COMMON FIXED POINTS OF SINGLE-VALUED AND MULTIVALUED MAPS

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We define a new property which contains the property (EA) for a hybrid pair of single- and multivalued maps and give some new common fixed point theorems under hybrid contractive conditions. Our results extend previous ones. As an application, we give a partial answer to the problem raised by Singh and Mishra.

1. Introduction and preliminaries

Let \((X, d)\) be a metric space. Then, for \(x \in X, A \subset X, d(x, A) = \inf\{d(x, y), y \in A\}. \) We denote \(CB(X)\) as the class of all nonempty bounded closed subsets of \(X.\) Let \(H\) be the Hausdorff metric with respect to \(d,\) that is,

\[
H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\},
\]

for every \(A, B \in CB(X).\) A self-map \(T\) defined on \(X\) satisfies Rhoades’ contractive definition in following sense: (see [19]) for all \(x, y \in X, x \neq y,\)

\[
d(Tx, Ty) < \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.
\]

The fixed points theorems for Rhoades-type contraction mapping were investigated by many authors [1, 5, 8, 10, 13, 16, 22] and the more results on this fields can be found in [2, 4, 9, 11, 15, 23]. Hybrid fixed point theory for nonlinear single-valued and multivalued maps is a new development in the domain of contraction-type multivalued theory (see [3, 7, 10, 12, 14, 17, 18, 20] and references therein). In 1998, Jungck and Rhoades [12] introduced the notion of weak compatibility to the setting of single-valued and multivalued maps. In [21], Singh and Mishra introduced the notion of (IT)-commutativity for hybrid pair of single-valued and multivalued maps which need not be weakly compatible. Recently, Aamri and El Moutawakil [1] defined a property (EA) for self-maps which contained the class of noncompatible maps. More recently, Kamran [13] extended the property (EA) for a hybrid pair of single- and multivalued maps and generalized the notion of (IT)-commutativity for such pair.
The aim of this paper is to define a new property which contains the property (EA) for a hybrid pair of single- and multivalued maps and give some new common fixed point theorems under hybrid contractive conditions. As an application, we give an affirmative (half-) answer (Theorem 2.8) to the open problem in [21].

Now we state some known definitions and facts.

**Definition 1.1** [12]. Maps $f: X \to X$ and $T: X \to CB(X)$ are weakly compatible if they commute at their coincidence points, that is, if $fTx = Tf x$ whenever $fx \in Tx$.

**Definition 1.2** [21]. Maps $f: X \to X$ and $T: X \to CB(X)$ are said to be (IT)-commuting at $x \in X$ if $fTx \subset Tfx$ whenever $fx \in Tx$.

**Definition 1.3** [1]. Maps $f, g: X \to X$ are said to satisfy the property (EA) if there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} g x_n = t \in X$.

**Definition 1.4** [13]. Maps $f: X \to X$ and $T: X \to CB(X)$ are said to satisfy the property (EA) if there exist a sequence $\{x_n\}$ in $X$, some $t$ in $X$, and $A$ in $CB(X)$ such that

$$\lim_{n \to \infty} fx_n = t \in A = \lim_{n \to \infty} Tx_n.$$  \hspace{1cm} (1.3)

**Definition 1.5** [13]. Let $T: X \to CB(X)$. The map $f: X \to X$ is said to be $T$-weakly commuting at $x \in X$ if $ffx \in Tfx$.

For the rest of the introduction, we state the following theorem as the prototype in this paper.

**Theorem 1.6** (see [13]). Let $f$ be a self-map of the metric space $(X, d)$ and let $F$ be a map from $X$ into $CB(X)$ such that

1. $(f, F)$ satisfies the property (EA);
2. for all $x \neq y$ in $X$,

$$H(Fx, Fy) < \max \left\{ d(fx, fy), \frac{d(fx, Fx) + d(fy, Fy)}{2}, \frac{d(fx, Fy) + d(fy, Fx)}{2} \right\}. \hspace{1cm} (1.4)$$

If $fX$ is closed subset of $X$, then

(a) $f$ and $F$ have a coincidence point;

(b) $f$ and $F$ have a common fixed point provided that $f$ is $F$-weakly commuting at $v$ and $ffv = fv$ for $v \in C(f, F)$, where $C(f, F) = \{ x : x$ is a coincidence point of $f$ and $F \}$.

2. **Main results**

We begin with the following definition.

