We deal with a locally conformal cosymplectic manifold \( M(\phi, \Omega, \xi, \eta, g) \) admitting a conformal contact quasi-torse-forming vector field \( T \). The presymplectic 2-form \( \Omega \) is a locally conformal cosymplectic 2-form. It is shown that \( T \) is a 3-exterior concurrent vector field. Infinitesimal transformations of the Lie algebra of \( \wedge M \) are investigated. The Gauss map of the hypersurface \( M_\xi \) normal to \( \xi \) is conformal and \( M_\xi \times M_\xi \) is a Chen submanifold of \( M \times M \).

1. Introduction

Locally conformal cosymplectic manifolds have been investigated by Olszak and Rosca [7] (see also [6]).

In the present paper, we consider a \((2m + 1)\)-dimensional Riemannian manifold \( M(\phi, \Omega, \xi, \eta, g) \) endowed with a locally conformal cosymplectic structure. We assume that \( M \) admits a principal vector field (or a conformal contact quasi-torse-forming), that is,

\[
\nabla T = sdp + T \wedge \xi = sdp + \eta \otimes T - T^\flat \otimes \xi, \tag{1.1}
\]

with \( ds = s\eta \).

First, we prove certain geometrical properties of the vector fields \( T \) and \( \phi T \). The existence of \( T \) and \( \phi T \) is determined by an exterior differential system in involution (in the sense of Cartan [3]).

The principal vector field \( T \) is 3-exterior concurrent (see also [8]), it defines a Lie relative contact transformation of the co-Reeb form \( \eta \), and the Lie differential of \( T^\flat \) with respect to \( T \) is conformal to \( T^\flat \). The vector field \( \phi T \) is an infinitesimal transformation of generators \( T \) and \( \xi \). The vector fields \( \xi, T, \) and \( \phi T \) commute and the distribution \( DT = \{ T, \phi T, \xi \} \) is involutive. The divergence and the Ricci curvature of \( T \) are computed.

Next, we investigate infinitesimal transformations on the Lie algebra of \( \wedge M \).
In the last section, we study the hypersurface $M_\xi$ normal to $\xi$. We prove that $M_\xi$ is Einsteinian, its Gauss map is conformal, and the product submanifold $M_\xi \times M_\xi$ in $M \times M$ is a $\mathcal{U}$-submanifold in the sense of Chen.

### 2. Preliminaries

Let $(M,g)$ be an $n$-dimensional Riemannian manifold endowed with a metric tensor $g$. Let $\Gamma TM$ and $\flat : TM \to T^* M$ be the set of sections of the tangent bundle $TM$ and the musical isomorphism defined by $g$, respectively. Following a standard notation, we set $A^q(M, TM) = \text{Hom}(\Lambda^q TM, TM)$ and notice that the elements of $A^q(M, TM)$ are the vector-valued $q$-forms ($q \leq n$) (see also [9]). Denote by $d\nabla$ the exterior covariant derivative operator with respect to the Levi-Civita connection $\nabla$. It should be noticed that generally $d\nabla^2 = d\nabla \circ d\nabla \neq 0$, unlike $d^2 = d \circ d = 0$. If $dp \in A^1(M, TM)$ denotes the soldering form on $M$, one has $d\nabla(dp) = 0$.

The cohomology operator $d\omega$ acting on $\Lambda M$ is defined by $d\omega_\gamma = d\gamma + \omega \wedge \gamma$, where $\omega$ is a closed 1-form. If $d\omega_\gamma = 0$, $\gamma$ is said to be $d\omega$-closed.

