We give a necessary condition for a set in $L_p(\Omega)$ spaces ($1 < p < \infty$) to be self-extremal that partially extends our previous results to the case of $L_p$ spaces. Examples of self-extremal sets in $L_p(\Omega)$ ($1 < p < \infty$) are also given.

In [4, 5], we introduced the notion of (self-) extremal sets of a Banach space $(X, \| \cdot \|)$. For a nonempty bounded subset $A$ of $X$, we denote by $d(A)$ its diameter and by $r(A)$ the relative Chebyshev radius of $A$ with respect to the closed convex hull $\overline{\text{co}}A$ of $A$, that is, $r(A) := \inf_{y \in \overline{\text{co}}A} \sup_{x \in A} \| x - y \|$. The self-Jung constant of $X$ is defined by $J_s(X) := \sup \{ r(A) : A \subset X, \text{ with } d(A) = 1 \}$. If in this definition we replace $r(A)$ by the relative Chebyshev radius $r_X(A)$ of $A$ with respect to the whole $X$, we get the Jung constant $J(X)$ of $X$. Recall that a bounded subset $A$ of $X$ consisting of at least two points is said to be extremal (resp., self-extremal) if $r_X(A) = J(X)d(A)$ (resp., $r(A) = J_s(X)d(A)$).

Throughout the note, unless otherwise mentioned, we will work with the following assumption: $(\Omega, \mu)$ is a $\sigma$-finite measure space such that $L_p(\Omega)$ is infinite-dimensional. The Jung and self-Jung constants of $L_p(\Omega)$ ($1 \leq p < \infty$) were determined in [1, 3, 6, 7]:

$$J(L_p(\Omega)) = J_s(L_p(\Omega)) = \max \{ 2^{1/p-1}, 2^{-1/p} \}.$$  \hfill (1)

**Theorem 1.** If $1 < p < \infty$ and $A$ is self-extremal in $L_p(\Omega)$, then $\kappa(A) = d(A)$.

Here $\kappa(A) := \inf \{ \varepsilon > 0 : A \text{ can be covered by finitely many sets of diameter } \leq \varepsilon \}$—the Kuratowski measure of noncompactness of $A$ (for our convenience we use the notation $\kappa(A)$ in this note).

Before proving our theorem, we need the following results which for convenience we reformatulate in the form of Lemmas 2 and 3.

**Lemma 2** (see [1], Theorem 1.1). Let $X$ be a reflexive strictly convex Banach space and $A$ a finite subset of $X$. Then there exists a subset $B \subset A$ such that

(i) $r(B) \geq r(A)$;

(ii) $\| x - b \| = r(B)$ for every $x \in B$, where $b$ is the relative Chebyshev center of $B$, that is, $b \in \overline{\text{co}}B$ and $\sup_{x \in B} \| x - b \| = r(B)$. © 2005 Hindawi Publishing Corporation

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Lemma 3 (see [8], Theorem 15.1). Let \((\Omega, \mu)\) be a \(\sigma\)-finite measure space, \(1 < p < \infty\), \(x_1, \ldots, x_n\) vectors in \(L_p(\Omega)\), and \(t_1, \ldots, t_n\) nonnegative numbers such that \(\sum_{i=1}^n t_i = 1\). The following inequality holds:

\[
2 \sum_{i=1}^n t_i \left\| x_i - \sum_{j=1}^n t_j x_j \right\|^{\alpha} \leq \sum_{i,j=1}^n t_i t_j \left\| x_i - x_j \right\|^{\alpha},
\]

where

\[
\alpha = \begin{cases} 
\frac{p}{p-1} & \text{if } 1 < p < 2, \\
p & \text{if } p \geq 2.
\end{cases}
\]

Proof of Theorem 1. Since \(r(A)\) and \(d(A)\) remain the same with replacing \(A\) by \(\overline{A}\), we may assume that \(A\) is closed convex and \(r(A) = 1\). For each integer \(n \geq 2\), we have

\[
\bigcap_{x \in A} B\left(x, 1 - \frac{1}{n}\right) \cap A = \emptyset,
\]

where \(B(x, r)\) denotes the closed ball centered at \(x\) with radius \(r\) which is weakly compact since \(L_p(\Omega)\) is reflexive. Hence there exist \(x_{q_1-1+1}, x_{q_1-1+2}, \ldots, x_{q_n}\) in \(A\) (with convention \(q_1 = 0\)) such that

\[
\bigcap_{i=q_{n-1}+1}^{q_n} B\left(x_i, 1 - \frac{1}{n}\right) \cap A = \emptyset.
\]