**Definition 2.1.** (1) Let $f, g, F, G: X \to X$. The maps pair $(f, F)$ and $(g, G)$ are said to satisfy the common property (EA) if there exist two sequences $\{x_n\}$, $\{y_n\}$ in $X$ and some $t$ in $X$ such that

$$\lim_{n \to \infty} Gy_n = \lim_{n \to \infty} Fx_n = \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gy_n = t \in X.$$  \hspace{1cm} (2.1)
Let $f, g : X \to X$ and $F, G : X \to CB(X)$. The maps pair $(f, F)$ and $(g, G)$ are said to satisfy the common property (EA) if there exist two sequences $\{x_n\}, \{y_n\}$ in $X$, some $t$ in $X$, and $A, B$ in $CB(X)$ such that

$$\lim_{n \to \infty} Fx_n = A, \quad \lim_{n \to \infty} Gy_n = B, \quad \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g y_n = t \in A \cap B. \quad (2.2)$$

**Example 2.2.** Let $X = [1, +\infty)$ with the usual metric. Define $f, g : X \to X$ and $F, G : X \to CB(X)$ by $f(x) = 2 + x/3$, $g(x) = 2 + x/2$, and $F(x) = [1, 2 + x]$, $G(x) = [3, 3 + x/2]$ for all $x \in X$. Consider the sequences $\{x_n\} = \{3 + 1/n\}$, $\{y_n\} = \{2 + 1/n\}$. Clearly, $\lim_{n \to \infty} Fx_n = [1, 5] = A$, $\lim_{n \to \infty} Gy_n = [3, 4] = B$, $\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g y_n = 3 \in A \cap B$. Therefore, $(f, F)$ and $(g, G)$ are said to satisfy the common property (EA).

**Theorem 2.3.** Let $f, g$ be two self-maps of the metric space $(X, d)$ and let $F, G$ be two maps from $X$ into $CB(X)$ such that

1. $(f, F)$ and $(g, G)$ satisfy the common property (EA);
2. for all $x \neq y$ in $X$,

$$H(Fx, Gy) < \max \left\{ \frac{d(fx, gy) + d(gy, Fx)}{2}, \frac{d(fx, Gy) + d(gy, Fx)}{2} \right\}. \quad (2.3)$$

If $FX$ and $G X$ are closed subsets of $X$, then

(a) $f$ and $F$ have a coincidence point;
(b) $g$ and $G$ have a coincidence point;
(c) $f$ and $F$ have a common fixed point provided that $f$ is $F$-weakly commuting at $v$ and $fv = f v$ for $v \in C(f, F)$;
(d) $g$ and $G$ have a common fixed point provided that $g$ is $G$-weakly commuting at $v$ and $gv = Gv$ for $v \in C(g, G)$;
(e) $f, g, F,$ and $G$ have a common fixed point provided that both (c) and (d) are true.

**Proof.** Since $(f, F)$ and $(g, G)$ satisfy the common property (EA), there exist two sequences $\{x_n\}, \{y_n\}$ in $X$ and $u \in X, A, B \in CB(X)$ such that

$$\lim_{n \to \infty} Fx_n = A, \quad \lim_{n \to \infty} Gy_n = B, \quad \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g y_n = u \in A \cap B. \quad (2.4)$$

By virtue of $FX$ and $G X$ being closed, we have $u = fv$ and $u = gw$ for some $v, w \in X$. We claim that $fv \in Fv$ and $gw \in Gw$. Indeed, condition (2) implies that

$$H(Fx_n, Gw) < \max \left\{ \frac{d(fx_n, gw) + d(gw, Fx_n)}{2}, \frac{d(fx_n, Gw) + d(gw, Fx_n)}{2} \right\}. \quad (2.5)$$
Taking the limit as \( n \to \infty \), we obtain

\[
H(A, G_w) < \max \left\{ d(f_v, g_w), \frac{d(f_v, A) + d(g_w, G_w)}{2}, \frac{d(f_v, G_w) + d(g_w, A)}{2} \right\}
\]

\[(2.6)\]

\[
= \frac{d(g_w, G_w)}{2}.
\]

Since \( g_w = f_v \in A \), it follows from the definition of Hausdorff metric that

\[
d(g_w, G_w) \leq H(A, G_w) \leq \frac{d(g_w, G_w)}{2},
\]

\[(2.7)\]

which implies that \( g_w \in G_w \).

