\]
Now choose $0 < \alpha, \beta, \gamma$ such that $\alpha + \beta + \gamma = 2$, and $\mu = \delta_a, \nu = \delta_b, \text{and } \lambda = \delta_c$ for $a, b, c \in \mathbb{R}$. It is easily checked that

$$\int_{\mathbb{R}^2} \frac{1}{\varepsilon^a} \phi_a^{\alpha} \left( \frac{\xi - a}{\varepsilon} \right) \phi_b^{\beta} \left( \frac{\eta - b}{\varepsilon} \right) \phi_\gamma^{\gamma} \left( \frac{\xi + \eta - c}{\varepsilon} \right) m(\xi, \eta) d\xi d\eta$$

$$= \int_{\mathbb{R}^2} \phi_a^{\alpha}(\xi) \phi_b^{\beta}(\eta) \phi_\gamma^{\gamma}(\xi + \eta) m(a + \varepsilon \xi, b + \varepsilon \eta) d\xi d\eta$$

$$= \int_{\mathbb{R}^2} \mu * (\phi_a)^{\alpha}(\xi) \nu * (\phi_b)^{\beta}(\eta) \lambda * (\phi_\gamma)^{\gamma}(\xi + \eta) m(a, b) d\xi d\eta.$$  

Since

$$\lim_{\varepsilon \to 0} \phi_a^{\alpha}(\xi) \phi_b^{\beta}(\eta) \phi_\gamma^{\gamma}(\xi + \eta) m(a + \varepsilon \xi, b + \varepsilon \eta) = \delta_c(a + b) \phi_a^{\alpha}(\xi) \phi_b^{\beta}(\eta) \phi_\gamma^{\gamma}(\xi + \eta)m(a, b),$$

the Lebesgue convergence theorem implies that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \frac{1}{\varepsilon^a} \phi_a^{\alpha}(\xi) \phi_b^{\beta}(\eta) \phi_\gamma^{\gamma}(\xi + \eta) m(\xi, \eta) d\xi d\eta$$

$$= C \mu(a, b) \delta_c(a + b) = C \mu(a, b) \nu(\{a\}) \lambda(\{a + b\}),$$

where $C = \int_{\mathbb{R}^2} \phi_a^{\alpha}(\xi) \phi_b^{\beta}(\eta) \phi_\gamma^{\gamma}(\xi + \eta) d\xi d\eta$.

Therefore we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \mu * (\phi_a)^{\alpha}(\xi) \nu * (\phi_b)^{\beta}(\eta) \lambda * (\phi_\gamma)^{\gamma}(\xi + \eta) m(\xi, \eta) d\xi d\eta$$

$$= C \sum_{t \in \mathbb{R}, s \in \mathbb{R}} \sum m(t, s) \mu(\{t\}) \nu(\{s\}) \lambda(\{t + s\})$$

for all measures $\mu, \nu, \lambda$ having their supports on finite sets of points.

On the other hand, from (i) and Lemma 2.1, we have

$$\left| \int_{\mathbb{R}} \mu * (\phi_a)^{\alpha}(\xi) \nu * (\phi_b)^{\beta}(\eta) \lambda * (\phi_\gamma)^{\gamma}(\xi + \eta) m(\xi, \eta) d\xi d\eta \right|$$

$$\leq K \|\widehat{\mu}(\phi_a)^{\alpha}\|_{p_1} \|\nu(\phi_b)^{\beta}\|_{p_2} \|\lambda(\phi_\gamma)^{\gamma}\|_{p_3}.$$  

We now choose $\alpha = 1/p_1, \beta = 1/p_2$, and $\gamma = 1/p_3$. Since $(\phi_a)^{\alpha} = \varepsilon^{1-\alpha} / \alpha^{1/2} \phi_{\alpha a - 1/2}$, we get

$$\widehat{(\phi_a)^{\alpha}}(\xi) = C_a \varepsilon^{1/p_1} e^{-\xi^{\gamma}/2a},$$

$$\widehat{(\phi_b)^{\beta}}(\xi) = C_b \varepsilon^{1/p_2} e^{-\xi^{\gamma}/2b},$$

and $(\phi_\gamma)^{\gamma}(\xi) = C_\gamma \varepsilon^{1/p_3} e^{-\xi^{\gamma}/2\gamma}$ for some constants $C_a, C_b$, and $C_\gamma$.