Let $R$ be the curvature operator on $M$. Then, for any vector field $Z$ on $M$, the second covariant differential is defined as

$$\nabla^2 Z = d\nabla(\nabla Z) \in A^2(M, TM) \tag{2.1}$$

and satisfies

$$\nabla^2 Z(V, W) = R(V, W)Z, \quad Z, V, W \in \Gamma TM. \tag{2.2}$$

Let $O = \text{vect}\{e_A | A = 1, \ldots, n\}$ be an adapted local field of orthonormal frames over $M$ and let $O^* = \text{covect}\{\omega^A\}$ be its associated coframe. With respect to $O$ and $O^*$, É. Cartan’s structure equation can be written, in the indexless manner, as

$$\nabla e = \theta \otimes e \in A^1(M, TM),$$

$$d\omega = -\theta \wedge \omega, \tag{2.3}$$

$$d\theta = -\theta \wedge \theta + \Theta.$$ 

In the above equations, $\theta$, respectively, $\Theta$ are the local connection forms in the bundle $\mathcal{O}(M)$, respectively, the curvature forms on $M$.

### 3. Locally conformal cosymplectic structure

Let $M(\phi, \Omega, \xi, \eta, g)$ be a $(2m + 1)$-dimensional Riemannian manifold carrying a quintuple of structure tensor fields, where $\phi$ is an automorphism of the tangent bundle $TM$, $\Omega$ a presymplectic form of rank $2m$, $\xi$ the Reeb vector field, and $\eta = \xi^\flat$ the associated Reeb covector, $g$ the metric tensor.

We assume in the present paper that $\eta$ is closed and $\lambda$ is a scalar ($\lambda \in \Lambda^0 M$) such that $d\lambda = \lambda' \eta$, with $\lambda' \in \Lambda^0 M$. 

In the above equations, $\theta$, respectively, $\Theta$ are the local connection forms in the bundle $\mathcal{O}(M)$, respectively, the curvature forms on $M$. 

We agree to denominate the manifold $M$ a \textit{locally conformal cosymplectic manifold} if it satisfies
\begin{align*}
\phi^2 &= -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1, \\
\nabla \xi &= \lambda (dp - \eta \otimes \xi), \\
d\lambda &= \lambda' \eta, \\
\Omega(Z, Z') &= g(\phi Z, Z'), \quad \Omega^m \wedge \eta \neq 0,
\end{align*}
(3.1)
where $dp \in A^1(M, TM)$ denotes the canonical vector-valued 1-form (or the soldering form [5]) on $M$. Then $\Omega$ is called the fundamental 2-form on $M$ and is expressed by
\begin{equation}
\Omega = \sum_{i=1}^{m} \omega^i \wedge \omega^i, \quad i^* = i + m.
\end{equation}
(3.2)

By the well-known relations
\begin{equation}
\theta^i_j = \theta^{i*}_j, \quad \theta^i_{j*} = \theta^{i*}_j, \quad i^* = i + m,
\end{equation}
(3.3)
on one derives by differentiation of $\Omega$
\begin{equation}
d^{-2\lambda \eta} \Omega = 0 \quad (d\Omega = 2\lambda \eta \wedge \Omega),
\end{equation}
(3.4)
which shows that the presymplectic 2-form $\Omega$ is a locally conformal cosymplectic form. Operating on $\phi dp$ by $d^V$, it follows that
\begin{equation}
d^V (\phi dp) = 2\lambda \Omega \otimes \xi + 2\eta \wedge \phi dp.
\end{equation}
(3.5)

On the other hand, we agree to call a vector field $T$, such that
\begin{equation}
\nabla T = s dp + T \wedge \xi = s dp + \eta \otimes T - T^v \otimes \xi,
\end{equation}
(3.6)
a \textit{principal vector field} on $M$, or a \textit{conformal contact quasi-torse-forming} if
\begin{equation}
ds = s\eta.
\end{equation}
(3.7)

In these conditions, since the $q$th covariant differential $\nabla^q$ of a vector field $Z \in \Gamma TM$ is defined inductively, that is, $\nabla^q Z = d^V (\nabla^{q-1} Z)$, one derives from (3.6)
\begin{equation}
\nabla^4 T = -\lambda^3 \eta \wedge T^v \otimes dp.
\end{equation}
(3.8)

As a natural concept of concurrent vector fields and by reference to [8], this proves that $T$ is a 3-exterior concurrent vector field.