On the other hand, by condition (2) again, we have

\[
H(F_v, G_{y_n}) < \max \left\{ d(f_v, g_{y_n}), \frac{d(f_v, F_v) + d(g_{y_n}, G_{y_n})}{2}, \frac{d(f_v, G_{y_n}) + d(g_{y_n}, F_v)}{2} \right\}.
\]

\[(2.8)\]

Similarly, we obtain

\[
d(f_v, F_v) \leq H(F_v, B) \leq \frac{d(f_v, F_v)}{2}.
\]

\[(2.9)\]

Hence \( f_v \in F_v \). Thus \( f \) and \( F \) have a coincidence point \( v \), \( g \) and \( G \) have a coincidence point \( w \). This ends the proofs of part (a) and part (b).

Furthermore, by virtue of condition (c), we obtain \( f f_{v} = f v \) and \( f f_v \in F f v \). Thus \( u = f u \in F u \). This proves (c). A similar argument proves (d). Then (e) holds immediately.

**Remark 2.4.** In Theorem 2.3, if \( F, G \) are two maps from \( K \) into \( CB(X) \), where \( K \) is a closed subset of \( X \). In this case, it is necessary to assume that \( (X, d) \) is a metrically convex metric space. In this direction, many excellent works have appeared (see [5, 21]).

**Corollary 2.5** (see [13, Theorem 3.10]). Let \( f \) be a self-map of the metric space \( (X, d) \) and let \( F \) be a map from \( X \) into \( CB(X) \) such that

1. \((f, F)\) satisfies the property (EA);
2. for all \( x \neq y \) in \( X \),

\[
H(F_x, F_y) < \max \left\{ d(x, y), \frac{d(f_x, F_y) + d(f_y, F_x)}{2}, \frac{d(f_x, F_y) + d(f_y, F_x)}{2} \right\}.
\]

\[(2.10)\]

If \( f X \) is closed subset of \( X \), then

1. \( f \) and \( F \) have a coincidence point;
2. \( f \) and \( F \) have a common fixed point provided that \( f \) is \( F \)-weakly commuting at \( v \) and \( f f v = f v \) for \( v \in C(f, F) \).
Proof. Let $F = G$ and $f = g$, then the results follow from Theorem 2.3 immediately. \qed

If $f = g$, we can conclude the following corollary.

**Corollary 2.6.** Let $f$ be a self-map of the metric space $(X,d)$ and let $F, G$ be two maps from $X$ into $CB(X)$ such that
1. $(f, F)$ and $(f, G)$ satisfy the common property (EA);
2. for all $x \neq y$ in $X$,
   \[ H(Fx, Gy) < \max \left\{ \frac{d(fx, fy) + d(fy, Gy)}{2}, \frac{d(fx, Gy) + d(fy, Fx)}{2} \right\}. \] (2.11)

If $fX$ is closed subset of $X$, then
(a) $f, G$ and $F$ have a coincidence point;
(b) $f, G$ and $F$ have a common fixed point provided that $f$ is both $F$-weakly commuting and $G$-weakly commuting at $v$ and $ffv = fv$ for $v \in C(f, F)$.

If both $F$ and $G$ are single-valued maps in Theorem 2.3, then we have the following corollary.

**Corollary 2.7.** Let $f, g, F, G$ be four self-maps of the metric space $(X,d)$ such that
1. $(f, F)$ and $(g, G)$ satisfy the common property (EA);
2. for all $x \neq y$ in $X$,
   \[ d(Fx, Gy) < \max \left\{ \frac{d(fx, gy) + d(gy, Gy)}{2}, \frac{d(fx, Gy) + d(gy, Fx)}{2} \right\}. \] (2.12)

If $fX$ and $gX$ are closed subsets of $X$, then
(a) $f$ and $F$ have a coincidence point;
(b) $g$ and $G$ have a coincidence point;
(c) $f$ and $F$ have a common fixed point provided that $f$ is $F$-weakly commuting at $v$ and $ffv = fv$ for $v \in C(f, F)$;
(d) $g$ and $G$ have a common fixed point provided that $g$ is $G$-weakly commuting at $v$ and $ggv = gv$ for $v \in C(g, G)$;
(e) $f, g, F, G$ have a common fixed point provided that both (c) and (d) are true.

