Some of the properties of the completely regular fuzzifying topological spaces are investigated. It is shown that a fuzzifying topology $\tau$ is completely regular if and only if it is induced by some fuzzy uniformity or equivalently by some fuzzifying proximity. Also, $\tau$ is completely regular if and only if it is generated by a family of probabilistic pseudometrics.

1. Introduction

The concept of a fuzzifying topology was given in [1] under the name $L$-fuzzy topology. Ying studied in [9, 10, 11] the fuzzifying topologies in the case of $L = [0, 1]$. A classical topology is a special case of a fuzzifying topology. In a fuzzifying topology $\tau$ on a set $X$, every subset $A$ of $X$ has a degree $\tau(A)$ of belonging to $\tau$, $0 \leq \tau(A) \leq 1$. In [4], we defined the degrees of compactness, of local compactness, Hausdorffness, and so forth in a fuzzifying topological space $(X, \tau)$. We also introduced the fuzzifying proximities. Every fuzzifying proximity $\delta$ induces a fuzzifying topology $\tau_\delta$. In [6], we studied the level classical topologies $\tau^\theta$, $0 \leq \theta < 1$, corresponding to a fuzzifying topology $\tau$. In the same paper, we studied connectedness and local connectedness in fuzzifying topological spaces as well as the so-called sequential fuzzifying topologies. In [5], we introduced the fuzzifying syntopogenous structures. We also proved that every fuzzy uniformity $\mathcal{U}$, as it is defined by Lowen in [7], induces a fuzzifying proximity $\delta_\mathcal{U}$, and that for every fuzzifying proximity $\delta$, there exists at least one fuzzy uniformity $\mathcal{U}$ with $\delta = \delta_\mathcal{U}$. Some of the results contained in papers [4, 6] are closely related to those which appeared in the papers [12, 13].

In this paper, we continue with the investigation of fuzzifying topologies. In particular, we study the completely regular fuzzifying topologies, that is, those fuzzifying topologies $\tau$ for which each level topology $\tau^\theta$ is completely regular. As in the classical case, we prove that for a fuzzifying topology $\tau$ on $X$, the following properties are equivalent: (1) $\tau$ is completely regular; (2) $\tau$ is uniformizable, that is, it is induced by some fuzzy uniformity; (3) $\tau$ is proximizable, that is, it is induced by some fuzzifying proximity; and (4) $\tau$ is generated by a family of so-called probabilistic pseudometrics on $X$. We also give a characterization of completely regular fuzzifying spaces in terms of continuous functions. Many Theorems on classical topologies follow as special cases of results obtained in the paper.
2. Preliminaries

A fuzzifying topology on a set $X$ (see [1, 9, 10, 11]) is a map $\tau: 2^X \to [0, 1]$ (where $2^X$ is the power set of $X$) satisfying the following conditions:

- (FT1) $\tau(X) = \tau(\emptyset) = 1$;
- (FT2) $\tau(A_1 \cap A_2) \geq \tau(A_1) \wedge \tau(A_2)$;
- (FT3) $\tau(\bigcup A_i) \geq \inf_i \tau(A_i)$.

If $\tau$ is a fuzzifying topology on $X$ and $x \in X$, then the $\tau$-neighborhood system of $x$ is the function

$$N_x = N_x^\tau: 2^X \to [0, 1], \quad N_x(A) = \sup \{ \tau(B) : x \in B \subseteq A \}. \quad (2.1)$$

By [9, Theorem 3.2], we have that $\tau(A) = \inf_{x \in A} N_x(A)$.

The following theorem is contained in [9] (see also [3, 13]).

**Theorem 2.1.** If $\tau$ is a fuzzifying topology on a set $X$, then the map $x \to N_x = N_x^\tau$, from $X$ to the fuzzy power set $F(2^X)$ of $2^X$, has the following properties:

- (FN1) $N_x(X) = 1$ and $N_x(A) = 0$ if $x \notin A$;
- (FN2) $N_x(A_1 \cap A_2) = N_x(A_1) \wedge N_x(A_2)$;
- (FN3) $N_x(A) = \sup_{x \in D \subseteq A} \inf_{y \in D} N_y(D)$.

Conversely, if a map $x \to N_x$, from $X$ to $F(2^X)$, satisfies (FN1)–(FN3), then the map

$$\tau: 2^X \to [0, 1], \quad \tau(A) = \inf_{x \in A} N_x(A), \quad (2.2)$$

is a fuzzifying topology and $N_x = N_x^\tau$ for every $x \in X$.

Let now $(X, \tau)$ be a fuzzifying topological space. To every subset $A$ of $X$ corrsponds a fuzzy subset $\bar{A} = \bar{A}^\tau$ of $X$ defined by $\bar{A}(x) = 1 - N_x(A^c)$ (see [13, Remark 3.16]). A function $f$, from a fuzzifying topological space $(X, \tau_1)$ to another one $(Y, \tau_2)$, is said to be continuous at some $x \in X$ (see [4, 13]) if $N_x(f^{-1}(A)) \geq N_{\bar{f}(x)}(A)$ for every subset $A$ of $Y$. If $f$ is continuous at every point of $X$, then it is said that $(\tau_1, \tau_2)$-continuous. As it is shown in [4], $f$ is continuous if and only if $\tau_2(A) \leq \tau_1(f^{-1}(A))$ for every subset $A$ of $Y$. For $f: X \to Y$ a function and $\tau$ a fuzzifying topology on $Y$, $f^{-1}(\tau)$ is defined to be the weakest fuzzifying topology on $X$ for which $f$ is continuous. By [4], $f^{-1}(\tau)$ is given by the neighborhood structure $N_x(A) = N_{\bar{f}(x)}(Y \setminus f(A^c))$. If $(\tau_i)_{i \in I}$ is a family of fuzzifying topologies on $X$, we will denote by $\bigvee_{i \in I} \tau_i$ or by $\sup \tau_i$, the weakest of all fuzzifying topologies on $X$ which are finer than each $\tau_i$. As it is proved in [4], $\bigvee_{i \in I} \tau_i$ is given by the neighborhood structure

$$N_x(A) = \sup \left\{ \inf_{i \in I} N_{\bar{A}^\tau_i}(A_i) : x \in \bigcap_{i \in I} A_i \subseteq A \right\}, \quad (2.3)$$

where the infimum is taken over the family of all finite subsets $J$ of $I$ and all $A_i \subseteq X$, $i \in J$. For $Y$ a subset of a fuzzifying topological space $(X, \tau)$, $\tau|_Y$ will be the fuzzifying topology induced on $Y$ by $\tau$, that is, the fuzzifying topology $f^{-1}(\tau)$, where $f: Y \to X$ is the inclusion map. For a family $(X_i, \tau_i)_{i \in I}$ of fuzzifying topological spaces, the product
fuzzifying topology $\tau = \prod \tau_i$ on $X = \prod X_i$ is the weakest fuzzifying topology on $X$ for which each projection $\pi_i : X \to X_i$ is continuous. Thus, $\tau = \bigvee_i \pi_i^{-1}(\tau_i)$ and it is given by the neighborhood structure

$$N_x(A) = \sup \left\{ \inf_{i \in I} N_{x_i}(A_i) : x \in \bigcap_{i \in I} \pi_i^{-1}(A_i) \subset A \right\},$$

(2.4)

where the supremum is taken over the family of all finite subsets $J$ of $I$ and $A_i \subset X_i$, for $i \in J$ (see [4]).

The degree of convergence to an $x \in X$, of a net $(x_\delta)$ in a fuzzifying topological space $(X, \tau)$, is the number $c(x_\delta \to x) = c^\tau(x_\delta \to x)$ defined by

$$c(x_\delta \to x) = \inf \{ 1 - N_x(A) : A \subset X, \ (x_\delta) \text{ frequently in } A^c \}.$$  

(2.5)

As it is shown in [6], for $A \subset X$ and $x \in X$, we have

$$\bar{A}(x) = \max \{ c(x_\delta \to x) : (x_\delta) \text{ net in } A \}.$$  

(2.6)

The degree of Hausdorffness of $X$ (see [4]) is defined by

$$T_2(X) = 1 - \sup_{x \neq y} \inf \sup \{ c(x_\delta \to x) \wedge c(x_\delta \to y) : (x_\delta) \text{ net in } X \}.$$  

(2.7)

Also, the degree of $X$ being $T_1$ is defined by

$$T_1(X) = \inf \inf \sup_{x \neq y} \{ N_x(B) : y \notin B \}.$$  

(2.8)

Let now $(X, \tau)$ be a fuzzifying topological space. For each $0 \leq \theta < 1$, the family $B^\theta_\tau = \{ A \subset X : \tau(A) > \theta \}$ is a base for a classical topology $\tau^\theta$ on $X$ (see [5]). It is easy to see that a subset $B$ of $X$ is a $\tau^\theta$-neighborhood of $x$ if and only if $N_x(B) \geq \theta$. By [6], $T_2(X)$ (resp., $T_1(X)$) is the supremum of all $0 \leq \theta < 1$ for which $\tau^\theta$ is $T_2$ (resp., $T_1$). Also, for $\tau = \bigvee \tau_i$, we have that $\tau^\theta = \bigcup i \tau^\theta_i$ (see [6, Theorem 3.5]). If $\tau = \prod \tau_i$ is a product fuzzifying topology, then $\tau^\theta = \prod \tau^\theta_i$ (see [6, Theorem 3.5]). If $Y$ is a subspace of $(X, \tau)$ and $\tau_1 = \tau|Y$, then $\tau_{1}^\theta = \tau^\theta|Y$. By [6, Theorem 3.10], for a fuzzifying topological space $(X, \tau)$, co($X$) coincides with the supremum of all $0 < \theta < 1$ for which $\tau^{1-\theta}$ is compact.