Since, as it is known, the divergence of a vector field $Z$ is defined by
\begin{equation}
\text{div} Z = \sum_A g(\nabla e_A Z, e_A),
\end{equation}
(3.9)
Locally conformal cosymplectic structure

one derives from (3.2) and (3.6)

\[
\begin{align*}
\text{div} \xi &= 2m\lambda, \\
\text{div} T &= T^0 + (2m + 1)s,
\end{align*}
\]

where \( T^0 = \eta(T) \). On the other hand, from (3.6), we derive

\[
\begin{align*}
dT^a + T^b \theta^a_b + \lambda T^0 \omega^a &= sa^a + T^a \eta, & a, b \in \{1, \ldots, 2m\}, \\
dT^0 &= -(1 + \lambda)T^b + [s + (1 + \lambda)T^0] \eta.
\end{align*}
\]

After some calculations, one gets

\[
dT^b = \lambda dT^0 \land \eta = \lambda(1 + \lambda) \eta \land T^b,
\]

which proves that \( T^b \) is an exterior recurrent form [1].

Taking the Lie differential of \( \eta \) with respect of \( T \), one gets

\[
\mathcal{L}_T \eta = dT^0,
\]

and so it turns out that

\[
d(\mathcal{L}_T \eta) = 0.
\]

Following a known terminology, \( T \) defines a relative contact transformation of the co-Reeb form \( \eta \).

Next, we will point out some properties of the vector field \( \phi T \).

By virtue of (3.11), one derives

\[
\nabla \phi T = (s - \lambda T^0) \phi dp + \phi T \land \eta,
\]

and so, by (3.6) and (3.2), one gets

\[
\begin{align*}
[\phi T, T] &= -\lambda T^0 \phi T, \\
[\phi T, \xi] &= (1 - \lambda) \phi T, \\
[T, \xi] &= 0.
\end{align*}
\]

The above relations prove that \( \phi T \) admits an infinitesimal transformation of generators \( T \) and \( \xi \). In addition, it is seen that \( \xi \) and the principal vector field \( T \) commute and that the distribution \( DT = \{T, \phi T, \xi\} \) is involutive.

By Orsted lemma [1], if one takes

\[
\mathcal{L}_T T^b = \rho T^b + [T, \xi]^b,
\]

one gets at once by (3.16)

\[
\mathcal{L}_T T^b = \rho T^b,
\]
which shows that the Lie differential of $T^b$ with respect to the principal vector field $T$ is conformal to $T^b$.

Moreover, making use of the contact $\phi$-Lie derivative operator $(\mathcal{L}_\xi \phi)Z = [\xi, \phi] - \phi[\xi, Z]$, one gets in the case under discussion

$$ (\mathcal{L}_\xi \phi) T = (\lambda - 1) \phi T. \quad (3.19) $$

Hence, $\xi$ defines a $\phi$-Lie transformation of the principal vector field $T$.

It is worth to point out that the existence of $T$ and $\phi T$ is determined by an exterior differential system $\Sigma$ whose characteristic numbers are $r = 3, s_0 = 1, s_1 = 2$ ($r = s_0 + s_1$). Consequently, the system $\Sigma$ is in involution (in the sense of Cartan [3]) and so $T$ and $\phi T$ depend on 1 arbitrary function of 2 arguments (É. Cartan's test).

Recall Yano's formula for any vector field $Z$, that is,

$$ \text{div} (\nabla_Z Z) - \text{div} (\text{div} Z) Z = \mathcal{R}(Z, Z) - (\text{div} Z)^2 + \sum_{A,B} (\nabla e_A Z, e_B) (\nabla e_B Z, e_A), \quad (3.20) $$

where $\mathcal{R}$ denotes the Ricci tensor.