Now taking into account that $\int_{\mathbb{R}} \varepsilon^{-p_1 \xi^{\gamma}/2a} d\xi = C_a e^{-1}$, we have

$$\|\widehat{\mu}(\phi_a)^{\alpha}\|_{p_1} = C \left( \frac{1}{A(\varepsilon)} \int_{\mathbb{R}} \|\widehat{\mu}(\xi)\|_{p_1} \varepsilon^{-p_1 \xi^{\gamma}/2a} d\xi \right)^{1/p_1}.$$  

for $A(\varepsilon) = \int_{\mathbb{R}} \varepsilon^{-p_1 \xi^{\gamma}/2a} d\xi$. Hence $C \|\widehat{\mu}\|_{p_1} = \lim_{\varepsilon \to 0} \|\widehat{\phi_a^{\alpha}}\|_{p_1}$.

Applying a similar procedure for $\nu$ and $\lambda$, we finish this implication.
(ii) ⇒ (i). From (ii) we can get that the inequality holds for all finite measures \( \mu, \nu, \lambda \), with countable supports. We take \( \phi, \psi, \) and \( \rho \) such that \( \hat{\phi}, \hat{\psi}, \hat{\rho} \) have compact support contained in \([-N/2, N/2]\) for \( N \) big enough. Now consider \( \mu_N, \nu_N, \lambda_N \) the measures with support in \((1/N)\mathbb{Z}\) whose Fourier transform coincides with the periodic extensions of \( \hat{\phi}, \hat{\psi}, \hat{\rho} \). In particular, we have

\[
\mu_N \left( \left\{ \frac{n}{N} \right\} \right) = \frac{1}{N} \phi \left( \frac{n}{N} \right), \quad \nu_N \left( \left\{ \frac{n}{N} \right\} \right) = \frac{1}{N} \psi \left( \frac{n}{N} \right), \quad \lambda_N \left( \left\{ \frac{n}{N} \right\} \right) = \frac{1}{N} \rho \left( \frac{n}{N} \right).
\]

Therefore we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{(t,s) \in \mathbb{R} \times \mathbb{R}} m(t,s) \mu_N (\{t\}) \nu_N (\{s\}) \lambda_N (\{t + s\}) = \lim_{N \to \infty} \frac{1}{N} \sum_{(n,m) \in \mathbb{Z} \times \mathbb{Z}} m \left( \frac{n}{N}, \frac{m}{N} \right) \phi \left( \frac{n}{N} \right) \psi \left( \frac{m}{N} \right) \rho \left( \frac{n + m}{N} \right) \frac{1}{N^2}
\]

\[
= \int_{\mathbb{R}^2} m(\xi, \nu) \phi(\xi) \psi(\eta) \rho(\xi + \eta) d\xi d\eta.
\]

Now observe that \( \|\hat{\mu}_N\|_{B_{p_1}} = ((1/2N) \int_{-N}^{N} |\hat{\phi}(\xi)|^{p_1} d\xi)^{1/p_1} = (1/2N)^{1/p_1} \|\hat{\phi}\|_{p_1} \) and the same for the others.

Using that \( \|\hat{\mu}_N\|_{B_{p_1}} \cdot \|\hat{\nu}_N\|_{B_{p_2}} \|\hat{\lambda}_N\|_{B'_{p_3}} = 1/2N \) and passing to the limit, we get the result.

\[\square\]

**Remark 2.5.** We point out that condition (ii) in Theorem 2.4 is simply a way to say that \( m \) defines a multiplier on \( \mathbb{D} \times \mathbb{D} \) where \( \mathbb{D} \) is the group \( \mathbb{R} \) with the discrete topology (see [6]).

Recall that a function \( m \) is called regulated if

\[
\lim_{\varepsilon \to 0} \frac{1}{4\varepsilon^2} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} m(x - s, y - t) ds dt = m(x, y)
\]

for all \( (x, y) \in \mathbb{R}^2 \).

