A double inequality involving the constant $e$ is proved by using an inequality between the logarithmic mean and arithmetic mean. As an application, we generalize the weighted Carleman-type inequality.

1. Introduction

Let $p > 1$ and $a_n \geq 0$ with $0 < \sum_{n=1}^{\infty} a_n^p < \infty$. Then

$$\sum_{n=1}^{\infty} \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)^p < \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p. \quad (1.1)$$

The constant $\left(\frac{p}{p-1}\right)^p$ is the best possible.

Inequality (1.1) is due to Hardy [6, page 239].

Replacing $a_n$ in (1.1) by $a_n^{1/p}$ for $n \in \mathbb{N}$, we obtain

$$\sum_{n=1}^{\infty} \left( \frac{a_1^{1/p} + a_2^{1/p} + \cdots + a_n^{1/p}}{n} \right)^p < \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n. \quad (1.2)$$

In (1.2), letting $p \to \infty$, then the following Carleman inequality [6, page 249] is deduced:

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n, \quad (1.3)$$

where $a_n \geq 0$ for $n \in \mathbb{N}$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$. The constant $e$ is the best possible.

Carleman's inequality (1.3) was generalized in [6, page 256] by Hardy as follows. Let $a_n \geq 0$, $\lambda_n > 0$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$ for $n \in \mathbb{N}$, and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, then

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n a_n. \quad (1.4)$$
Note that inequality (1.4) is usually referred to as a Carleman-type inequality or weighted Carleman-type inequality. In his original paper [5], Hardy himself said that it was Pólya who pointed out this inequality to him.

In several recent papers [2, 4, 11, 12, 13, 14, 15], some strengthened and generalized results of (1.3) and (1.4) have been given by estimating the weight coefficient \((1 + \frac{1}{n})^n\).

For information about the history of both Hardy’s inequality and Carleman-type inequalities, please refer to [7, 9].

In this note, we will give a generalization of (1.4) as follows.

**Theorem 1.1.** Let \(0 < \lambda_{n+1} \leq \lambda_n\) with \(\Lambda = \sum_{m=1}^{n} \lambda_m \geq 1\) and \(\lim_{n \to \infty} \Lambda_n = \infty\), and let \(a_n \geq 0\) for \(n \in \mathbb{N}\) satisfying \(0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty\). Then for \(0 < p \leq 1\),

\[
\sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \right)^{1/\Lambda_n} \leq \frac{1}{p} \sum_{n=1}^{\infty} \left[ \left( 1 + \frac{1}{\Lambda_n/\lambda_n} \right)^{p\Lambda_n/\lambda_n} \lambda_n a_n^{p-1} \left( \sum_{k=1}^{n} \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \right],
\]

in particular,

\[
\sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \right)^{1/\Lambda_n} < e^p \sum_{n=1}^{\infty} \left[ \left( 1 - \frac{1 - 2/e}{\Lambda_n/\lambda_n} \right)^{p\Lambda_n/\lambda_n} \lambda_n a_n^{p-1} \left( \sum_{k=1}^{n} \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \right],
\]

where

\[
c_k = \frac{(\Lambda_{k+1})^{\Lambda_k}}{(\Lambda_k)^{\Lambda_{k-1}}}.
\]

**Remark 1.2.** In particular, taking in (1.6) \(p = 1\), we obtain the following strengthened Hardy’s inequality:

\[
\sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \right)^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \left( 1 - \frac{1 - 2/e}{\Lambda_n/\lambda_n} \right)^{\lambda_n a_n}.
\]

Taking in (1.8) \(\lambda_n \equiv 1\), we obtain the following strengthened Carleman’s inequality:

\[
\sum_{n=1}^{\infty} \left( a_1 a_2 \cdots a_n \right)^{1/n} < e \sum_{n=1}^{\infty} \left( 1 - \frac{1 - 2/e}{n} \right)^{a_n}.
\]

**2. Lemma**

The well-known arithmetic mean \(A(a,b)\) and logarithmic mean \(L(a,b)\) of two positive numbers \(a\) and \(b\) are defined, respectively, for \(a = b\) by \(A(a,b) = L(a,b) = a\) and for \(a \neq b\)
by
\[ A(a,b) = \frac{a + b}{2}, \quad L(a,b) = \frac{b - a}{\ln b - \ln a}. \] (2.1)

For \( a \neq b \), we have
\[ L(a,b) < A(a,b). \] (2.2)

See [1] and the references therein.