Then, since one has

$$ \text{div} T = T^0 + (2m + 1)s, $$
$$ \nabla_T T = (s + T^0) T - \|T\|^2 \xi, \quad (3.21) $$

it follows by (3.20) that the Ricci tensor corresponding to $T$ is expressed by

$$ \mathcal{R}(T, T) = (s + T^0) (T^0 + (2m + 1)s) - 4m^2 - s^2. \quad (3.22) $$

Finally, in the same order of ideas, since one has $i_{\phi T} T^b = 0$, then, by the Lie differentiation, one derives $\mathcal{L}_{\phi T} T^b = 0$, which shows that $\phi T$ defines a Lie Pfaffian transformation of the dual form of the vector field $T$.

Besides, by the Ricci identity involving the triple $T, \phi T, \xi$, that is,

$$ (\mathcal{L}_\xi g)(T, \phi T) = g(\nabla_\xi T, \phi T) + g(T, \nabla_\xi \phi T), \quad (3.23) $$

one gets $(\mathcal{L}_\xi g)(T, \phi T) = 0$.

Hence, one may say that the Lie structure vanishes.

Thus, we have the following.

**Theorem 3.1.** Let $M(\phi, \Omega, \xi, \eta, g)$ be a $(2m + 1)$-dimensional Riemannian manifold endowed with a locally conformal cosymplectic structure and a principal vector field $T$ defined as a conformal contact quasi-torsor-forming and structure scalar $\lambda$.

The following properties hold.

(i) $\Omega$ is a locally conformal cosymplectic 2-form.

(ii) The principal vector field $T$ is 3-exterior concurrent, that is,

$$ \nabla^4 T = -\lambda^3 \eta \wedge T^b \otimes dp. \quad (3.24) $$

(iii) $T$ defines a Lie relative contact transformation of the co-Reeb form $\eta$. 


(iv) $\phi T$ is an infinitesimal transformation of generators $T$ and $\xi$. The vector fields $\xi$, $T$, and $\phi T$ commute and the distribution $D_T = \{T, \phi T, \xi\}$ is involutive.

(v) The Lie differential of $T^\flat$ with respect to $T$ is conformal to $T^\flat$.

(vi) $\text{div} T = T^0 + (2m + 1)s$.

(vii) The Ricci tensor corresponding to $T$ is expressed by

$$\mathcal{R}(T, T) = (s + T^0)(T^0 + (2m + 1)s) - 4m^2 - s^2. \quad (3.25)$$

(viii) The dual form $T^\flat$ of $T$ is an exterior recurrent form.

4. Conformal symplectic form

We will point out some problems regarding the conformal symplectic form $\Omega$. Taking the Lie differential of $\Omega$ with respect to the Reeb vector field $\xi$, we quickly get

$$d(\mathcal{L}_\xi \Omega) = 2\lambda \Omega. \quad (4.1)$$

Hence, we may say that $\xi$ defines a conformal Lie derivative of $\Omega$.

Next, taking the Lie differential of $\Omega$ with respect to the vector field $\phi T$, one gets in two steps

$$\mathcal{L}_{\phi T} \Omega = d(T^0 \eta - T^\flat), \quad (4.2)$$

and, by (3.12), one derives at once

$$d(\mathcal{L}_{\phi T} \Omega) = 0. \quad (4.3)$$

Consequently, from above, we may state that the vector field $\phi T$ defines a relative almost-Pfaffian transformation of the form $\Omega$ (see [6]).

In the same order of ideas, one derives after some longer calculations

$$d(\mathcal{L}_T \Omega) = 2\lambda \eta \wedge d(\phi T)^\flat - 2\lambda(1 + \lambda)T^\flat \wedge \Omega + [s + (1 + s)T^0 + 4\lambda^2 T^0] \eta \wedge \Omega, \quad (4.4)$$

and we may say that the principal vector field $T$ defines a Lie almost-conformal transformation of $\Omega$.

Finally, we agree to define the 3-form

$$\psi = T^\flat \wedge \Omega, \quad (4.5)$$

the principal 3-form on the manifold $M$ under consideration.