Set \(A_n := \{x_{q_{n-1}+1}, x_{q_{n-1}+2}, \ldots, x_{q_n}\}\). By Lemma 2, there exists a subset \(B_n = \{y_{s_{n-1}+1}, y_{s_{n-1}+2}, \ldots, y_{s_n}\}\) of \(A_n\) satisfying properties (i)-(ii) of the lemma. Let us denote the relative Chebyshev center of \(B_n\) by \(b_n\), and let \(r_n := r(B_n)\). By what we said above, we have \(r_n > 1 - 1/n\) and \(\|y_i - b_n\| = r_n\) for every \(i \in I_n := \{s_{n-1} + 1, s_{n-1} + 2, \ldots, s_n\}\). Since \(B_n\) is a finite set, there exist non-negative numbers \(t_{s_{n-1}+1}, t_{s_{n-1}+2}, \ldots, t_{s_n}\) with \(\sum_{i \in I_n} t_i = 1\) such that \(b_n = \sum_{i \in I_n} t_i y_i\). Applying Lemma 3, one gets

\[
2 r_n^{\alpha} = 2 \sum_{i \in I_n} t_i \left\| y_i - \sum_{j \in I_n} t_j y_j \right\|^{\alpha} \leq \sum_{i,j \in I_n} t_i t_j \left\| y_i - y_j \right\|^{\alpha},
\]

where \(\alpha\) is as in (3).

Setting \(B_\infty := \{y_{s_{n-1}+1}, y_{s_{n-1}+2}, \ldots, y_{s_n}\}_{n=2}^\infty\), we claim that \(\kappa(B_\infty) = d(A)\). Evidently \(\kappa(B_\infty) \leq d(A)\) by definition. If \(\kappa(A_\infty) < d(A)\), so there exist \(\varepsilon_0 \in (0, d(A))\) satisfying \(\kappa(B_\infty) \leq d(A) - \varepsilon_0\), and subsets \(D_1, D_2, \ldots, D_m\) of \(L_p(\Omega)\) with \(d(D_i) \leq d(A) - \varepsilon_0\) for every \(i = 1, 2, \ldots, m\)
such that $B_\infty \subset \bigcup_{i=1}^m D_i$. Then one can find at least one set among $D_1, D_2, \ldots, D_m$, say $D_1$, with the property that there are infinitely many $n$ satisfying

$$\sum_{i \in I_n} t_i \geq \frac{1}{m},$$

where

$$J_n := \{ i \in I_n : y_i \in D_1 \}. \tag{8}$$

From (1), it follows that $(d(A))^\alpha = (1/J_\alpha(L_p(\Omega)))^\alpha = 2$. In view of (6), we have, for all $n$ satisfying (7),

$$2 \cdot r_n^\alpha \leq \sum_{i,j \in I_n} t_i t_j \|y_i - y_j\|^\alpha \leq (d(A) - \varepsilon_0)^\alpha \cdot \left( \sum_{i,j \in I_n} t_i t_j \right) + (d(A))^\alpha \cdot \left( 1 - \sum_{i,j \in I_n} t_i t_j \right) \leq 2 - \left( (d(A))^\alpha - (d(A) - \varepsilon_0)^\alpha \right) \cdot \frac{1}{m^2}. \tag{9}$$

On the other hand, obviously $1 - 1/n \leq r_n \leq 1$, therefore $\lim_{n \to \infty} r_n = 1$. We get a contradiction with (9) since there are infinitely many $n$ satisfying (7).

One concludes that $\kappa(B_\infty) = d(A)$, and hence $\kappa(A) = d(A)$.

The proof of Theorem 1 is complete. \hfill \Box

Observe that no relatively compact set $A$ in $L_p(\Omega)$ $(1 < p < \infty)$ is self-extremal by Theorem 1. Hence we obtain an immediate extension of Gulevich's result for $L_p(\Omega)$ spaces.