**Lemma 2.1.** Let \( x \geq 1 \) be a real number. Then
\[ e \left(1 - \frac{1/2}{x}\right) < \left(1 + \frac{1}{x}\right)^x \leq e \left(1 - \frac{1 - 2/e}{x}\right). \] (2.3)

The constants \( 1/2 \) and \( 1 - 2/e \) are best possible.

**Proof.** Inequality (2.3) is equivalent to
\[ 1 - \frac{2}{e} \leq x \left[1 - \frac{1}{e} \left(1 + \frac{1}{x}\right)^x\right] < \frac{1}{2}. \] (2.4)

Define a function \( f \) for \( x > 0 \) by
\[ f(x) = x \left[1 - \frac{1}{e} \left(1 + \frac{1}{x}\right)^x\right]. \] (2.5)

In order to prove (2.4), it is sufficient to show that the function \( f \) is strictly increasing on \([1, \infty)\) and with
\[ f(1) = 1 - \frac{2}{e}, \quad \lim_{x \to \infty} f(x) = \frac{1}{2}. \] (2.6)

The following proof shows that in fact \( f'(x) > 0 \) holds on \((0, \infty)\).

Easy computation yields
\[ e f'(x) = e - \left[1 + x g(x)\right] \left(1 + \frac{1}{x}\right)^x, \] (2.7)

where
\[ g(x) = \ln \left(1 + \frac{1}{x}\right) - \frac{1}{x+1} = \frac{1}{L(x,x+1)} - \frac{1}{x+1}. \] (2.8)

Now we are in a position to prove \( f'(x) > 0 \), which is equivalent to
\[ h(x) = \left[1 + x g(x)\right] \left(1 + \frac{1}{x}\right)^x < e. \] (2.9)
Differentiation yields

\[ h'(x) = \left[ xg^2(x) + 2g(x) - \frac{1}{(x+1)^2} \right] \left( 1 + \frac{1}{x} \right)^x. \tag{2.10} \]

In the following we show \( h'(x) > 0 \). Clearly, the equation

\[ xt^2 + 2t - \frac{1}{(x+1)^2} = 0 \tag{2.11} \]

has two roots

\[ t_{1,2} = \frac{-(x+1) \pm \sqrt{(x+1)^2 + x}}{x(x+1)}. \tag{2.12} \]

To prove \( h'(x) > 0 \), it is sufficient to show that

\[ \frac{-(x+1) + \sqrt{(x+1)^2 + x}}{x(x+1)} = t_2 < g(x) = \frac{1}{L(x,x+1)} - \frac{1}{x+1}, \tag{2.13} \]

which is equivalent to

\[ \frac{\sqrt{(x+1)^2 + x} - 1}{x(x+1)} < \frac{1}{L(x,x+1)}. \tag{2.14} \]

Inequality (2.14) holds based on the following fact:

\[ \frac{\sqrt{(x+1)^2 + x} - 1}{x(x+1)} < \frac{2}{2x+1} = \frac{1}{A(x,x+1)} < \frac{1}{L(x,x+1)}. \tag{2.15} \]

Hence, the function \( h \) is increasing on \((0, \infty)\), and then \( h(x) < \lim_{x \to \infty} h(x) = e \). This means \( f'(x) > 0 \), and then

\[ 1 - \frac{2}{e} = f(1) < \lim_{x \to \infty} f(x). \tag{2.16} \]