Next, we will recall the notion of a fuzzifying proximity given in [4]. A fuzzifying proximity on a set $X$ is a map $\delta : 2^X \times 2^X \to [0, 1]$ satisfying the following conditions:

(FP1) $\delta(A, B) = 1$ if the $A, B$ are not disjoint;
(FP2) $\delta(A, B) = \delta(B, A)$;
(FP3) $\delta(\emptyset, B) = 0$;
(FP4) $\delta(A_1 \cup A_2, B) = \delta(A_1, B) \vee \delta(A_2, B)$;
(FP5) $\delta(A, B) = \inf \{ \delta(A, D) \vee \delta(D^c, B) : D \subset X \}$. 

Every fuzzifying proximity $\delta$ induces a fuzzifying topology $\tau_\delta$ given by the neighborhood structure $N(x)(A) = 1 - \delta(x,A)$. A fuzzifying proximity $\delta_1$ is said to be finer than another one $\delta_2$ if $\delta_1(A,B) \leq \delta_2(A,B)$ for all subsets $A, B$ of $X$. For $f : X \to Y$ a function and $\delta$ a fuzzifying proximity on $X$, the function

$$f^{-1}(\delta) : 2^X \times 2^X \to [0,1], \quad f^{-1}(\delta)(A,B) = \delta(f(A),f(B)), \quad (2.9)$$

is a fuzzifying proximity on $X$ (see [4]) and it is the weakest of all fuzzifying proximities $\delta_1$ on $X$ for which $f$ is $(\delta_1,\delta)$-proximally continuous, that is, it satisfies $\delta_1(A,B) \leq \delta(f(A),f(B))$ for all subsets $A, B$ of $X$. As it is shown in [4], $\tau_{f^{-1}(\delta)} = f^{-1}(\tau_\delta)$.

Let now $(\delta_1)_{\lambda \in \Lambda}$ be a family of fuzzifying proximities on a set $X$. We will denote by $\delta = \bigvee_{\lambda} \delta_1$, or by $\sup \delta_1$, the weakest fuzzifying proximity on $X$ which is finer than each $\delta_1$. By [4, Theorem 8.10], $\delta$ is given by

$$\delta(A,B) = \inf \left\{ \sup \inf \delta_1(A_i,B_j) \right\}, \quad (2.10)$$

where the infimum is taken over all finite collections $(A_i)$, $(B_j)$ of subsets of $X$ with $A = \bigcup A_i$, $B = \bigcup B_j$. Moreover, $\tau_\delta = \bigvee_{\lambda} \tau_{\delta_1}$ (see [4]).

Finally, we will recall the definition of a fuzzy uniformity introduced by Lowen in [7]. For a set $X$, let $\Omega_X$ be the collection of all functions $\alpha : X \times X \to [0,1]$ such that $\alpha(x,x) = 1$ for all $x \in X$. For $\alpha, \beta \in \Omega_X$, the $\alpha \wedge \beta$, $\alpha \circ \beta$ and $\alpha^{-1}$ are defined by $\alpha \wedge \beta(x,y) = \alpha(x,y) \wedge \beta(x,y)$, $\alpha \circ \beta(x,y) = \sup_{z} \beta(x,z) \wedge \alpha(z,y)$, $\alpha^{-1}(x,y) = \alpha(y,x)$. If $\alpha = \alpha^{-1}$, then $\alpha$ is called symmetric. A fuzzy uniformity on $X$ is a nonempty subset $\mathcal{U}$ of $\Omega_X$ satisfying the following conditions.

(FU1) If $\alpha, \beta \in \mathcal{U}$, then $\alpha \wedge \beta \in \mathcal{U}$.

(FU2) If $\alpha \in \mathcal{U}$ is such that, for every $\epsilon > 0$, there exists a $\beta \in \mathcal{U}$ with $\beta \leq \alpha + \epsilon$, then $\alpha \in \mathcal{U}$.

(FU3) For each $\alpha \in \mathcal{U}$ and each $\epsilon > 0$, there exists a $\beta \in \mathcal{U}$ with $\beta \circ \alpha \leq \alpha + \epsilon$.

(FU4) If $\alpha \in \mathcal{U}$, then $\alpha^{-1} \in \mathcal{U}$.

A subset $\mathcal{B}$, of a fuzzy uniformity $\mathcal{U}$, is a base for $\mathcal{U}$ if for each $\alpha \in \mathcal{U}$ and each $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with $\beta \leq \alpha + \epsilon$. It is easy to see that for a subset $\mathcal{B}$ of $\Omega_X$, the following are equivalent.

(1) $\mathcal{B}$ is a base for a fuzzy uniformity on $X$.

(2) (a) If $\alpha, \beta \in \mathcal{B}$ and $\epsilon > 0$, then there exists $\gamma \in \mathcal{B}$ with $\gamma \leq \alpha \wedge \beta + \epsilon$.

(b) For each $\alpha \in \mathcal{B}$ and each $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with $\beta \circ \alpha \leq \alpha + \epsilon$.

(c) For each $\alpha \in \mathcal{B}$ and each $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with $\beta \leq \alpha^{-1} + \epsilon$.

In case (2) is satisfied, the fuzzy uniformity $\mathcal{U}$ for which $\mathcal{B}$ is a base consists of all $\alpha \in \Omega_X$ such that for each $\epsilon > 0$, there exists a $\beta \in \mathcal{B}$ with $\beta \leq \alpha + \epsilon$.

By [5], every fuzzy uniformity $\mathcal{U}$ on $X$ induces a fuzzifying proximity $\delta_\mathcal{U}$ defined by

$$\delta_\mathcal{U}(A,B) = \inf_{\alpha \in \mathcal{U}} \sup_{x \in A, y \in B} \alpha(x,y). \quad (2.11)$$
In case \( \mathcal{B} \) is a base for \( \mathcal{U} \), then
\[
\delta_{\mathcal{U}}(A,B) = \inf_{a \in \mathcal{B}} \sup_{x \in A, y \in B} \alpha(x,y). \tag{2.12}
\]

Every fuzzy uniformity \( \mathcal{U} \) induces a fuzzifying topology \( \tau_{\mathcal{U}} \) given by the neighborhood structure
\[
N_x(A) = 1 - \delta_{\mathcal{U}}(x,A^c) = 1 - \inf_{a \in \mathcal{U}} \sup_{y \notin A} \alpha(x,y). \tag{2.13}
\]

For every fuzzifying proximity \( \delta \), there exists at least one compatible fuzzy uniformity, that is, a fuzzy uniformity \( \mathcal{U} \) with \( \delta_{\mathcal{U}} = \delta \) (see [5, Theorem 11.4]).

3. Probabilistic pseudometrics

A fuzzy real number is a fuzzy subset \( u \) of the real numbers \( \mathbb{R} \) which is increasing, left continuous, and such that \( \lim_{t \to -\infty} u(t) = 1 \), \( \lim_{t \to -\infty} u(t) = 0 \). A fuzzy real number \( u \) is said to be nonnegative if \( u(t) = 0 \) if \( t \leq 0 \). We will denote by \( \mathbb{R}_+^f \) the collection of all nonnegative fuzzy real numbers. To every real number \( r \) corresponds a fuzzy real number \( \bar{r} \), where \( \bar{r}(t) = 0 \) if \( t \leq r \) and \( \bar{r}(t) = 1 \) if \( t > r \). For \( u, v \in \mathbb{R}_+^f \), we define \( u \leq v \) if and only if \( v(t) \leq u(t) \) for all \( t \in \mathbb{R} \). If \( \mathcal{A} \) is a nonempty subset of \( \mathbb{R}_+^f \) and if \( u_o \in \mathbb{R}_+^f \) is defined by \( u_o(t) = \sup_{v \in \mathcal{A}} v(t) \), then \( u_o \) is the biggest of all \( u \in \mathbb{R}_+^f \) with \( u \leq v \) for all \( v \in \mathcal{A} \). We will denote \( u_o \) by \( \inf \mathcal{A} \) or by \( \bigwedge \mathcal{A} \). For \( u_1, u_2 \in \mathbb{R}_+^f \), we define \( u = u_1 \oplus u_2 \in \mathbb{R}_+^f \) by \( u(t) = \sup\{u_1(t_1) \wedge u_2(t_2) : t = t_1 + t_2\} \). Also, for \( u \in \mathbb{R}_+^f \) and \( \lambda > 0 \), we define \( \lambda u \) by \( (\lambda u)(t) = u(\lambda^{-1}t) \). It is easy to see that for \( u \in \mathbb{R}_+^f \) and \( \lambda > 0 \), we have \( (\lambda \oplus u)(t) = u(t - \lambda) \).