**Corollary 4 (cf. [2]).** Suppose that $1 < p < \infty$ and that $A$ is a relatively compact set in $L_p(\Omega)$ with $d(A) > 0$. Then $r(A) < (1/\sqrt{2})d(A)$, where $\alpha$ is as in (3).

The following theorem gives a necessary condition for a set in $L_p(\Omega)$ $(1 < p < \infty)$ to be self-extremal.

**Theorem 5.** Under the assumptions of Theorem 1, for every $\varepsilon \in (0, d(A))$, every positive integer $m$, there exists an $m$-simplex $\Delta(\varepsilon, m)$ with vertices in $A$ such that each edge of $\Delta(\varepsilon, m)$ has length not less than $d(A) - \varepsilon$.

**Proof.** We will assume $A$ is closed convex and $r(A) = 1$. From the proof of Theorem 1, we derived a sequence $\{y_{s_{n-1}+1}, y_{s_{n-1}+2}, \ldots, y_{s_n}\}_{n=2}^\infty$ in $A$ and a sequence of positive numbers $\{t_{s_{n-1}+1}, t_{s_{n-1}+2}, \ldots, t_{s_n}\}_{n=2}^\infty$ (with convention $s_1 = 0$) such that

$$2 \cdot r_n^\alpha \leq \sum_{i,j \in I_n} t_i t_j \|y_i - y_j\|^\alpha, \quad \sum_{i \in I_n} t_i = 1, \tag{10}$$

where $r_n \in (1 - 1/n, 1]$, $\alpha$ is as in (3), and $I_n := \{s_{n-1} + 1, s_{n-1} + 2, \ldots, s_n\}$. 

We denote

\[ T_{nj} := \sum_{i \in I_n} t_i \| y_i - y_j \|^a, \]

\[ S_n := \left\{ j \in I_n : T_{nj} \geq 2 \cdot r_n^a \left( 1 - \sqrt{1 - r_n^a} \right) \right\}, \]

\[ S_n(y_j) := \left\{ i \in I_n : \| y_i - y_j \|^a \geq 2 \cdot \left( 1 - \frac{1}{\sqrt{n}} \right) \right\}, \quad j \in S_n, \]

\[ \hat{S}_n(y_j) := \left\{ i \in I_n : \sum_{i \in I_n} t_i \| y_i - y_j \|^a \geq 2 \cdot \left( 1 - \frac{1}{4} \sqrt{n} \right) \right\}, \quad j \in S_n, \]

\[ \lambda_n := \sum_{i \in I_n \setminus S_n} t_i = 1 - \sum_{i \in S_n} t_i. \]

One can proceed furthermore as follows. We have

\[ 2r_n^a \leq \sum_{i,j \in I_n} t_i t_j \| y_i - y_j \|^a \]

\[ = \sum_{j \in S_n} t_j \sum_{i \in I_n} t_i \| y_i - y_j \|^a + \sum_{j \in I_n \setminus S_n} t_j \sum_{i \in I_n} t_i \| y_i - y_j \|^a \]

\[ \leq 2 \sum_{j \in S_n} t_j + 2r_n^a \left( 1 - \sqrt{1 - r_n^a} \right) \sum_{j \in I_n \setminus S_n} t_j \]

\[ = 2 - 2\lambda_n \left( 1 - r_n^a + r_n^a \sqrt{1 - r_n^a} \right) \]

\[ \leq 2 - 2\lambda_n \sqrt{1 - r_n^a}. \] (12)

Hence \( \lambda_n \leq \sqrt{1 - r_n^a} \to 0 \), as \( n \to \infty \). Thus \( \lim_{n \to \infty} (\sum_{i \in S_n} t_i) = \lim_{n \to \infty} (1 - \lambda_n) = 1 \).