Using Maclaurin formula

\[ (1 + t)^{1/t} = e - \frac{e}{2} t + o(t), \tag{2.17} \]

we have

\[ \lim_{n \to \infty} f(n) = \lim_{x \to \infty} f(x) = \lim_{t \to 0^+} f \left( \frac{1}{t} \right) = \lim_{t \to 0^+} \left( \frac{et/2 + o(t)}{et} \right) = \frac{1}{2}. \tag{2.18} \]

The proof of Lemma 2.1 is complete. \( \square \)

**Remark 2.2.** There are other very sharp estimates of the crucial factor \((1 + 1/n)^n\) in [8] and the references therein.
3. Proof of Theorem 1.1

By the power mean inequality, we have

\[ \prod_{m=1}^{n} a_{m}^{\alpha_{m}} \leq \left( \sum_{m=1}^{n} q_{m} a_{m}^{\alpha_{m}} \right)^{1/p}, \tag{3.1} \]

where \( p \geq 0 \), \( \alpha_{m} \geq 0 \), and \( q_{m} > 0 \) for \( m \in \mathbb{N} \) with \( \sum_{m=1}^{n} q_{m} = 1 \).

Let \( c_{m} > 0 \), \( \alpha_{m} = c_{m} a_{m} \), and \( q_{m} = \lambda_{m}/\Lambda_{m} \), then we obtain

\[ (c_{1} a_{1})^{\lambda_{1}/\Lambda_{n}} (c_{2} a_{2})^{\lambda_{2}/\Lambda_{n}} \cdots (c_{n} a_{n})^{\lambda_{n}/\Lambda_{n}} \leq \left( \frac{1}{\Lambda_{n}} \sum_{m=1}^{n} \lambda_{m} (c_{m} a_{m})^{p} \right)^{1/p}. \tag{3.2} \]

Further, we have

\[ \sum_{n=1}^{\infty} \lambda_{n+1} \left( a_{1}^{\lambda_{1}} a_{2}^{\lambda_{2}} \cdots a_{n}^{\lambda_{n}} \right)^{1/\Lambda_{n}} \leq \sum_{n=1}^{\infty} \frac{\lambda_{n+1}}{\left( c_{1}^{\lambda_{1}} c_{2}^{\lambda_{2}} \cdots c_{n}^{\lambda_{n}} \right)^{1/\Lambda_{n}}} \left( \frac{1}{\Lambda_{n}} \sum_{m=1}^{n} \lambda_{m} (c_{m} a_{m})^{p} \right)^{1/p}. \tag{3.3} \]

By the following inequality (see \([3, 10] \))

\[ \left( \sum_{m=1}^{n} z_{m} \right)^{t} \leq t \sum_{m=1}^{n} z_{m} \left( \sum_{k=1}^{m} z_{k} \right)^{t-1}, \tag{3.4} \]

where \( t \geq 1 \) is constant and \( z_{m} \geq 0 \) for \( m \in \mathbb{N} \), it is easy to see that

\[ \left( \frac{1}{\Lambda_{n}} \sum_{m=1}^{n} \lambda_{m} (c_{m} a_{m})^{p} \right)^{1/p} \leq \frac{1}{\Lambda_{n}} \left( \sum_{m=1}^{n} \lambda_{m} (c_{m} a_{m})^{p} \right)^{1/p} \leq \frac{1}{p \Lambda_{n}} \sum_{m=1}^{n} \lambda_{m} (c_{m} a_{m})^{p} \left( \sum_{k=1}^{m} \lambda_{k} (c_{k} a_{k})^{p} \right)^{(1-1/p)/p}, \tag{3.5} \]

where \( \Lambda_{n} \geq 1 \) and \( 0 < p \leq 1 \). Thus, we obtain from (3.3) and (3.5) that