**Theorem 2.8.** Let $f, g$ be two self-maps of the complete metric space $(X,d)$, let $\lambda \in (0,1)$ be a constant, and let $F, G$ be two maps from $X$ into $CB(X)$ such that for all $x \neq y$ in $X$,
\[ H(Fx, Gy) \leq \lambda \max \left\{ d(fx, gy), d(fx, Fx), d(gy, Gy), \frac{d(fx, Gy) + d(gy, Fx)}{2} \right\}. \] (2.13)

If $fX$ and $gX$ are closed subsets of $X$ and $FX \subset gX$, $GX \subset fX$, then
(a) $f$ and $F$ have a coincidence point;
(b) $g$ and $G$ have a coincidence point;
(c) $f$ and $F$ have a common fixed point provided that $f$ is $F$-weakly commuting at $v$ and $ffv = fv$ for $v \in C(f, F)$;
Hence, we obtain

\[ d(y_1, y_2) \leq H(Fx_0, Gx_1) + \lambda. \] (2.14)

Since \( GX \subset fX \), there exists \( x_2 \) such that \( f(x_2) = y_2 \in Gx_1 \), then we choose \( y_3 \in Fx_2 \) satisfying

\[ d(y_2, y_3) \leq H(Gx_1, Fx_2) + \lambda^2, \] (2.15)

and \( y_3 = gx_3 \) for some \( x_3 \in X \).

We continue this process to obtain a sequence \( \{y_n\} \) in \( X \) such that

\[
y_{2n} = f(x_{2n}) \in Gx_{2n-1}, \quad y_{2n+1} = g(x_{2n+1}) \in Fx_{2n},
\]

\[
d(y_{2n}, y_{2n+1}) \leq H(Gx_{2n-1}, Fx_{2n}) + \lambda^{2n}, \] (2.16)

\[
d(y_{2n-1}, y_{2n}) \leq H(Fx_{2n-2}, Gx_{2n-1}) + \lambda^{2n-1}, \quad n = 1, 2, \ldots.
\]

Let \( a_n = d(y_n, y_{n+1}) \), then

\[
a_{2n} = d(y_{2n}, y_{2n+1}) \leq H(Gx_{2n-1}, Fx_{2n}) + \lambda^{2n}
\]

\[
\leq \lambda \max \left\{ d(fx_{2n}, gx_{2n-1}), d(fx_{2n}, Fx_{2n}), d(gx_{2n-1}, Gx_{2n-1}), \right. \]

\[
\left. \frac{d(fx_{2n}, Gx_{2n-1}) + d(gx_{2n-1}, Fx_{2n})}{2} \right\} + \lambda^{2n}.
\] (2.17)

By \( fx_{2n} \in Gx_{2n-1} \), we have

\[
d(gx_{2n-1}, Gx_{2n-1}) \leq d(gx_{2n-1}, fx_{2n}), \quad d(fx_{2n}, Fx_{2n}) \leq H(Gx_{2n-1}, Fx_{2n}).
\] (2.18)

Thus, we rewrite (2.17) as

\[
a_{2n} \leq \lambda \max \left\{ d(fx_{2n}, gx_{2n-1}), \frac{d(gx_{2n-1}, Fx_{2n})}{2} \right\} + \lambda^{2n}.
\] (2.19)

Hence, we obtain

\[
a_{2n} \leq \lambda \max \left\{ a_{2n-1}, \frac{a_{2n-1} + a_{2n}}{2} \right\} + \lambda^{2n}.
\] (2.20)

If \( a_{2n-1} \leq a_{2n} \) for some \( n \), we have \( a_{2n} \leq \lambda^{2n}/(1 - \lambda) \). Otherwise, we get

\[
a_{2n} \leq \lambda a_{2n-1} + \lambda^{2n}.
\] (2.21)
Therefore, by (2.20), we achieve
\[ a_{2n} \leq \max \left\{ \lambda a_{2n-1} + \lambda^{2n}, \frac{\lambda^{2n}}{1 - \lambda} \right\}. \tag{2.22} \]

On the other hand,
\[ a_{2n-1} \leq H(Gx_{2n-1}, Fx_{2n-2}) + \lambda^{2n-1} \]
\[ \leq \lambda \max \left\{ d(fx_{2n-2}, gx_{2n-1}), d(fx_{2n-2}, Fx_{2n-2}), d(gx_{2n-1}, Gx_{2n-1}) \right\}, \tag{2.23} \]
\[ + \frac{d(fx_{2n-2}, Gx_{2n-1}) + d(gx_{2n-1}, Fx_{2n-2})}{2} + \lambda^{2n-1}. \]