**Definition 3.1.** A probabilistic pseudometric on a set \( X \) (see [2]) is a mapping \( F : X \times X \to \mathbb{R}_+^f \) such that for all \( x, y, z \in X \),
\[
F(x,x) = \bar{0}, \quad F(x,y) = F(y,x), \quad F(x,z) \leq F(x,y) \oplus F(y,z). \tag{3.1}
\]

If in addition \( F(x,y)(0^+) = 0 \) when \( x \neq y \), then \( F \) is called a probabilistic metric.

If \( r_1, r_2 \) are nonnegative real numbers, then \( \bar{r}_1 \leq \bar{r}_2 \) if and only if \( r_1 \leq r_2 \). Also, for \( r = |r_1 - r_2| \), we have that
\[
\bar{r} = \bigwedge \{ u \in \mathbb{R}_+^f : r_2 \leq u \oplus \bar{r}_1, \bar{r}_1 \leq u \oplus r_2 \}. \tag{3.2}
\]

In fact, let \( u_o = \bigwedge \{ u \in \mathbb{R}_+^f : r_2 \leq u \oplus \bar{r}_1 \text{ and } \bar{r}_1 \leq u \oplus r_2 \} \) and assume that (say) \( r_1 \geq r_2 \). Let \( u \in \mathbb{R}_+^f \) be such that \( r_2 \leq u \oplus \bar{r}_1 \text{ and } \bar{r}_1 \leq u \oplus r_2 \). Then \( \bar{r}_1(t) \geq (u \oplus \bar{r}_2)(t) = u(t - r_2) \) for all \( t \). If \( s < r_1 \), then \( 0 = \bar{r}_1(s) \geq u(s - r_2) \) and so \( u(r_1 - r_2) = \sup_{s \leq r_1} u(s - r_2) = 0 \) which implies that \( \bar{r} \leq u_o \). On the other hand, we have \( \bar{r} \oplus \bar{r}_2 = \bar{r}_1 \) and \( \bar{r} \oplus \bar{r}_1 = 2r_1 - r_2 \). Since \( \bar{r}_2 \leq 2r_1 - r_2 \), it follows that \( u_o \leq \bar{r} \), and hence \( \bar{r} = u_o \). Motivated by the above, we define the following distance function on \( \mathbb{R}_+^f \):
\[
D : \mathbb{R}_+^f \times \mathbb{R}_+^f \to \mathbb{R}_+^f, \quad D(u_1, u_2) = \bigwedge \{ u \in \mathbb{R}_+^f : u_1 \leq u_2 \oplus u, u_2 \leq u \oplus u_1 \}. \tag{3.3}
\]
Then \( D \) is a probabilistic pseudometric on \( \mathcal{R}_\phi^+ \). In fact, it is clear that \( D(u_1,u_2) = D(u_2,u_1) \). Also, since \( u = u \circ \theta \), when \( u \in \mathcal{R}_\phi^+ \), we have that \( D(u,u) = 0 \). Finally, let \( D(u_1,u_2)(t_1) \land D(u_2,u_3)(t_2) > \theta > 0 \). There are \( v_1,v_2 \in \mathcal{R}_\phi^+ \) with \( u_1 \leq v_1 \circ u_2 \), \( u_2 \leq v_1 \circ u_1 \), \( u_3 \leq v_2 \circ u_2 \), \( u_2 \leq v_2 \circ u_3 \), \( v_1(t_1) > \theta \), \( v_2(t_2) > \theta \). Now \( u_1 \leq v_1 \circ u_2 \leq v_1 \circ (v_2 \circ u_3) = (v_1 \circ v_2) \circ u_3 \) and \( u_3 \leq v_2 \circ u_2 \leq v_2 \circ (v_1 \circ u_1) = (v_1 \circ v_2) \circ u_1 \). Thus, \( D(u_1,u_3) \leq v_1 \circ v_2 \) and \( D(u_1,u_3)(t_1 + t_2) \geq v_1(t_1) \land v_2(t_2) > \theta \). This proves that \( D(u_1,u_3) \leq D(u_1,u_2) \circ D(u_2,u_3) \) and the claim follows. We will refer to \( D \) as the usual probabilistic pseudometric on \( \mathcal{R}_\phi^+ \).

Let now \( F \) be a probabilistic pseudometric on \( X \). For \( t > 0 \), let \( u_{F,t} \) be defined on \( X^2 \) by

\[
u(x,y) = F(x,y)(t).
\]

The family \( \mathcal{B}_F = \{ u_{F,t} : t > 0 \} \) is a base for a fuzzy uniformity \( \mathcal{U}_F \) on \( X \). Let \( \tau_F \) be the fuzzifying topology induced by \( \mathcal{U}_F \).

In the rest of the paper, we will consider on \( \mathcal{R}_\phi^+ \) the fuzzifying topology induced by the usual probabilistic pseudometric \( D \).

**Theorem 3.2.** A probabilistic pseudometric \( F \), on a fuzzifying topological space \( (X, \tau) \), is \( \tau \times \tau \) continuous if and only if \( \tau_F \leq \tau \).

**Proof.** Assume that \( \tau_F \leq \tau \) and let \( G \) be a subset of \( \mathbb{R}_\phi^+ \) and \( u = F(x_o,y_o) \) with \( N_{\phi}(G) > \theta > 0 \). There exists a \( t > 0 \) such that \( 1 - \sup_{v \in G} D(v,u)(t) > \theta \). For \( x,y \in X \), we have

\[
F(x,y) \leq F(x,x_o) \circ (x_o,y_o) \circ F(y_o,y) = [F(x,x_o) \circ F(y,y_o)] \circ F(x_o,y_o).
\]

Similarly, \( F(x_o,y_o) \leq [F(x,x_o) \circ F(y,y_o)] \circ F(x,y) \). Thus,

\[
D(F(x,y),F(x_o,y_o)) \leq F(x,x_o) \circ F(y,y_o).
\]

Let

\[
A_1 = \left\{ x \in X : F(x,x_o) \left( \frac{t}{2} \right) \geq 1 - \theta \right\}, \quad A_2 = \left\{ x \in X : F(y,y_o) \left( \frac{t}{2} \right) \geq 1 - \theta \right\}.
\]

If \( x \in A_1, \ y \in A_2 \), then

\[
D(F(x,y),F(x_o,y_o))(t) \geq F(x,x_o) \left( \frac{t}{2} \right) \land F(y,y_o) \left( \frac{t}{2} \right) \geq 1 - \theta,
\]

and so \( F(x,y) \in G \). Also, \( N_{x_o}^\tau(A_1) \geq N_{x_o}^\tau(A_1) \geq 1 - \sup_{x \notin A_1} F(x,x_o)(t/2) \geq \theta \) and \( N_{y_o}^\tau(A_2) \geq \theta \). Therefore,

\[
N_{x_o,y_o}^{\tau \times \tau}(F^{-1}(G)) \geq N_{x_o}^\tau(A_1) \land N_{y_o}^\tau(A_1) \geq \theta,
\]

which proves that \( N_{x_o,y_o}^{\tau \times \tau}(F^{-1}(G)) \geq N_{F(x_o,y_o)}^\tau(G) \) and so \( F \) is \( \tau \times \tau \) continuous. Conversely, assume that \( F \) is \( \tau \times \tau \) continuous and let \( N_{x_o}^\tau(A) > \theta > 0 \). Choose \( \epsilon > 0 \) such that \( N_{x_o}^\tau(A) > \theta + \epsilon \). There exists a \( t > 0 \) such that \( 1 - \sup_{x \notin A} F(x,x_o)(t) > \theta + \epsilon \). If

\[
Z = \left\{ u \in \mathcal{R}_\phi^+ : D(u,\tilde{0})(t) = u(t) > 1 - \theta - \epsilon \right\},
\]

then

\[
N_{\phi}(Z) \geq 1 - \sup_{u \notin Z} D(u,\tilde{0})(t) \geq \theta + \epsilon > \theta.
\]
Since $F$ is $\tau \times \tau$ continuous and $F(x_0, x_0) = 0$, there exists a subset $A_1$ of $X$ containing $x_0$ such that $A_1 \times A_1 \subseteq F^{-1}(Z)$ and $N_{x_0}(A_1) > \theta$. If $x \in A_1$, then $F(x, x_0) \in Z$ and so 
\[ F(x, x_0)(t) > 1 - \theta - \epsilon, \] which implies that $x \in A$. Thus, $A_1 \subseteq A$ and so $N_{x_0}(A) \geq N_{x_0}^{r_F}(A)$ for every subset $A$ of $X$ and every $x_0 \in X$. Hence, $\tau_F \leq \tau$ and the result follows.