**Theorem 2.6.** Let \( m(\xi, \eta) \) be a bounded regulated function on \( \mathbb{R} \times \mathbb{R} \). Then \( m \) is a \( (p_1, p_2) \)-multiplier on \( \mathbb{R} \times \mathbb{R} \) if and only if there exists a constant \( K \) so that

\[
\left| \sum_{t \in \mathbb{R}} \sum_{s \in \mathbb{R}} m(t, s) \mu(\{t\}) \nu(\{s\}) \lambda(\{t + s\}) \right| \leq K \|\hat{\mu}\|_{B_{p_1}} \|\hat{\nu}\|_{B_{p_2}} \|\hat{\lambda}\|_{B'_{p_3}}
\]

for all measures \( \mu, \nu, \lambda \) having their supports on finite sets of points.

**Proof.** Assume that \( m \) is a \( (p_1, p_2) \)-multiplier. Let \( \Phi(s, t) = (1/4)\chi_{[-1, 1]}(s)\chi_{[-1, 1]}(t) \) and \( \Phi_\varepsilon(\xi, \eta) = (1/\varepsilon^2)\Phi(\xi/\varepsilon, \eta/\varepsilon) \) for \( \varepsilon > 0 \). Now Lemma 2.2, Theorem 2.4, and the fact that \( m(x, y) = \lim_{\varepsilon \to 0} m * \Phi_\varepsilon(x, y) \) give the direct implication.
Conversely, assume (2.16) for \( \mu, \nu, \lambda \) having finite supports. Then

\[
\begin{align*}
\sum_{t \in \mathbb{R}} \sum_{s \in \mathbb{R}} (m \ast \Phi_\epsilon)(t, s) & \mu(\{t\}) \nu(\{s\}) \lambda(\{t + s\}) \\
= & \int_{\mathbb{R}^2} \left( \sum_{t \in \mathbb{R}} \sum_{s \in \mathbb{R}} m(t - u, s - v) \mu(\{t\}) \nu(\{s\}) \lambda(\{t + s\}) \right) \Phi_\epsilon(u, v) \, du \, dv \\
= & \int_{\mathbb{R}^2} \left( \sum_{t \in \mathbb{R}} \sum_{s \in \mathbb{R}} m(t, s) \mu(\{t + u\}) \nu(\{s + v\}) \lambda(\{t + s + u + v\}) \right) \Phi_\epsilon(u, v) \, du \, dv.
\end{align*}
\]

(2.17)

This shows that \( m \ast \Phi_\epsilon \) verifies (2.16) with a uniform constant for all \( \epsilon > 0 \). Now apply Theorem 2.4 to get that \( m \ast \Phi_\epsilon \) are \((p_1, p_2)\)-multipliers with uniform norm.

Finally we have that for \( \phi, \psi, \gamma \in \mathcal{S}' \),

\[
\begin{align*}
\left| \int_{\mathbb{R}^2} \phi(\xi) \psi(\eta) \nu(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| & = \left| \lim_{\epsilon \to 0} \int_{\mathbb{R}^2} \phi(\xi) \psi(\eta) \nu(\xi + \eta) (m \ast \Phi_\epsilon)(\xi, \eta) d\xi d\eta \right| \\
& \leq C \| \phi \|_{p_1} \| \psi \|_{p_2} \| \nu \|_{p_3}.
\end{align*}
\]

(2.18)

The result now follows from Lemma 2.1. \( \square \)

### 3. Transference theorems

We mention the formulations for \((p_1, p_2)\)-multipliers on the groups \( \mathbb{T} \) and \( \mathbb{Z} \) which follow directly from duality.

**Lemma 3.1.** Let \( \hat{m}(t, s) \) be a bounded measurable function on \( \mathbb{T} \times \mathbb{T} \). Then \( m \) is a \((p_1, p_2)\)-multiplier on \( \mathbb{T} \times \mathbb{T} \) if and only if there exists a constant \( K \) so that

\[
\left| \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} P_n(t) P_b(s) P_c(t + s) \hat{m}(t, s) \, dt \, ds \right| \leq K \| a \|_{p_1} \| b \|_{p_2} \| c \|_{p_3}
\]

(3.1)

for all finite sequences \( (a(n))_n, (b(n))_n, (c(n))_n \), where \( P_a(t) = \sum_n a(n) e^{2\pi i n t} \).