Making use of (3.4) and (3.12), one derives

$$d\psi = \lambda(1 + \lambda)\eta \wedge \psi. \quad (4.6)$$

This shows that $\psi$ is a recurrent 3-form. Consequently, since one gets

$$i_{\phi T} T^\flat = 0, \quad i_{\phi T} \Omega = T^0 \eta - T^\flat, \quad (4.7)$$
one derives
\[ i_{\phi^T} \psi = T^0 \eta \wedge T^\flat, \] (4.8)
and so one obtains
\[ \mathcal{L}_{\phi^T} \psi = 0. \] (4.9)

Hence, we may say that the Lie derivative defines \( \phi^T \) as a Pfaffian transformation of \( \psi \). Thus, we may state the following theorem.

**Theorem 4.1.** Let \( M(\phi, \Omega, \xi, \eta, g) \) be a locally conformal cosymplectic manifold. Then, the following hold.

(i) The Reeb vector field \( \xi \) defines a conformal Lie derivative of \( \Omega \).

(ii) The vector field \( \phi^T \) defines a relative almost-Pfaffian transformation of the 2-form \( \Omega \).

(iii) The principal vector field \( T \) defines a Lie almost-conformal transformation of \( \Omega \).

(iv) Let \( \psi = T^\flat \wedge \Omega \) be the principal 3-form on the manifold \( M \). Then \( \psi \) is a recurrent 2-form and the Lie derivative defines \( \phi^T \) as a Pfaffian transformation of \( \psi \).

5. **Hypersurface \( M_\xi \) normal to \( \xi \)**

We denote by \( M_\xi \) the hypersurface of \( M \) normal to \( \xi \). Since \( d\eta = 0 \) (\( \eta = \xi^\flat \)), one may consider the \( 2m \)-dimensional manifold \( M_\xi \) and the 1-dimensional foliation in the direction of \( \xi \) is totally geodesic.

Recall that the Weingarten map
\[ A : T_P(M_\xi) \rightarrow T_P(M_\xi), \quad \forall P \in M_\xi, \] (5.1)
is a linear and selfadjoint application and \( \Omega_\eta \) is symplectic.

Then, if \( Z^T \) is any horizontal vector field, one gets by \( d\eta = 0 \)
\[ AZ^T = \nabla_{Z^T} \xi = -Z^T, \] (5.2)
and this shows that \( Z^T \) is a principal vector field of \( M_\xi \).

Recall that \( II = \langle dP, dP \rangle \) and \( III = \langle \nabla \xi, \nabla \xi \rangle \) denote the second and the third fundamental forms associated with the immersion \( x : M_\xi \rightarrow M \).

Then, by the expression of \( \nabla \xi \), one finds that \( II = g^T \) and \( III = g^T \), where \( g^T \) means the horizontal component of \( g \). Hence, we may conclude that the immersion \( x : M_\xi \rightarrow M \) is horizontally umbilical and has \( 2m \) principal curvatures equal to 1.

The expression of \( III \) proves that the Gauss map is conformal and it can also be seen that \( M_\xi \) is Einsteinian.

On the other hand, since obviously the mean curvature field \( \xi \) is nowhere zero, by reference to [4], it follows that the product submanifold \( M_\xi \times M_\xi \) in \( M \times M \) is a \( \mathcal{U} \)-submanifold (i.e., its allied mean curvature vanishes), or a Chen submanifold.

We may summarize the above by the following.
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**Theorem 5.1.** Let $M(\phi, \Omega, \xi, \eta, g)$ be a locally conformal cosymplectic manifold and $x : M_\xi \to M$ the immersion of one hypersurface normal to $\xi$. Then, the following hold.

(i) The Gauss map associated to the immersion $x : M_\xi \to M$ is conformal.

(ii) The product submanifold $M_\xi \times M_\xi$ in $M \times M$ is a $\mathfrak{U}$-submanifold.

**Acknowledgment**

The third author was supported by the CEEX-ET 2968/2005 Grant of the Romanian Ministry of Education and Research.

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