**Theorem 3.3.** Let $F$ be a probabilistic pseudometric on a set $X$, $\tau = \tau_F$, $(x_\delta)_{\delta \in \Delta}$ a net in $X$, and $x \in X$. Then
\[ c(x_\delta \rightarrow x) = \inf_{\delta > 0} \lim \inf_{t} F(x_\delta, x)(t). \] 

**Proof.** Let $d = \inf_{t > 0} \lim \inf F(x_\delta, x)(t)$ and assume that $d < \theta < 1$. There exists a $t > 0$ such that $\lim \inf F(x_\delta, x)(t) < \theta$. Let $A = \{ y : F(y, x)(t) > \theta \}$. Then $(x_\delta)$ is not eventually in $A$, and so $c(x_\delta \rightarrow x) \leq 1 - N_x(x) \leq \sup_{y \in X} F(y, x)(t) \leq \theta$, which proves that $c(x_\delta \rightarrow x) \leq d$. On the other hand, let $c(x_\delta \rightarrow x) < r < 1$. There exists a subset $B$ of $X$ such that $(x_\delta)$ is not eventually in $B$ and $1 - N_x(B) < r$. Let $s > 0$ be such that $1 - \sup_{y \in B} F(y, x)(s) > 1 - r$. For each $\delta \in \Delta$, there exists $\delta' \geq \delta$ with $x_{\delta'} \not\in B$, and so $F(x_{\delta'}, x)(s) \leq \sup_{y \in B} F(y, x)(s)$. Thus, $d \leq \lim \inf F(x_\delta, x)(s) < r$, which proves that $d \leq c(x_\delta \rightarrow x)$ and the result follows.

**Theorem 3.4.** Let $F_1, F_2, \ldots, F_n$ be probabilistic pseudometrics on $X$ and define $F$ by
\[ F(x, y)(t) = \min_{1 \leq k \leq n} F_k(x, y)(t). \] 
Then $F$ is a probabilistic pseudometric and $\tau_F = \bigvee_{k+1} \tau_{F_k}$.

**Proof.** Using induction on $n$, it suffices to prove the result in the case of $n = 2$. It follows easily that $F$ is a probabilistic pseudometric. Since $F_1, F_2 \leq F$, it follows that $\tau_{F_1}, \tau_{F_2} \leq \tau_F$ and so $\tau_F = \tau_{F_1} \vee \tau_{F_2} \leq \tau_F$. On the other hand, let $N_x^{T_F}(A) > \theta > 0$. There exists a $t > 0$ such that $1 - \sup_{y \in A} F(y, x)(t) > \theta$. Let $B_i = \{ y \in A^i : F_i(y, x)(t) < 1 - \theta \}$, $i = 1, 2$. Then $A^i = B_1 \cup B_2$ and so $A = A_1 \cap A_2, A_i = B_i$. Moreover $N_x^{T_F}(A_i) \geq 1 - \sup_{y \in B_i} F_i(y, x)(t) \geq \theta$, and thus
\[ N_x^{T_F}(A) \geq N_x^{T_F}(A_1) \cap N_x^{T_F}(A_2) \geq N_x^{T_F}(A_1) \bigcap N_x^{T_F}(A_2) \geq \theta. \] 

This proves that $N_x^{T_F}(A) \geq N_x^{T_F}(A)$ and the result follows.

For $\mathcal{F}$ a family of probabilistic pseudometrics on a set $X$, we will denote by $\tau_{\mathcal{F}}$ the supremum of the fuzzifying topologies $\tau_{F}, F \in \mathcal{F}$, that is, $\tau_{\mathcal{F}} = \bigvee_{F \in \mathcal{F}} \tau_F$.

**Theorem 3.5.** If $\tau = \tau_{\mathcal{F}}$, where $\mathcal{F}$ is a family of probabilistic pseudometrics on a set $X$, then $T_2(X) = T_1(X) = 1 - \sup_{y \neq x} \lim \inf_{F \in \mathcal{F}} F(x, y)(0+)$. 

**Proof.** Let $d = 1 - \sup_{y \neq x} \lim \inf_{F \in \mathcal{F}} F(x, y)(0+)$. It is always true that $T_2(X) \leq T_1(X)$. Suppose that $T_1(X) > r > 0$ and let $x \neq y$. Since $\tau'$ is $T_1$, there exists a $\tau'$-neighborhood $A$ of $x$ not containing $y$. Now $N_x(A) > r$, and hence there are subsets $A_1, \ldots, A_n$ of $X$ and $F_1, \ldots, F_n \in \overline{\mathcal{F}}$ such that $\bigcap A_k \subseteq A, N_x^{T_F}(A_k) > r$. Since $y$ is not in $A$, there exists a $k$ with
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\( y \not\in A_k \). Let \( t > 0 \) be such that

\[
1 - \sup_{z \not\in A_k} F_k(z, x)(t) > r \quad \text{and so} \quad \inf_{F \in \mathcal{F}} F(x, y)(t)(0+) \leq F_k(x, y)(t) < 1 - r,
\]

(3.14)

which proves that \( d \geq r \). Thus \( d \geq T_1(X) \). On the other hand, assume that \( d > \theta > 0 \) and let \( x \neq y \). Choose \( \epsilon > 0 \) such that \( d > \theta + \epsilon \). There exists \( F \in \mathcal{F} \) with \( F(x, y)(0+) < 1 - \theta - \epsilon \), and hence \( F(x, y)(t) < 1 - \theta - \epsilon \) for some \( t > 0 \). Let

\[
A = \left\{ z : F(z, x) \left( \frac{t}{2} \right) > 1 - \theta - \epsilon \right\}, \quad B = \left\{ z : F(z, y) \left( \frac{t}{2} \right) > 1 - \theta - \epsilon \right\}.
\]

(3.15)

Clearly \( x \in A \), \( y \in B \). If \( z \in A \cap B \), then

\[
F(x, y)(t) \geq F(x, z) \left( \frac{t}{2} \right) \land F(z, y) \left( \frac{t}{2} \right) > 1 - \theta - \epsilon,
\]

(3.16)

a contradiction. Thus \( A \cap B = \emptyset \). Moreover

\[
N_x(A) \geq N_x^\tau(A) \geq 1 - \sup_{z \not\in A} F(x, z) \left( \frac{t}{2} \right) \geq \theta + \epsilon > \theta, \quad N_y(A) > \theta.
\]

(3.17)

It follows that \( T_2(X) \geq d \) and the proof is complete.

\[ \square \]

Let us say that a fuzzifying topology \( \tau \) on a set \( X \) is pseudometrizable if there exists a probabilistic pseudometric \( F \) on \( X \) with \( \tau = \tau_F \).

**Theorem 3.6.** A fuzzifying topology \( \tau \) on \( X \) is pseudometrizable if and only if each level topology \( \tau^\theta \), \( 0 \leq \theta < 1 \), is pseudometrizable.

**Proof.** Assume that \( \tau = \tau_F \) for some probabilistic pseudometric \( F \) and let \( 0 \leq \theta < 1 \). For each positive integer \( n \), with \( n > 1/(1 - \theta) \), let

\[
A_n = \left\{ (x, y) \in X^2 : F(x, y) \left( \frac{1}{n} \right) > 1 - \theta - \frac{1}{n} \right\}.
\]

(3.18)

Then \( A_{n+1} \subset A_n \) and the family \( \mathcal{U} = \{ A_n : n \in \mathbb{N}, \ n > 1/(1 - \theta) \} \) is a base for a uniformity \( \mathcal{U} \) on \( X \). The topology \( \sigma_\theta \) induced by \( \mathcal{U} \) is pseudometric since \( \mathcal{U} \) is countable. Moreover \( \sigma_\theta = \tau^\theta \). Indeed, let \( A \) be a \( \sigma_\theta \)-neighborhood of \( x \). There exists \( n \in \mathbb{N}, \ n > 1/(1 - \theta) \), such that \( B = \{ y : F(x, y)(1/n) > 1 - \theta - 1/n \} \subset A \). Now

\[
N_x^\tau(A) \geq N_x^\tau(B) \geq 1 - \sup_{y \in B} F(x, y) \left( \frac{1}{n} \right) \geq \theta + \frac{1}{n} > \theta,
\]