**Lemma 3.2.** Let \( (m_{k,k'}) \) be a bounded sequence on \( \mathbb{Z} \times \mathbb{Z} \). Then \( m \) is a \((p_1, p_2)\)-multiplier on \( \mathbb{Z} \times \mathbb{Z} \) if and only if there exists a constant \( K \) so that

\[
\left| \sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} m_{k,k'} \hat{P}(k) \hat{Q}(k') \hat{R}(k + k') \right| \leq K \| P \|_{p_1} \| Q \|_{p_2} \| R \|_{p_3}
\]

(3.2)

for all trigonometric polynomials \( P, Q, \) and \( R \).

**Theorem 3.3** (see [7, Theorem 1]). Let \( m(\xi, \eta) \) be a regulated bounded function on \( \mathbb{R} \times \mathbb{R} \). If \( m(\xi, \eta) \) is a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \), then \( (m(k, k'))_{k,k'} \) is a \((p_1, p_2)\)-multiplier on \( \mathbb{Z} \times \mathbb{Z} \).
Proof. According to Lemma 3.2, we have to show that there exists a constant $K$ so that
\[
\left| \sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} m(k,k') \hat{P}(k) \hat{Q}(k') \hat{R}(k+k') \right| \leq K \|P\|_{p_1} \|Q\|_{p_2} \|R\|_{p_3} \tag{3.3}
\]
for all trigonometric polynomials $P$, $Q$, and $R$.

This follows by selecting the measures $\mu$, $\nu$, $\lambda$ in Theorem 2.6 such that $\hat{\mu} = P$, $\hat{\nu} = Q$, and $\hat{\lambda} = R$. \hfill \Box

Theorem 3.4. Let $m(\xi,\eta)$ be a bounded regulated function on $\mathbb{R} \times \mathbb{R}$. The following are equivalent:

(i) $m(\xi,\eta)$ is a $(p_1,p_2)$-multiplier on $\mathbb{R} \times \mathbb{R}$;

(ii) $m(e^i \cdot, e^i \cdot) \chi_{[-1/2e,1/2e]}$ (extended by periodicity) are uniformly bounded $(p_1,p_2)$-multipliers on $\mathbb{T} \times \mathbb{T}$.

Proof. (i) \Rightarrow (ii). Using Lemma 3.1, it suffices to show that for any finite sequences $(a(n))_n$, $(b(n))_n$, and $(c(n))_n$ with $\|a\|_{p_1} = \|b\|_{p_2} = \|c\|_{p_3} = 1$, there exists a constant $K > 0$ such that
\[
\left| \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} m(\xi,\eta) P_a(\xi) P_b(\eta) P_c(\xi+\eta) d\xi d\eta \right| \leq K, \tag{3.4}
\]
where $P_a(\xi) = \sum_n a(n) e^{2\pi i n \xi}$.

Since $P_a(x) \chi_{[-1/2,1/2]}(x) = \hat{\phi}_a(x)$, where $\phi_a(x) = \sum_n a(n) (\sin(\pi(x-n))/\pi(x-n))$, and $P_c(x) \chi_{[-1,1]}(x) = \hat{\psi}_c(x)$, where $\psi_c(x) = \sum_n c(n) (\sin(2\pi(x-n))/\pi(x-n))$, we can write
\[
\int_{-1/2}^{1/2} \int_{-1/2}^{1/2} m(\xi,\eta) P_a(\xi) P_b(\eta) P_c(\xi+\eta) d\xi d\eta = \int_{\mathbb{R}} \int_{\mathbb{R}} m(\xi,\eta) \hat{\phi}_a(\xi) \hat{\phi}_b(\eta) \hat{\psi}_c(\xi+\eta) d\xi d\eta. \tag{3.5}
\]

Using now the assumption and Shannon’s sampling theorem, one gets $\|\psi_a\|_{L^p(\mathbb{R})} \leq C_1 \|\phi_a\|_{L^p(\mathbb{R})} \leq C_2 \|a\|_{p_3} \leq C_3 \|a\|_{L^p(\mathbb{R})}$ for some constants $C_i$ for $i = 1, 2, 3$. Hence the desired inequality follows.

Now we apply Lemma 2.3 to get the result for each $\epsilon$.