On the other hand,

\[ 2r_n^a \leq \sum_{i,j \in I_n} t_i t_j \| y_i - y_j \|^a \leq 2 \left( 1 - \left( \sum_{i \in I_n} t_i^2 \right) \right) \leq 2(1 - t_i^2) \] (13)

for every \( i \in I_n \). Therefore \( t_i \leq \sqrt{1 - r_n^a} \to 0 \) as \( n \to \infty \). One concludes that the cardinality \( |S_n| \) of \( S_n \) tends to \( \infty \) as \( n \to \infty \). In a similar manner (cf. [5, the proof of Theorem 3.4]), for every \( \epsilon \in (0,d(A)) \) and a given positive integer \( m \), we choose \( n \) sufficiently large satisfying

\[ |S_n| > m, \quad \frac{2am}{\sqrt{n}} < 1, \quad 2 \left( 1 - \frac{1}{\sqrt{n}} \right) \geq (d(A) - \epsilon)^a \] (14)

such that for every \( 1 \leq k \leq m \) and every choice of \( i_1, i_2, \ldots, i_k \in S_n \), we have

\[ \bigcap_{\gamma=1}^{k} \hat{S}_n(y_{i_\gamma}) \neq \emptyset. \] (15)
With \( m \) and \( n \) as above and a fixed \( j \in S_n \), setting \( z_1 := y_j \), we take consecutively \( z_2 \in \hat{S}_n(z_1), z_3 \in \hat{S}_n(z_1) \cap \hat{S}_n(z_2), \ldots, z_{m+1} \in \bigcap_{k=1}^{m} \hat{S}_n(z_k) \). One sees that
\[
\|z_i - z_j\|^{a} \geq 2 \left( 1 - \frac{1}{\sqrt{n}} \right) \geq (d(A) - \epsilon)^{a}
\] (16)
for all \( i \neq j \) in \( \{1, 2, \ldots, m + 1\} \), with \( n \) sufficiently large. We obtain an \( m \)-simplex formed by \( z_1, z_2, \ldots, z_{m+1} \), whose edges have length not less than \( d(A) - \epsilon \), as claimed.

The proof of Theorem 5 is complete. \( \Box \)

Remark 6. (i) Since for \( L_p(\Omega) \) spaces \( J_s = J \), the extremal sets in \( L_p(\Omega) \) are also self-extremal. Thus we obtain a similar result for extremal sets in \( L_p(\Omega) \) via Theorem 5 above.

(ii) In particular, \( \Omega = \mathbb{N}, \mu(A) := \text{card}(A), A \subset \mathbb{N} \) leads to the \( \ell_p \) space case [5, Theorem 3.4].

Example 7. (i) Let \( p \geq 2 \), consider a sequence \( \{\Omega_n\}_{n=1}^{\infty} \) consisting of measurable subsets of \( \Omega \) such that
\[
0 < \mu(\Omega_i) < \infty, \quad i = 1, 2, \ldots; \quad \Omega_i \cap \Omega_j = \emptyset \quad \forall i \neq j; \quad \bigcup_{i=1}^{\infty} \Omega_i = \Omega. \quad (17)
\]
Let \( \chi_{\Omega_i} \) denote the characteristic function of \( \Omega_i \), and set
\[
A := \{f_i\}_{i=1}^{\infty}, \quad f_i := \frac{\chi_{\Omega_i}}{[\mu(\Omega_i)]^{1/p}}. \quad (18)
\]
One can check easily that \( r(A) = 1, \ d(A) = 2^{1/p} \), hence \( A \) is a self-extremal set in \( L_p(\Omega) \).

(ii) In the case \( 1 < p < 2 \), we set \( B := \{r_i\}_{i=0}^{\infty} \), where \( \{r_i\}_{i=0}^{\infty} \) is the sequence of Rademacher functions in \( L_p[0,1] \). If \( r \in \text{co}\{r_0,r_1,\ldots,r_n\} \) and \( k \geq n + 1 \), then it is easy to see that
\[
d(B) = 2^{1-1/p}
\]
and
\[
\|r - r_k\|_p^{1/p} := \left( \int_{0}^{1} |r - r_k|^p d\mu \right)^{1/p} \geq \left| \int_{0}^{1} (r - r_k) r_k d\mu \right| = 1,
\]
(19)
hence \( r(B) = 1 \). Thus \( B \) is a self-extremal set in \( L_p[0,1] \) with \( 1 < p < 2 \). This is in contrast to the \( \ell_p \) case [5], where we conjectured that there are no (self)-extremal sets in \( \ell_p \) spaces with \( 1 < p < 2 \).

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A note on self-extremal sets in $L_p(\Omega)$ spaces


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