(3.19)

and so \( A \) is a \( \tau^\theta \)-neighborhood of \( x \). Conversely, assume that \( A \) is a \( \tau^\theta \)-neighborhood of \( x \). There exists \( \epsilon > 0 \) with \( N_x(A) > \theta + \epsilon \). Now there exists a positive integer \( n > 1/\epsilon \) such
that \( 1 - \sup_{y \in A} F(x, y)(1/n) > \theta + 1/n. \) Hence

\[
\left\{ y : F(x, y) \left( \frac{1}{n} \right) > 1 - \theta - \frac{1}{n} \right\} \subset A, \tag{3.20}
\]

which implies that \( A \) is a \( \sigma_\theta \)-neighborhood of \( x \). Thus \( \tau^\theta = \sigma_\theta \), and therefore each \( \tau^\theta \) is pseudometrizable. Conversely, suppose that each \( \tau^\theta \) is pseudometrizable. By an argument analogous to the one used in the proof of [6, Theorem 3.3], we show that there exists a family \( \{ d_\theta : 0 \leq \theta < 1 \} \) of pseudometrics on \( X \) such that \( d_\theta = \sup_{\theta > 0} d_{\theta_1} \), for each \( 0 \leq \theta < 1 \), and \( \tau^\theta \) coincides with the topology induced by the pseudometric \( d_\theta \). Now, for \( x, y \in X \), define \( F(x, y) : \mathbb{R} \rightarrow [0, 1] \) by \( F(x, y)(t) = 0 \) if \( t \leq 0 \) and \( F(x, y)(t) = \sup\{ \theta : 0 < \theta \leq 1, d_{1-\theta}(x, y) < t \} \) if \( t > 0 \). It is clear that \( F(x, y) \) is increasing and left continuous.

Thus, \( \tau F \) and the result follows.

**Theorem 3.7.** Let \((X, F)\) be a probabilistic pseudometric space, \( A \subset X \), and \( x \in X \).

Let

\[
\alpha = \sup \left\{ \inf_{n} \liminf_{t \uparrow 0} F(x_n, x)(t) : (x_n) \text{ sequence in } A \right\},
\]

\[
\beta = \sup \left\{ \liminf_{n} F(x_n, x)(t_n) : t_n \rightarrow 0+, (x_n) \text{ sequence in } A \right\}, \tag{3.22}
\]

\[
y = \sup \left\{ \liminf_{n} F(x_n, x)(1/n) : (x_n) \text{ sequence in } A \right\}.
\]

Then \( \alpha = \beta = y = \bar{A}(x) \).

**Proof.** If \( (x_n) \subset A \), then

\[
\bar{A}(x) \geq c(x_n \rightarrow x) = \inf_{t > 0} \liminf_{n} F(x_n, x)(t), \tag{3.23}
\]

and so \( \bar{A}(x) \geq \alpha \). Assume that \( \beta > \theta > 0 \). There exist a sequence \( (x_n) \in A \) and a sequence \( (t_n) \) of positive real numbers, with \( t_n \rightarrow 0+ \), such that \( \liminf_{n} F(x_n, x)(t_n) > \theta \). Let \( t > 0 \)
and choose \( k \) such that \( t_n < t \) when \( n \geq k \). For \( m \geq k \), we have \( \inf_{n \geq m} F(x_n, x)(t) \geq \inf_{n \geq m} F(x_n, x)(t_n) > \theta \). Thus \( \liminf_{n} F(x_n, x)(t) > \theta \) for each \( t > 0 \) and so \( \alpha \geq \theta \), which proves that \( \alpha \geq \beta \). Clearly \( \beta \geq \gamma \). Finally, \( N_x(A^c) \geq 1 - \sup_{y \in A} F(y, x)(1/n) \) and so \( \sup_{y \in A} F(y, x)(1/n) \geq 1 - N_x(A^c) = \bar{A}(x) > \bar{A}(x) - 1/n \). Hence, for each \( n \in \mathbb{N} \), there exists \( x_n \in A \) with \( F(x_n, x)(1/n) > \bar{A}(x) - 1/n \). Consequently,

\[
y \geq \liminf_{n} F(x_n, x) \left( \frac{1}{n} \right) \geq \liminf_{n} \left( \bar{A}(x) - \frac{1}{n} \right) = \bar{A}(x),
\]

and so \( y \geq \bar{A}(x) \geq \alpha \geq \beta \geq \gamma \), which completes the proof. \( \square \)

In view of [6, Theorem 4.14], we have the following corollary.

**Corollary 3.8.** Every pseudometrizable fuzzifying topological space is \( \mathbb{N} \)-sequential and hence sequential.

**Theorem 3.9.** If \( (F_n) \) is a sequence of probabilistic pseudometrics on a set \( X \), then there exists a probabilistic pseudometric \( F \) such that \( \tau_F = \bigvee_n \tau_{F_n} \).

**Proof.** If \( F \) is a probabilistic pseudometric on \( X \) and if \( \tilde{F} \) is defined by \( \tilde{F}(x, y)(t) = F(x, y)(t) \) if \( t \leq 1 \) and \( \tilde{F}(x, y)(t) = 1 \) if \( t > 1 \), then \( \tilde{F} \) is a probabilistic pseudometric on \( X \) and \( \tau_{\tilde{F}} = \tau_F \). Hence, we may assume that \( F_n(x, y)(t) = 1 \), for all \( n \), if \( t > 1 \). For \( x, y \in X \), define \( F(x, y) \) on \( \mathbb{R} \) by \( F(x, y)(t) = 0 \) if \( t \leq 0 \) and \( F(x, y)(t) = \inf_n [(1/n)F_n(x, y)](t) \) if \( t > 0 \). Clearly, \( F(x, y) \) is increasing and \( F(x, y)(t) = 1 \) if \( t > 1 \). Also, \( F(x, y) \) is left continuous. In fact, let \( F(x, y)(t) > \theta > 0 \) and choose \( n \) such that \( (n + 1)t > 1 \). There exists \( 0 < s_1 < t \) such that \( F_k(x, y)(ks_1) > \theta \) for \( k = 1, \ldots, n \). Choose \( s_1 < s < t \) such that \( (n + 1)s > 1 \). Now, \( F_m(x, y)(ms) = 1 \) if \( m > n \). Thus

\[
F(x, y)(s) = \min_{1 \leq k \leq n} \left[ \frac{1}{k} F_k(x, y) \right](s) > \theta,
\]

which proves that \( F(x, y) \) is in \( \mathbb{R}^+_\theta \). It is clear that \( F(x, x) = 0 \). We need to prove that \( F \) satisfies the triangle inequality. So assume that \( F(x, y)(t_1) \wedge F(y, z)(t_2) > \theta > 0 \). If \( m \) is such that \( (m + 1)(t_1 + t_2) > 1 \), then

\[
F(x, z)(t_1 + t_2) = \min_{1 \leq k \leq m} F_k(x, z)(k(t_1 + t_2)).
\]

Since

\[
F_k(x, z)(k(t_1 + t_2)) \geq F_k(x, y)(kt_1) \wedge F_k(y, z)(kt_2) > \theta,
\]

it follows that \( F(x, z)(t_1 + t_2) > \theta \), and so \( F \) satisfies the triangle inequality. We will finish the proof by showing that \( \tau_F = \sqrt{\tau_{F_n}} \). To see this, we first observe that \( (1/n)F_n \leq F \), which implies that \( \tau_{F_n} = \tau_{(1/n)F_n} \leq \tau_F \), and so \( \tau_F = \sqrt{n \tau_{(1/n)F_n}} \leq \tau_F \). On the other hand, let \( N^\tau_{\bar{A}}(A) > \theta \) and choose \( \epsilon > 0 \) such that \( N^\tau_{\bar{A}}(A) > \theta + \epsilon \). Let \( t > 0 \) be such that \( 1 - \sup_{y \in A} F(y, x)(t) > \theta + \epsilon \). If \( (m + 1)t > 1 \), then

\[
F(y, z)(t) = \min_{1 \leq k \leq m} F_k(y, z)(kt).
\]
Let \( A_k = \{ y : F_k(y, x)(kt) \geq 1 - \theta - \epsilon \} \). Then

\[
N_x^{r_k}(A_k) \geq N_x^{r_k}(A_k) \geq 1 - \sup_{z \notin A_k} F_k(z, x)(kt) \geq \theta + \epsilon > \theta
\]  

(3.29)

and \( \bigcap_{k=1}^m A_k \subset A \). Hence, \( N_x^{r_k}(A) \geq \min_{1 \leq k \leq m} N_x^{r_k}(A_k) > \theta \). This proves that \( \tau_F \leq \tau_o \) and the result follows. \( \square \)

\textbf{Theorem 3.10.} Let \( f : X \to Y \) be a function and let \( F \) be a probabilistic pseudometric on \( Y \). Then the function

\[
f^{-1}(F) : X^2 \to \mathbb{R}_+^{+1}, \quad f^{-1}(F)(x, y) = F(f(x), f(y)),
\]

is a probabilistic pseudometric on \( X \) and \( \tau_{f^{-1}(F)} = f^{-1}(\tau_F) \).