\[ \sum_{n=1}^{\infty} \lambda_{n+1} \left( a_{1}^{\lambda_{1}} a_{2}^{\lambda_{2}} \cdots a_{n}^{\lambda_{n}} \right)^{1/\Lambda_{n}} \leq \frac{1}{p \Lambda_{n}} \sum_{n=1}^{\infty} \frac{\lambda_{n+1}}{\left( c_{1}^{\lambda_{1}} c_{2}^{\lambda_{2}} \cdots c_{n}^{\lambda_{n}} \right)^{1/\Lambda_{n}}} \sum_{m=1}^{n} \lambda_{m} (c_{m} a_{m})^{p} \left( \sum_{k=1}^{m} \lambda_{k} (c_{k} a_{k})^{p} \right)^{(1-1/p)/p} \tag{3.6} \]

\[ = \frac{1}{p} \sum_{m=1}^{\infty} \lambda_{m} (c_{m} a_{m})^{p} \sum_{n=m}^{\infty} \left( \frac{\lambda_{n+1}}{\Lambda_{n} \left( c_{1}^{\lambda_{1}} c_{2}^{\lambda_{2}} \cdots c_{n}^{\lambda_{n}} \right)^{1/\Lambda_{n}}} \right) \left( \sum_{k=1}^{m} \lambda_{k} (c_{k} a_{k})^{p} \right)^{(1-1/p)/p}. \]
Choosing $c_1^\lambda_1 c_2^\lambda_2 \cdots c_n^\lambda_n = (\Lambda_{n+1})^{\lambda_n}$ for $n \in \mathbb{N}$ and setting $\Lambda_0 = 0$, we get from $0 < \lambda_{n+1} \leq \lambda_n$ that

$$c_n = \left[ \frac{(\Lambda_{n+1})^{\lambda_n}}{(\Lambda_{n})^{\lambda_{n-1}}} \right]^{1/\lambda_n} = \left( 1 + \frac{\lambda_{n+1}}{\Lambda_{n}} \right)^{\Lambda_n/\lambda_n} \Lambda_n \leq \left( 1 + \frac{\lambda_n}{\Lambda_{n}} \right)^{\Lambda_n/\lambda_n} \Lambda_n. \quad (3.7)$$

This implies that

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \leq \frac{1}{p} \sum_{m=1}^{\infty} \lambda_m (c_m a_m)^p \sum_{n=m}^{\infty} \frac{\lambda_{n+1}}{\Lambda_n} \left( \sum_{k=1}^{m} \lambda_k (c_k a_k)^p \right)^{(1-p)/p}$$

$$= \frac{1}{p} \sum_{m=1}^{\infty} \lambda_m (c_m a_m)^p \sum_{n=m}^{\infty} \left( \frac{1}{\Lambda_n} - \frac{1}{\Lambda_{n+1}} \right) \left( \sum_{k=1}^{m} \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \quad (3.8)$$

$$= \frac{1}{p} \sum_{m=1}^{\infty} \lambda_m (c_m a_m)^p \frac{1}{\Lambda_m} \left( \sum_{k=1}^{m} \lambda_k (c_k a_k)^p \right)^{(1-p)/p}$$

$$\leq \frac{1}{p} \sum_{m=1}^{\infty} \left( 1 + \frac{1}{\Lambda_m/\Lambda_{m+1}} \right)^{p/\lambda_m} \lambda_m a_m^p \Lambda_m^{p-1} \left( \sum_{k=1}^{m} \lambda_k (c_k a_k)^p \right)^{(1-p)/p}. \quad (3.9)$$

Hence, we obtain from the above inequality and Lemma 2.1 that

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < \frac{e^p}{p} \sum_{n=1}^{\infty} \left( 1 - \frac{1-2/e}{\Lambda_n/\Lambda_{n+1}} \right)^{p} \lambda_n a_n^p \Lambda_n^{p-1} \left( \sum_{k=1}^{n} \lambda_k (c_k a_k)^p \right)^{(1-p)/p}. \quad (3.9)$$

The last inequality holds strictly since the right-hand inequality of (2.3) is valid if and only if $n = 1$. The proof is complete.

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