Since \( gx_{2n-1} \in Fx_{2n-2} \), we have
\[ d(gx_{2n-1}, Gx_{2n-1}) \leq H(Gx_{2n-1}, Fx_{2n-2}), \]
\[ d(fx_{2n-2}, Fx_{2n-2}) \leq d(gx_{2n-1}, fx_{2n-2}). \tag{2.24} \]

Thus, we obtain
\[ a_{2n-1} \leq \lambda \max \left\{ a_{2n-2}, \frac{a_{2n-2} + a_{2n-1}}{2} \right\} + \lambda^{2n-1}. \tag{2.25} \]

Similarly, we get
\[ a_{2n-1} \leq \max \left\{ \lambda a_{2n-2} + a_{2n-1}, \frac{\lambda^{2n-1}}{1 - \lambda} \right\}. \tag{2.26} \]

By (2.22) and (2.26), we obtain
\[ a_n \leq \max \left\{ \lambda a_{n-1} + \lambda^n, \frac{\lambda^n}{1 - \lambda} \right\}, \quad n = 1, 2, \ldots \tag{2.27} \]

It is easy to see that
\[ a_n \leq \max \left\{ \lambda^n(a_0 + n), \frac{\lambda^n}{1 - \lambda} \right\}, \quad n = 1, 2, \ldots \tag{2.28} \]

Thus, there exists \( n_0 > 0 \) such that for \( n \geq n_0 \),
\[ a_n \leq \lambda^n(a_0 + n). \tag{2.29} \]

Hence \( \lim_{n \to \infty} a_n = 0. \)

In order to prove that \( \{ y_n \} \) is Cauchy sequence, for any \( \varepsilon > 0 \), we choose a sufficiently large number \( N \) such that
\[ \lambda^N(a_0 + N) \leq \frac{\varepsilon(1 - \lambda)}{2}, \quad \lambda^N \leq \frac{\varepsilon(1 - \lambda)^2}{4}. \tag{2.30} \]
Thus, for any positive integer $k$, we obtain
\[
d(y_N, y_{N+k}) \leq \sum_{i=0}^{k-1} a_{N+i} \leq \sum_{i=0}^{k-1} \lambda^{N+i}(a_0 + N + i)
\]
\[
< \lambda^N (a_0 + N) \frac{1}{1-\lambda} + \lambda^N \left( \sum_{i=0}^{k-1} i\lambda^i \right)
\]
\[
< \lambda^N (a_0 + N) \frac{1}{1-\lambda} + \lambda^N \frac{2}{(1-\lambda)^2} \leq \epsilon.
\]

This implies that $\{y_n\}$ is a Cauchy sequence. Thus there is $u$ satisfying
\[
\lim_{n \to \infty} y_n = u = \lim_{n \to \infty} f x_{2n} = \lim_{n \to \infty} g x_{2n+1}.
\]
Since $FX$ and $GX$ are closed, there exist $a, b$ such that $fa = u = gb$. A similar argument proves that
\[
\lim_{n \to \infty} Fx_{2n} = \lim_{n \to \infty} Gx_{2n+1},
\]
\[
u \in \lim_{n \to \infty} Fx_{2n} = \lim_{n \to \infty} Gx_{2n+1}.
\]

Then $(f, F)$ and $(g, G)$ satisfy the common property (EA). The rest of the proof follows Theorem 2.3 immediately, then the proof of Theorem 2.8 is complete. $\square$

**Corollary 2.9.** Let $f, g$ be two self-maps of the complete metric space $(X, d)$, let $\lambda \in (0, 1)$ be a constant, and let $F, G$ be two maps from $X$ into $CB(X)$ such that for all $x \neq y$ in $X$,
\[
H(Fx, Gy) \leq \alpha d(fx, gy) + \beta \max \{d(fx, Fx), d(gy, Gy)\}
\]
\[
+ \gamma \max \{d(fx, Gy) + d(gy, Fx), d(fx, Fx) + d(gy, Gy)\},
\]
and $\alpha + \beta + 2\gamma < 1$. If $FX$ and $GX$ are closed subsets of $X$ and $FX \subset GX, GX \subset Fx$, then
(a) $f$ and $F$ have a coincidence point;
(b) $g$ and $G$ have a coincidence point;
(c) $f$ and $F$ have a common fixed point provided that $f$ is $F$-weakly commuting at $v$ and $fv = f v$ for $v \in C(f,F)$;
(d) $g$ and $G$ have a common fixed point provided that $g$ is $G$-weakly commuting at $v$ and $gv = gv$ for $v \in C(g,G)$;
(e) $f, g, F, and G$ have a common fixed point provided that both (c) and (d) are true.