\textbf{Proof.} It follows easily that \( f^1(F) \) is a probabilistic pseudometric on \( X \). Let \( x \in X \) and \( B \subset X \). If \( D = Y \setminus f(B^c) \), then

\[
N_x^{r_{f^{-1}(F)}}(B) = \inf_{t > 0} \left[ 1 - \sup_{y \notin B} F(f(y), f(x))(t) \right]
\]

\[
= \inf_{t > 0} \left[ 1 - \sup_{z \in D^c} F(z, f(x))(t) \right]
\]

\[
= N_x^{f^{-1}(\tau)}(D) = N_x^{f^{-1}(\tau)}(B),
\]

which clearly completes the proof. \( \square \)

\textbf{Corollary 3.11.} If \( F \) is a probabilistic pseudometric on a set \( X \) and \( Y \subset X \), then \( \tau_F \mid_Y \) is induced by the probabilistic pseudometric \( G = F \mid_{Y \times Y} \), \( G(x, y) = F(x, y) \).

\textbf{Corollary 3.12.} If \((X_n, \tau_n)\) is a sequence of pseudometrizable fuzzifying topological spaces, then the Cartesian product \((X, \tau) = (\prod X_n, \prod \tau_n)\) is pseudometrizable.

\textbf{Proof.} Let \( F_n \) be a probabilistic pseudometric on \( X_n \) inducing \( \tau_n \). If \( G_n = \pi_n^{-1}(F_n) \), then \( \tau_{G_n} = \pi_n^{-1}(\tau_n) \), and so \( \tau = \bigvee_n \pi_n^{-1}(\tau_n) \) is pseudometrizable. \( \square \)

\textbf{4. Level proximities}

Let \( \delta \) be a fuzzifying proximity on a set \( X \). For each \( 0 < d \leq 1 \), let \( \delta^d \) be the binary relation on \( 2^X \) defined by \( A \delta^d B \) if and only if \( \delta(A, B) \geq d \). It is easy to see that \( \delta^d \) is a classical proximity on \( X \). We will show that the classical topology \( \sigma_d \) induced by \( \delta^d \) coincides with \( \tau^{1-d} \). In fact, let \( x \in A \in \sigma_d \). Then, \( x \) is not in the \( \sigma_d \)-closure of \( A^c \), which implies that \( x \) \( \delta^d(A) \), that is, \( \delta(x, A^c) < d \), and so \( N_x^{\tau^d}(A) = 1 - \delta(x, A^c) > 1 - d \). This proves that \( A \in \tau^{1-d} \). Conversely, if \( x \in B \in \tau^{1-d} \), then \( N_x^{\tau^d}(A) > 1 - d \), and thus \( \delta(x, A^c) < d \), which implies that \( x \) is not in the \( \sigma_d \)-closure of \( B^c \). Hence \( B^c \) is \( \sigma_d \)-closed, and so \( B \) is \( \sigma_d \)-open.
Theorem 4.1. If $\delta$ is a fuzzifying proximity on a set $X$ and $0 < d \leq 1$, then

$$\delta^d = \bigvee_{0 < \theta < d} \delta^\theta. \quad (4.1)$$

Proof. If $0 < \theta < d$, then $\delta^\theta$ is coarser than $\delta^d$, and so $\delta_0 = \bigvee_{0 < \theta < d} \delta^\theta$ is coarser than $\delta^d$. On the other hand, let $A \sigma B$. Since $\delta_0$ is finer than $\delta^d$ (for $0 < \theta < d$), we have that $A \delta^\theta B$ and so $\delta(A, B) \geq \theta$, for each $0 < \theta < d$, which implies that $\delta(A, B) \geq d$, that is, $A \delta dB$. So $\delta_0$ is finer than $\delta^d$ and the result follows.

Theorem 4.2. For a family $\{\gamma_d : 0 < d \leq 1\}$ of classical proximities on a set $X$, the following are equivalent.

1. There exists a fuzzifying proximity $\delta$ on $X$ such that $\delta^d = \gamma_d$ for all $d$.
2. $\gamma_d = \bigvee_{0 < \theta < d} \gamma_\theta$ for each $0 < d \leq 1$.

Proof. In view of the preceding theorem, (1) implies (2). Assume now that (2) is satisfied and define $\delta$ on $2^X \times 2^X$ by $\delta(A, B) = \sup \{d : A \gamma_d B\}$ (the supremum over the empty family is taken to be zero). It is clear that $\delta(A, B) = 1$ if the $A, B$ are not disjoint. Also, $\delta(A, B) = \delta(A, B)$ and $\delta(A, B) \geq \delta(A_i, B_j)$ if $A_i \subset A, B_j \subset B$. Let now $\delta(A, B) < d < 1$. Then $A \not\subset B$, and there exists a subset $D$ of $X$ such that $A \not\subset D$ and $D^c \not\subset B$. Since $A \not\subset D$, we have that $\delta(A, D) \leq d$. Similarly $\delta(D^c, B) \leq d$, and so $\inf \{\delta(A, D) \land \delta(D^c, B)\} = \delta(A, B)$. On the other hand, if $\delta(A, D) \land \delta(D^c, B) < \theta < 1$, then $A \subset D^c$, and so $\delta(A, B) \leq \delta(D^c, B) < \theta$. This proves that $\delta$ is a fuzzifying proximity on $X$. We will finish the proof by showing that $\delta^d = \gamma_d$ for all $d$. Indeed, if $A \gamma_d B$, then $\delta(A, B) \geq d$, that is, $A \delta dB$. On the other hand, let $A \delta dB$ and let $(A_i), (B_j)$ be finite families of subsets of $X$ such that $A = \bigcup_i A_i, B = \bigcup_j B_j$. Since $\delta(A, B) = \bigvee_{i, j} \delta(A_i, B_j) \geq d$, there exists a pair $(i, j)$ such that $\delta(A_i, B_j) \geq d$. If now $0 < \theta < d$, then there exists $r > \theta$ with $A_i \gamma_r B_j$, and so $A \gamma^0 B_j$. This proves that $A \gamma_d B$ since $\gamma_d = \bigvee_{0 < \theta < d} \gamma^\theta$. This completes the proof.

Theorem 4.3. Let $(X, \delta_1), (Y, \delta_2)$ be fuzzifying proximity spaces and let $f : X \to Y$ be a function. Then $f$ is proximally continuous if and only if $f : (X, \delta_1^d) \to (Y, \delta_2^d)$ is proximally continuous for each $0 < d \leq 1$.

Proof. It follows immediately from the definitions.

Theorem 4.4. Let $(X_\lambda, \delta_\lambda)_{\lambda \in \Lambda}$ be a family of fuzzifying proximity spaces and let $(X, \delta) = (\prod X_\lambda, \prod \delta_\lambda)$ be the product fuzzifying proximity space. Then $\delta^d = \prod \delta_\lambda^d$ for all $0 < d \leq 1$.

Proof. Since each projection $\pi_\lambda : (X, \delta) \to (X_\lambda, \delta_\lambda^d)$ is proximally continuous, it follows that $\delta^d$ is finer than $\sigma = \prod \delta_\lambda^d$. On the other hand, let $A \sigma B$. We need to show that $\delta(A, B) \geq d$. In fact, let $(A_i), (B_j)$ be finite families of subsets of $X$ such that $A = \bigcup_i A_i, B = \bigcup_j B_j$. Since $A \sigma B$ and $\sigma = \bigvee_i \pi^{-1}_\lambda(\delta_\lambda^d)$, there exists a pair $(i, j)$ such that $A_i \pi^{-1}_\lambda(\delta_\lambda) B_j$, that is, $\delta_\lambda(\pi_\lambda(A_i), \pi_\lambda(B_j)) \geq d$. In view of [4, Theorem 8.9], we conclude that $\delta(A, B) \geq d$. Hence, $\sigma = \delta^d$ and the proof is complete.

We have the following easily established theorem.

Theorem 4.5. Let $(Y, \delta)$ be a fuzzifying proximity space and let $f : X \to Y$. Then $f^{-1}(\delta)^d = f^{-1}(\delta^d)$ for each $0 < d \leq 1$. 
Theorem 4.6. Let \((\delta_{\lambda})_{\lambda \in A}\) be a family of fuzzifying proximities on a set \(X\) and \(\delta = \vee_{\lambda} \delta_{\lambda}\). Then \(\delta^d = \bigvee_{\lambda} \delta_{\lambda}^d\) for each \(0 < d \leq 1\).

