We prove that the sequence \( \{ b_n^{-1} \sum_{i=1}^{n} (X_i - EX_i) \}_{n \geq 1} \) converges a.e. to zero if \( \{ X_n, n \geq 1 \} \) is an associated sequence of random variables with \( \sum_{n=1}^{\infty} b_n^{-2} \text{Var}(\sum_{i=k_n}^{k_{n+1}} X_i) < \infty \) where \( \{ b_n, n \geq 1 \} \) is a positive nondecreasing sequence and \( \{ k_n, n \geq 1 \} \) is a strictly increasing sequence, both tending to infinity as \( n \) tends to infinity and \( 0 < a = \inf_{n \geq 1} b_n b_{n+1} \leq \sup_{n \geq 1} b_n b_{n+1} = c < 1. \)

1. Introduction

Let \((\Omega, F, P)\) be a probability space and \( \{ X_n, n \geq 1 \} \) a sequence of random variables defined on \((\Omega, F, P)\). We start with definitions. A finite sequence \( \{ X_1, \ldots, X_n \} \) is said to be associated if for any two componentwise nondecreasing functions \( f \) and \( g \) on \( \mathbb{R}^n \),

\[
\text{Cov}(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0,
\]

assuming of course that the covariance exists. The infinite sequence \( \{ X_n, n \geq 1 \} \) is said to be associated if every finite subfamily is associated. The concept of association was introduced by Esary et al. [1]. There are some results on the strong law of large numbers for associated sequences. Rao [4] developed the Hajek-Renyi inequality for associated sequences and proved the following theorem. Let \( \{ X_n, n \geq 1 \} \) be an associated sequence of random variables with

\[
\sum_{j=1}^{\infty} \frac{\text{Var}(X_j)}{b_j^2} + \sum_{1 \leq j \neq k}^{\infty} \frac{\text{Cov}(X_j, X_k)}{b_j b_k} < \infty,
\]

where \( \{ b_n, n \geq 1 \} \) is a positive nondecreasing sequence of real numbers. Then \( b_n^{-1} \sum_{j=1}^{n} (X_j - EX_j) \) converges to zero almost everywhere as \( n \to \infty \). In this note we will prove the strong law of large numbers for associated sequences with new conditions.
2. Result

Theorem 2.1. Let \( \{X_n, n \geq 1\} \) be an associated sequence of random variables. If

\[
\sum_{n=1}^{\infty} b_n^{-2} \text{Var} (S_{kn} - S_{kn-1}) < \infty,
\]

where \( S_n = \sum_{i=1}^{n} X_i \) and \( \{b_n, n \geq 1\} \) is a positive nondecreasing sequence and \( \{k_n, n \geq 1\} \) is a strictly increasing sequence, both tending to infinity as \( n \) tends to infinity and

\[
0 < a = \inf_{n \geq 1} b_k b_{kn+1}^{-1} \leq \sup_{n \geq 1} b_k b_{kn+1}^{-1} = c < 1.
\]

Then

\[
\lim_{n \to \infty} \frac{1}{b_n} \sum_{k=1}^{n} (X_k - EX_k) = 0 \quad \text{a.e.}
\]

Proof. We set \( k_0 = 0, b_0 = 0, \) and \( T_n = b_k^{-1} \sum_{j=k_n+1}^{k_{n+1}} Y_j, \) where \( Y_j = X_j - EX_j. \) For any positive integer \( n, \) there exists a positive integer \( m \) such that \( k_{m-1} < n \leq k_m. \) Note that \( m \to \infty \) as \( n \to \infty. \) Without loss of generality, we assume that \( n > k_1 \) and, therefore, \( k_{m-1} \geq 1 \) and \( b_n \geq b_{k_{m+1}} > 0. \) We can show that

\[
\frac{1}{b_n} \sum_{j=1}^{n} Y_j = \frac{b_{k_m}}{b_n} \sum_{j=1}^{m-1} \frac{b_{k_j} T_j + \frac{1}{b_n} \sum_{j=k_{m+1}+1}^{n} Y_j}{b_{k_m+1}}.
\]

Since \( b_{k_{m+1}} \geq ab_{k_m}, \) we conclude that

\[
\left| \frac{1}{b_n} \sum_{j=1}^{n} Y_j \right| \leq \left| \sum_{j=1}^{m-1} \frac{b_{k_j} T_j}{b_{k_m+1}} \right| + \frac{1}{b_n} \sum_{j=k_{m+1}+1}^{n} \left| Y_j \right|.
\]

In order to prove (2.3) it suffices to demonstrate that each of the two terms in the right-hand side of (2.5) converges to zero almost everywhere as \( n \to \infty. \) The first term on the right-hand side does so due to the Toeplitz lemma (see Loève [2]) provided that

\[
\lim_{j \to \infty} T_j = 0 \quad \text{a.e.,} \quad \sup_{m \geq 2} \sum_{j=1}^{m-1} \frac{b_{k_j}}{b_{k_m+1}} < \infty, \quad \lim_{n \to \infty} \frac{b_{k_j}}{b_{k_{m+1}}} = 0 \quad \text{for every } j.
\]