(ii) \Rightarrow (i). We take $\phi$ and $\psi$ such that $\text{supp} \phi$ and $\text{supp} \psi$ are contained in $[-1/4,1/4]$. For a fixed $u \in [-1/2,1/2]$, consider the periodic extensions of the functions $\hat{\phi}(\xi) e^{2\pi i u \xi}$, $\hat{\psi}(\eta) e^{2\pi i u \eta}$ to be denoted $\tilde{P}_u$ and $\tilde{Q}_u$, respectively.

If $a^u(n) = \int_{-1/2}^{1/2} \tilde{P}_u(\xi) e^{-2\pi n \xi} d\xi$, $b^u(n) = \int_{-1/2}^{1/2} \tilde{Q}_u(\xi) e^{-2\pi n \xi} d\xi$ for all $n \in \mathbb{Z}$, we have that if $x = k + u$ for some $k \in \mathbb{Z}$ and $u \in [-1/2,1/2]$,
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} m(\xi,\eta) \tilde{\phi}(\xi) \tilde{\psi}(\eta) e^{2\pi i (x+u)} d\xi d\eta = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} m(\xi,\eta) \tilde{P}_u(\xi) \tilde{Q}_u(\eta) e^{2\pi i (x+u)} d\xi d\eta. \tag{3.6}
\]

Let $\tilde{m}(\xi,\eta) = m(\xi,\eta) \chi_{[-1/2,1/2]}(\xi) \chi_{[-1/2,1/2]}(\eta)$. Hence for $x = u + k$,
\[
\langle E_m(\phi,\psi)(x) = \mathcal{D}_m(a^u, b^u)(k). \tag{3.7}
\]
Now
\[ \int_{\mathbb{R}} |\mathcal{C}_m(\phi, \psi)(x)|^{p_3} dx \]
\[ = \sum_{k} \int_{-1/2}^{1/2} |\mathcal{C}_m(\phi, \psi)(k + u)|^{p_3} du \]
\[ = \int_{-1/2}^{1/2} \sum_{k} |\mathcal{D}_m(a^u, b^v)(k)|^{p_3} du \]
\[ \leq \left\| \mathcal{D}_m \right\|^{p_3} \left( \int_{-1/2}^{1/2} \left( \sum_{k} |a^u(k)|^{p_1} \right)^{p_{3/p_1}} \left( \sum_{k} |b^v(k)|^{p_2} \right)^{p_{3/p_2}} du \right)^{1/2} \]
\[ \leq \left\| \mathcal{D}_m \right\|^{p_3} \left( \int_{-1/2}^{1/2} \sum_{k} |a^u(k)|^{p_1} du \right)^{p_{3/p_1}} \left( \int_{-1/2}^{1/2} \sum_{k} |b^v(k)|^{p_2} du \right)^{p_{3/p_2}} \]
\[ = \left\| \mathcal{D}_m \right\|^{p_3} \left\| \phi \right\|^{p_1}_{p_1} \left\| \psi \right\|^{p_2}_{p_2}. \]  

(3.8)

In the general case if \( \phi, \psi \) are such that \( \hat{\phi}, \hat{\psi} \) have compact support, then there exists \( \varepsilon > 0 \) so that \( \hat{\phi}_\varepsilon, \hat{\psi}_\varepsilon \) have their support in \([-1/4, 1/4]\). Now observe that
\[ \mathcal{C}_m(\phi, \psi)(x) = \varepsilon^2 C_{m(\varepsilon, x)}(\hat{\phi}_\varepsilon, \hat{\psi}_\varepsilon)(\varepsilon x). \]

(3.9)

Applying the previous case and the assumption, we obtain
\[ \left\| \mathcal{C}_m(\phi, \psi) \right\|_{p_3} = \varepsilon^{2-1/p_1} \left\| C_{m(\varepsilon, x)}(\hat{\phi}_\varepsilon, \hat{\psi}_\varepsilon) \right\|_{p_3} \]
\[ \leq K \varepsilon^{2-1/p} \left\| \phi \right\|_{p_1} \left\| \psi \right\|_{p_2} \]
\[ = K \varepsilon^{2-1/p} \left\| \phi \right\|_{p_1} \varepsilon^{-1/p_1} \left\| \psi \right\|_{p_1} \varepsilon^{-1/p_2} \]
\[ = K \left\| \phi \right\|_{p_1} \left\| \psi \right\|_{p_1}. \]  

(3.10)

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References


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