Proof. Let \(\sigma = \bigvee_{\lambda} \delta_{\lambda}^d\). Since \(\delta\) is finer than each \(\delta_{\lambda}\), it follows that \(\delta^d\) is finer than each \(\delta_{\lambda}^d\), and so \(\delta^d\) is finer than \(\sigma\). On the other hand, let \(A\sigma B\) and let \((A_i), (B_j)\) be finite families of subsets of \(X\) such that \(A = \bigcup A_i, B = \bigcup B_j\). There exists a pair \((i, j)\) such that \(A_i\sigma B_j\). Since \(\sigma\) is finer than each \(\delta_{\lambda}^d\), we have that \(A_i\delta_{\lambda}^d B_j\), that is, \(\delta_{\lambda}(A_i, B_j) \geq d\). In view of [4, Theorem 8.10], we get that \(\delta(A, B) \geq d\), that is, \(A\delta^d B\). So \(\sigma\) is finer than \(\delta^d\) and the proof is complete. \(\square\)

5. Completely regular fuzzifying spaces

Definition 5.1. A fuzzifying topological space \((X, \tau)\) is called completely regular if each of the classical level topologies \(\tau^d\), \(0 \leq d < 1\), is completely regular.

Definition 5.2. A fuzzifying proximity \(\delta\) on a set \(X\) is said to be compatible with a fuzzifying topology \(\tau\) if \(\tau\) coincides with the fuzzifying topology \(\tau_\delta\) induced by \(\delta\).

We have the following easily established theorem.

Theorem 5.3. Subspaces and Cartesian products of completely regular fuzzifying spaces are completely regular.

Theorem 5.4. Let \((X, \tau)\) be a completely regular fuzzifying topological space and define \(\delta = \delta(\tau) : 2^X \times 2^X \to [0,1]\) by

\[
\delta(A, B) = 1 - \left( \sup \{d : 0 \leq d < 1, \exists f : (X, \tau^d) \to [0,1] \text{ continuous } f(A) = 0, f(B) = 1\} \right).
\]

(5.1)

Then, (1) \(\delta\) is a fuzzifying proximity on \(X\) compatible with \(\tau\);

(2) if \(\delta_1\) is any fuzzifying proximity on \(X\) compatible with \(\tau\), then \(\delta\) is finer than \(\delta_1\).

Proof. It is easy to see that \(\delta\) satisfies (FP1), (FP2), (FP3), and (FP5). We will prove that \(\delta\) satisfies (FP4). Let

\[
\alpha = \inf \{\delta(A, D) \vee \delta(D^c, B) : D \subset X\}.
\]

(5.2)

If \(\delta(A, D) \vee \delta(D^c, B) < \theta\), then \(A \subset D^c\), and so \(\delta(A, B) \leq \delta(D^c, B) < \theta\), which proves that \(\delta(A, B) \leq \alpha\). On the other hand, assume that \(\delta(A, B) < r < 1\). There exist a \(d, 1 - r < d < 1\), and \(f : X \to [0,1]\) \(\tau^d\)-continuous such that \(f(A) = 0, f(B) = 1\). Let \(D = \{x \in X : 1/2 \leq f(x) \leq 1\}\) and define \(h_1, h_2 : [0,1] \to [0,1], h_1(t) = 2t, h_2(t) = 0\) if \(0 \leq t \leq 1/2\) and \(h_1(t) = 1, h_2(t) = 2t - 1\) if \(1/2 < t \leq 1\). If \(g_i = h_i \circ f, i = 1,2\), then \(g_1(A) = 0, g_1(D) = 1, g_2(D^c) = 0, g_2(B) = 1\). Thus, \(\delta(A, D) \leq 1 - d < r, \delta(D^c, B) < r\), which proves that \(\alpha \leq \delta(A, B)\). Hence, \(\delta\) is a fuzzifying proximity on \(X\). We need to show that \(\tau = \tau_\delta\). So, let \(\tau(A) > \theta > 0\). Since \(\tau^\theta\) is completely regular, given \(x \in A\), there exist \(f_x : X \to [0,1]\), \(\tau^\theta\)-continuous, \(f_x(A^c) = 0, f_x(A) = 1\). Thus \(\delta(x, A^c) \leq 1 - \theta\), and so \(N^\theta_x(A) = 1 - \delta(x, A^c) \geq \theta\). It follows that \(\tau_\delta(A) = \inf_{x \in A} N^\theta_x(A) \geq \theta\), which proves that \(\tau_\delta \geq \tau\). On the other hand, assume that \(\tau_\delta(A) > r > 0\). If \(x \in A\), then \(\delta(x, A^c) = 1 - N^\theta_x(A) < 1 - r\), and therefore there exist a \(d\), \(0 < 1 - d < 1 - r\) and \(f : X \to [0,1]\) \(\tau^d\)-continuous such that \(f(x) = 0, f(A^c) = 1\). The set
Let now \( (by \[8, \text{Remark 3.15}\]) an

This completes the proof.

(1) Proof.

Clearly, \( F \) define \( \tau_d \) a classical proximity compatible with \( F \). So \( \delta \) satisfies the triangle inequality. By \[5\], (2) is equivalent to (3). Also (1) implies (2) in view of the preceding theorem. Assume now that \( \tau = \tau_d \) for some fuzzifying proximity \( \delta \). For each \( 0 < d \leq 1 \), \( \delta^d \) is a classical proximity compatible with \( \tau^{1-d} \), and so \( \tau^{1-d} \) is completely regular. This completes the proof.

Theorem 5.5. For a fuzzifying topological space \((X, \tau)\), the following are equivalent.

1. \((X, \tau)\) is completely regular.
2. There exists a fuzzifying proximity \( \delta \) on \( X \) compatible with \( \tau \).
3. \((X, \tau)\) is fuzzy uniformizable, that is, there exists a fuzzy uniformity \( \tau_u \) on \( X \) such that \( \tau \) coincides with the fuzzifying topology \( \tau_u \) induced by \( \tau_u \).

Proof. By [5], (2) is equivalent to (3). Also (1) implies (2) in view of the preceding theorem. Assume now that \( \tau = \tau_d \) for some fuzzifying proximity \( \delta \). For each \( 0 < d \leq 1 \), \( \delta^d \) is a classical proximity compatible with \( \tau^{1-d} \), and so \( \tau^{1-d} \) is completely regular. This completes the proof.

Theorem 5.6. Every pseudometrizable fuzzy topological space \((X, \tau)\) is completely regular.

Proof. If \( \tau \) is pseudometrizable, then each \( \tau^d, 0 \leq d < 1 \), is pseudometrizable, and hence \( \tau^d \) is completely regular.

Theorem 5.7. For a fuzzifying topological space \((X, \tau)\), the following are equivalent.

1. \((X, \tau)\) is completely regular.
2. If \( \mathcal{F} = \mathcal{F}_\tau \) is the family of all probabilistic pseudometrics on \( X \) which are \( \tau \times \tau \) continuous as functions from \( X^2 \) to \( \mathbb{R}_+^d \), then \( \tau = \tau_{\mathcal{F}} \).
3. There exists a family \( \mathcal{F} \) of probabilistic pseudometrics on \( X \) such that \( \tau = \tau_{\mathcal{F}} \).

Proof. (1) \(\Rightarrow\) (2). For each \( F \in \mathcal{F}_\tau \), we have that \( \tau_F \leq \tau \) (by Theorem 3.2), and so \( \tau_{\mathcal{F}} \leq \tau \). Let now \( A \subset X \) and \( x_o \in X \) with \( N^F_{x_o} (A) > \theta > 0 \). Since \( \tau^\theta \) is completely regular, there exists a \( \tau^\theta \)-continuous function \( f \) from \( X \) to \( [0,1] \) such that \( f(x_o) = 0, f(A^c) = 1 \). For \( x, y \in X \), define \( F(x, y) \) on \( \mathbb{R} \) by

\[
F(x, y)(t) = \begin{cases} 
0 & \text{if } t \leq 0, \\
1 - \theta & \text{if } |f(x) - f(y)| \geq t > 0, \\
1 & \text{if } |f(x) - f(y)| < t.
\end{cases}
\]  

(5.4)

Clearly, \( F(x, y) = F(y, x) \in \mathbb{R}_+^d \) and \( F(x, x) = 0 \). We will prove that \( F \) satisfies the triangle inequality. So, assume that \( F(x, y)(t_1) \land F(y, z)(t_2) > F(x, z)(t_1 + t_2) \). Then, \( t_1, t_2 > 0 \), \( F(x, z)(t_1 + t_2) = 1 - \theta, F(x, y)(t_1) = F(y, z)(t_2) = 1 \). Thus, \( t_1 > |f(x) - f(y)|, t_2 > |f(y) - f(z)|, \) and hence \( |f(x) - f(z)| < t_1 + t_2 \), which implies that \( F(x, z)(t_1 + t_2) = 1 \), a contradiction. So \( F \) is a probabilistic pseudometric on \( X \). Next we show that \( F \) is \( \tau \times \tau \) continuous, or equivalently that \( \tau_F \leq \tau \). So assume that \( N^F_{x_o} (B) > r > 0 \). Let \( \theta_1 > r \) be such
that $N^r_x(B) > \theta_1$. Choose $t > 0$ such that $1 - \sup_{y \in B} F(x,y)(t) > \theta_1$, and so $F(x,y)(t) = 1 - \theta$ and $|f(x) - f(y)| \geq t$ if $y \notin B$. Thus, $\{y : |f(x) - f(y)| < t\} \subset B$. This shows that $B$ is a $\tau^\theta$-neighborhood of $x$. As $r < \theta$, $B$ is a $\tau'$-neighborhood of $x$, that is, $N^r_x(B) > r$, and so $\tau_F \leq \tau$. Finally if $y \notin A$, then $|f(y) - f(x_o)| = 1$, and so $F(y,x_o)(1/2) = 1 - \theta$, which implies that

$$N^r_{x_o}(A) \geq N^r_{x_o}(A) \geq 1 - \sup_{y \in A} F(y,x_o) \left(\frac{1}{2}\right) \geq \theta. \quad (5.5)$$

This shows that $N^r_{x_o} \geq N^r_{x_o}$, and so $\tau \leq \tau_F$, which completes the proof of the implication $(1) \Rightarrow (2)$.