**Proof.** Let $\lambda = \alpha + \beta + 2\gamma$. Following (2.34) and $\max \{d(fx, Fx), d(gy, Gy)\} \geq (d(fx, Fx) + d(gy, Gy))/2$, it is easy to see that
\[
H(Fx, Gy) \leq \lambda \max \left\{ d(fx, gy), d(fx, Fx), d(gy, Gy), \frac{d(fx, Gy) + d(gy, Fx)}{2} \right\}.
\]
(2.35)

Thus by Theorem 2.8, we arrive to the conclusion in Corollary 2.9. $\square$
The next theorem involves a function $\phi$. Various conditions on $\phi$ have been investigated by different authors [4, 6, 15, 16]. Let $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ continue and satisfy the following conditions:

(A1) $\phi$ is nondecreasing on $\mathbb{R}^+$,
(A2) $0 < \phi(t) < t$, for each $t \in (0, +\infty)$.

Theorem 2.10. Let $f, g$ be two self-maps of the metric space $(X, d)$ and let $F, G: X \rightarrow X$ be two maps from $X$ into $\text{CB}(X)$ such that

1. $(f, F)$ and $(g, G)$ satisfy the common property (EA);
2. for all $x \neq y$ in $X$,
\[
H(Fx, Gy) \leq \phi\left(\max\{d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)\}\right). \tag{2.36}
\]

If $fX$ and $gX$ are closed subsets of $X$, then

(a) $f$ and $F$ have a coincidence point;
(b) $g$ and $G$ have a coincidence point;
(c) $f$ and $F$ have a common fixed point provided that $f$ is $F$-weakly commuting at $v$ and $ffv = fFv$ for $v \in C(f, F)$;
(d) $g$ and $G$ have a common fixed point provided that $g$ is $G$-weakly commuting at $v$ and $ggv = gGv$ for $v \in C(g, G)$;
(e) $f$, $g$, $F$, and $G$ have a common fixed point provided that both (c) and (d) are true.

Proof. Since $(f, F)$ and $(g, G)$ satisfy the common property (EA), there exist two sequences $\{x_n\}, \{y_n\}$ in $X$ and $u \in X, A, B \in \text{CB}(X)$ such that
\[
\lim_{n \to \infty} Fx_n = A, \lim_{n \to \infty} Gy_n = B, \quad \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gy_n = u \in A \cap B. \tag{2.37}
\]

By virtue of $fX$ and $gX$ being closed, we have $u = fv$ and $u = gw$ for some $v, w \in X$. We claim that $fv \in Fv$ and $gw \in Gw$. Indeed, condition (2) implies that
\[
H(Fx_n, Gw) \leq \phi\left(\max\{d(fx_n, gw), d(fx_n, Fx_n), d(gw, Gw), d(fx_n, Gw), d(gw, Fx_n)\}\right). \tag{2.38}
\]

Taking the limit as $n \to \infty$, we obtain
\[
H(A, Gw) \leq \phi\left(\max\{d(fv, gw), d(fv, A), d(gw, Gw), d(fv, Gw), d(gw, A)\}\right) \leq \phi(d(gw, Gw)) < d(gw, Gw). \tag{2.39}
\]

Since $gw = fv \in A$, it follows from the definition of Hausdorff metric that
\[
d(gw, Gw) \leq H(A, Gw) < d(gw, Gw), \tag{2.40}
\]

which implies that $gw \in Gw$. 

\[\text{}\]
On the other hand, by condition (2) again, we have

$$H(Fv, Gy_n) \leq \varphi(\max\{d(fv, gy_n), d(fv, Fv), d(gy_n, Gy_n), d(fv, Gy_n), d(gy_n, Fv)\}).$$

(2.41)

Similarly, we obtain

$$d(fv, Fv) \leq H(Fv, B) < d(fv, Fv).$$

(2.42)

Hence $fv \in Fv$. Thus $f$ and $F$ have a coincidence point $v$, $g$ and $G$ have a coincidence point $w$. This ends the proofs of part (a) and part (b). The rest of proof is similar to the argument of Theorem 2.3.

□

References


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