The third condition is satisfied because by the hypothesis the sequence \( \{b_n, n \geq 1\} \) monotonically increases without bounds. The second condition holds because

\[
\frac{b_{k_j}}{b_{k_{m+1}}} = \prod_{i=j}^{m-1} \frac{b_{k_i}}{b_{k_{i+1}}} \leq c^{m-j-1},
\]

\[
\sum_{j=1}^{m-1} \frac{b_{k_j}}{b_{k_{m+1}}} \leq \sum_{j=1}^{m-1} c^{m-j-1} = \frac{1 - c^m}{1 - c} < \frac{1}{1 - c}.
\]
since by the hypothesis $b_{kj} \leq cb_{kj+1}, c \in (0, 1)$. Thus, the first term in the right-hand side of (2.5) converges to zero almost everywhere as $m \to \infty$ if the sequence $\{T_n, n \geq 1\}$ also does so. By the hypothesis, let $\epsilon$ be an arbitrary positive number. With the use of the Markov inequality, we obtain

$$
\epsilon^2 \sum_{n=2}^{\infty} P(|T_n| > \epsilon) \leq \sum_{n=2}^{\infty} E|T_n|^2 \leq \sum_{n=2}^{\infty} b_{n}^2 \text{Var} (S_{kn} - S_{kn-1}) < \infty. \tag{2.8}
$$

The finiteness of the last series in the right-hand side is guaranteed by condition (2.1). In view of the Borel-Cantelli lemma, the sequence $\{T_n, n \geq 1\}$ converges to zero a.e. Let us turn to the second term in the right-hand side of (2.5). Applying Chebyshev’s inequality, we get that, for any $\epsilon > 0$,

$$
\epsilon^2 P\left( \frac{1}{b_{km}} \max_{k_{m-1}+1 \leq k \leq km} \sum_{j=k_{m-1}+1}^{l} Y_j > \epsilon \right) \leq \frac{1}{b_{km}^2} E\left( \max_{k_{m-1}+1 \leq k \leq km} \sum_{j=k_{m-1}+1}^{l} Y_j \right)^2. \tag{2.9}
$$

We now apply the Kolmogorov-type inequality, valid for partial sums of associated random variables $\{Y_j, k_{m-1}+1 \leq j \leq km\}$ with mean zero (cf. Newman and Wright [3, Theorem 2]). Hence, from (2.1), we have

$$
\epsilon^2 \sum_{m=2}^{\infty} P\left( \frac{1}{b_{km}} \max_{k_{m-1}+1 \leq k \leq km} \sum_{j=k_{m-1}+1}^{l} Y_j > \epsilon \right) \leq \sum_{m=2}^{\infty} \frac{1}{b_{km}^2} E\left( \frac{1}{b_{km}} \sum_{j=k_{m-1}+1}^{km} Y_j \right)^2 \leq \sum_{m=2}^{\infty} \frac{\text{Var} (\sum_{j=k_{m-1}+1}^{km} Y_j)}{b_{km}^2} \leq \sum_{m=2}^{\infty} \frac{\text{Var} (S_{km} - S_{km-1})}{b_{km}^2} < \infty. \tag{2.10}
$$

By virtue of the Borel-Cantelli lemma, the sequence

$$
\left\{ \left( \frac{1}{b_{km}} \max_{k_{m-1}+1 \leq k \leq km} \sum_{j=k_{m-1}+1}^{l} Y_j \right) \right\}_{m \geq 1}
$$

converges to zero almost everywhere. Thus, the theorem is proved. \hfill \Box

**Theorem 2.2.** Let $\{X_n, n \geq 1\}$ be an associated sequence of random variables with

$$
\text{Var} (X_j) + \sum_{1 \leq k \neq j}^{\infty} \text{Cov}(X_j, X_k) = O(1), \tag{2.12}
$$

for all $j \geq 1$. Then

$$
\frac{\sum_{j=1}^{n} (X_j - EX_j)}{(n \log n)^{1/2} \log \log n} \to 0 \quad \text{a.e. as } n \to \infty. \tag{2.13}
$$
Proof. Under condition (2.12), there exists the constant of $B$ such that

$$\text{Var} \left( S_{k_n} - S_{k_{n-1}} \right) \leq B(k_n - k_{n-1}) \leq Bk_n. \quad (2.14)$$

The sequence $b_n = (n \log n)^{1/2} \log \log n$ and $k_n = 2^{n+1}$, $n = 1, 2, \ldots$, satisfy the hypotheses of Theorem 2.1, which proves Theorem 2.2. \hfill \Box

Example 2.3. Let $\{X_n, n \geq 1\}$ be an associated sequence with $\text{Var}(X_i) = 1$ and $\text{Cov}(X_i,X_j) = \rho |i-j|$, $0 < \rho < 1$ for every $i$ and $j$. Then

$$\text{Var}(X_i) + \sum_{1 \leq j \not= i}^{\infty} \text{Cov}(X_i,X_j) \leq 1 + 2 \sum_{k=1}^{\infty} \rho^k < \infty. \quad (2.15)$$

Therefore, we can apply Theorem 2.2.

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References


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