$(3) \Rightarrow (1)$. Assume that $\tau = \tau_F$ for some family $\mathcal{F}$ of probabilistic pseudometrics on $X$. For each $F \in \mathcal{F}$, $\tau_F$ is completely regular and so $\tau_F$ is completely regular since $\tau^d_F = \bigvee_{F \in \mathcal{F}} \tau^d_F$ for each $0 \leq d < 1$. Hence the result follows. \hfill $\square$

We will denote by $[0,1]_\phi$ the subspace of $\mathbb{R}^+_\phi$ consisting of all $u \in \mathbb{R}^+_\phi$ with $u(t) = 1$ if $t > 1$.

**Theorem 5.8.** A fuzzifying topological space $(X, \tau)$ is completely regular if and only if the following condition is satisfied. If $N_{x_o}(A) > \theta > 0$, then there exists $f : X \rightarrow [0,1]_\phi$ continuous such that $f(x_o) = 0$ and $f(y)(t) = 1 - \theta$ if $y \notin A$ and $0 < t < 1$.

**Proof.** Assume that $(X, \tau)$ is completely regular and let $N_{x_o}(A) > \theta > 0$. Since $\tau^\theta$ is completely regular, there exists $h : (X, \tau^\theta) \rightarrow [0,1]$ continuous, $h(x_o) = 0$, $h(y) = 1$ if $y \notin A$. For $x, y \in X$, define $F(x,y)$ on $\mathbb{R}$ by

$$F(x,y)(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 - \theta & \text{if } |h(x) - h(x_o)| \geq \tau > 0, \\ 1 & \text{if } |h(x) - h(x_o)| < \tau. \end{cases} \quad (5.6)$$

Clearly, $F(x,y) \in [0,1]_\phi$. Also, $F(x,z) \leq F(x,y) \oplus F(y,z)$. In fact, assume that $F(x,y)(t_1) \wedge F(y,z)(t_2) > r > F(x,z)(t_1 + t_2)$. Then $t_1, t_2 > 0$, $F(x,y)(t_1) = F(y,z)(t_2) = 1$. Now, $|h(x) - h(y)| < t_1$, $|h(y) - h(z)| < t_2$, and so $|h(x) - h(z)| < t_1 + t_2$, which implies that $F(x,z)(t_1 + t_2) = 1$, a contradiction. So $F$ is a probabilistic pseudometric. Moreover, $F$ is $\tau \times \tau$ continuous, or equivalently $\tau^\theta \leq \tau$. In fact, let $N^r_{x_o}(B) > r > 0$. There exists a $t > 0$ such that $1 - \sup F(x,z)(t) > r$. If $z \notin B$, then $F(z,x)(t) < 1 - r < 1$, and so $F(z,x)(t) = 1 - \theta < 1 - r$, that is, $r < \theta$, and $|h(z) - h(x)| \geq \tau$. Hence

$$M = \{z : |h(z) - h(x)| < \tau\} \subset B. \quad (5.7)$$

The set $M$ is a $\tau^\theta$-neighborhood of $x$, and hence a $\tau'$-neighborhood, that is, $N^r_{x_o}(B) > r$. Thus $\tau \geq \tau_F$. Finally, define $f : X \rightarrow [0,1]_\phi$, $f(y) = F(y,x_o)$. Then $f$ is $\tau$-continuous, $f(x_o) = 0$. For $y \notin A$ and $0 < t < 1$, we have that $f(y)(t) = F(y,x_o)(t) = 1 - \theta$ (since $|h(x) - h(x_o)| = 1 \geq t$). Conversely, assume that the condition is satisfied and let $\mathcal{F}$ be the family of all $\tau \times \tau$ continuous pseudometrics on $X$. Then $\tau_F \leq \tau$. Let $N^r_{x_o}(A) > \theta$. There
exists a \( \theta_1 > \theta \) such that \( N^{x\phi}_{\tau}(A) > \theta_1 \). By our hypothesis, there exists \( f: X \to [0,1]_\phi \) continuous such that \( f(x_0) = 0 \) and \( f(y)(t) = 1 - \theta_1 \) if \( y \notin A \) and \( 0 < t < 1 \). Define \( F(x,y) = D(f(x),f(y)) \). Then \( F \) is \( \tau \times \tau \) continuous and

\[
N^{x\phi}_{x_0}(A) \geq N^{x\phi}_{x_0}(A) \geq 1 - \sup_{y \notin A} F(x_\phi, y)(1) = 1 - \sup_{y \notin A} D(\overline{0}, f(y))(1) = 1 - \sup_{y \notin A} f(y)(1) \geq \theta_1 > \theta.
\] (5.8)

Thus \( N^{x\phi}_{x_0}(A) \geq N^{x\phi}_{x_0}(A) \), for every subset \( A \) of \( X \), and so \( \tau \leq \tau_{\overline{\phi}} \). Therefore, \( \tau = \tau_{\overline{\phi}} \), and so \( \tau \) is completely regular.

For a fuzzifying topological space \( X \), we will denote by \( C(X,[0,1]_\phi) \) the family of all continuous functions from \( X \) to \([0,1]_\phi\).

**Theorem 5.9.** A fuzzifying topological space \((X,\tau)\) is completely regular if and only if \( \tau \) coincides with the weakest of all fuzzifying topologies \( \tau_1 \) on \( X \) for which each \( f \in C(X,[0,1]_\phi) \) is continuous.

**Proof.** Assume that \((X,\tau)\) is completely regular and let \( \tau_1 \) be the weakest of all fuzzifying topologies on \( X \) for which each \( f \in C(X,[0,1]_\phi) \) is continuous. Clearly \( \tau_1 \leq \tau \). On the other hand, let \( \tau_2 \) be a fuzzifying topology on \( X \) for which each \( f \in C(X,[0,1]_\phi) \) is continuous. Let \( N^{x\phi}_X(A) > \theta > 0 \). In view of the preceding theorem, there exists an \( f \in C(X,[0,1]_\phi) \) such that \( f(x) = \overline{0} \), \( f(y)(t) = 1 - \theta \) if \( y \notin A \) and \( 0 < t < 1 \). Let

\[
G = \left\{ u \in \mathbb{R}_\phi^+: D(f(x),u) \left( \frac{1}{2} \right) = u \left( \frac{1}{2} \right) > 1 - \theta \right\}.
\] (5.9)

Then

\[
N_0(G) \geq 1 - \sup_{u \in G} D(f(x),u) \left( \frac{1}{2} \right) \geq \theta.
\] (5.10)

Since \( f \) is \( \tau_2 \)-continuous, we have that \( N^{x\phi}_X(f^{-1}(G)) \geq \theta \). But \( f^{-1}(G) \subset A \) since, for \( y \notin A \), we have that \( f(y)(1/2) = 1 - \theta \). Thus \( N^{x\phi}_X(A) \geq \theta \). This proves that \( N^{x\phi}_X(A) \geq N^{x\phi}_X(A) \), for every subset \( A \) of \( X \), and so \( \tau_2 \geq \tau \). This clearly proves that \( \tau_1 = \tau \). Conversely, assume that \( \tau_1 = \tau \). If \( \sigma \) is the usual fuzzifying topology of \( \mathbb{R}_\phi^+ \), then

\[
\tau = \tau_1 = \bigvee_{f \in C(X,[0,1]_\phi)} f^{-1}(\sigma).
\] (5.11)

Since \( \sigma \) is completely regular, each \( f^{-1}(\sigma) \) is completely regular, and so \( \tau \) is completely regular. This completes the proof. \( \square \)
References


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