ITERATIVE APPROXIMATION OF FIXED POINT FOR \( \Phi \)-HEMICONTRACTIVE MAPPING WITHOUT LIPSCHITZ ASSUMPTION

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Let \( E \) be an arbitrary real Banach space and let \( K \) be a nonempty closed convex subset of \( E \) such that \( K + K \subset K \). Assume that \( T : K \to K \) is a uniformly continuous and \( \Phi \)-hemicontractive mapping. It is shown that the Ishikawa iterative sequence with errors converges strongly to the unique fixed point of \( T \).

1. Introduction

Let \( E \) be a real Banach space and let \( E^* \) be the dual space on \( E \). The normalized duality mapping \( J : E \to 2^{E^*} \) is defined by

\[
Jx = \{ f \in E^* : \langle x, f \rangle = \| x \| \cdot \| f \| = \| f \|^2 \}
\] (1.1)

for all \( x \in E \), where \( \langle \cdot, \cdot \rangle \) denotes the generalized duality pairing. It is well known that if \( E \) is a uniformly smooth Banach space, then \( J \) is single valued and such that \( J(-x) = -J(x) \), \( J(tx) = tJ(x) \) for all \( x \in E \) and \( t \geq 0 \); and \( J \) is uniformly continuous on any bounded subset of \( E \). In the sequel, we shall denote single-valued normalized duality mapping by \( j \) by means of the normalized duality mapping \( J \). In the following, we give some concepts.

Definition 1.1. A mapping \( T \) with domain \( D(T) \) and range \( R(T) \) is said to be strongly pseudocontractive if for any \( x, y \in D(T) \), there exists \( j(x - y) \in J(x - y) \) such that

\[
\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2
\] (1.2)

for some constant \( k \in (0,1) \). The mapping \( T \) is called \( \Phi \)-strongly pseudocontractive if there exists a strictly increasing function \( \Phi : [0, \infty) \to [0, \infty) \) with \( \Phi(0) = 0 \) such that the inequality

\[
\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|)\|x - y\|
\] (1.3)

holds for all \( x, y \in D(T) \). Let \( F(T) = \{ x \in D(T) : Tx = x \} \). A mapping \( T \) is called \( \Phi \)-hemicontractive if there exists a strictly increasing function \( \Phi : [0, \infty) \to [0, \infty) \) with
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\( \Phi(0) = 0 \) such that the inequality

\[
\langle Tx - Tq, j(x - q) \rangle \leq \|x - q\|^2 - \Phi(\|x - q\|)\|x - q\|
\]

holds for all \( x \in D(T) \) and \( q \in F(T) \).

It is shown in [5] that the class of strongly pseudocontractive mapping is a proper subclass of \( \Phi \)-strongly pseudocontractive mapping. Furthermore, the example in [2] shows that the class of \( \Phi \)-strongly pseudocontractive mapping with the nonempty fixed point set is a proper subclass of \( \Phi \)-hemicontractive mapping. The classes of mappings introduced above have been studied by several authors. In [1], Chidume proved that if \( E = L_p \) (or \( l^p \), \( p \geq 2 \)), \( K \) is a nonempty closed convex and bounded subset of \( E \), and \( T : K \to K \) is a Lipschitz strongly pseudocontractive mapping, then Mann iteration process converges strongly to the unique fixed point of \( T \). In [4], Deng extended the above result to the Ishikawa iteration process. After Tan and Xu [7] extended the results of both Chidume [1] and Deng [4] to \( q \)-uniformly smooth Banach spaces \((1 < q < \infty)\), Chidume and Osilike [3] extended to real \( q \)-uniformly smooth Banach spaces \((1 < q < \infty)\). Recently, these results above have been extended from Lipschitz strongly pseudocontractive mapping to Lipschitz \( \Phi \)-strongly pseudocontractive mapping in real \( q \)-uniformly smooth Banach spaces \((1 < q < \infty)\). More recently, Osilike [6] proved that if \( K \) is a nonempty closed convex subset of arbitrary real Banach space \( E \) and \( T : K \to K \) is a Lipschitz \( \Phi \)-hemicontractive mapping, then Ishikawa iteration sequence \( \{x_n\}_{n=1}^{\infty} \) converges strongly to the unique fixed point of \( T \). It is our purpose in this paper to examine the strong convergence theorems of the Ishikawa iterative sequences with errors for \( \Phi \)-hemicontractive mapping in arbitrary real Banach spaces.

Let \( E \) be a real Banach space, then for all \( x, y \in E \), there exists \( j(x + y) \in J(x + y) \) such that \( \|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle \).

**Proof.** By definition of duality mapping, we may obtain directly the results of Lemma 1.2. □

2. Main results

**Theorem 2.1.** Let \( E \) be a real Banach space, and let \( K \) be a nonempty closed convex subset of \( E \) such that \( K + K \subset K \). Assume that \( T : K \to K \) is a uniformly continuous \( \Phi \)-hemicontractive mapping. Let \( \{\alpha_n\}_{n=0}^{\infty} \) and \( \{\beta_n\}_{n=0}^{\infty} \) be two real sequences in \([0,1]\) satisfying the following conditions: (i) \( \alpha_n, \beta_n \to 0 \) as \( n \to \infty \); (ii) \( \sum_{n=0}^{\infty} \alpha_n = \infty \). Suppose that \( \{u_n\}_{n=0}^{\infty} \) and \( \{v_n\}_{n=0}^{\infty} \) are two sequences in \( K \) satisfying that \( \sum_{n=0}^{\infty} \|u_n\| < \infty \) and \( \sum_{n=0}^{\infty} \|v_n\| < \infty \). Define the Ishikawa iterative sequence \( \{x_n\}_{n=0}^{\infty} \) with errors in \( K \) by

\[
\begin{cases}
  x_0 \in K, \\
  y_n = (1 - \beta_n)x_n + \beta_nTx_n + v_n, \quad n \geq 0, \\
  x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n + u_n, \quad n \geq 0.
\end{cases}
\]
If \( \{ T y_n \}_{n=0}^{\infty} \) and \( \{ T x_n \}_{n=0}^{\infty} \) are bounded, then the sequence \( \{ x_n \}_{n=0}^{\infty} \) converges strongly to the unique fixed point of \( T \).

**Proof.** We first observe that the iterative sequence \( \{ x_n \} \) defined by (2.1) is well defined, since \( K \) is convex and \( T \) is a self-mapping from \( K \) to itself with \( K + K \subseteq K \). By the definition of \( T \), we know that if \( F(T) \neq \emptyset \), then \( F(T) \) must be a singleton, let \( q \in K \) denote the unique fixed point. And we also obtain that for any \( x \in K \), there exists \( j(x - q) \in j(x - y) \) such that

\[
\langle Tx - Tq, j(x - q) \rangle \leq \| x - q \|^2 - \Phi(\| x - q \|)\| x - q \|. \tag{2.2}
\]

Now set

\[
M = \sup_{n \geq 0} \| Ty_n - q \| + \| x_0 - q \|, \\
D = \sum_{n=0}^{\infty} \| u_n \| + M + 1. \tag{2.3}
\]

By using induction, we obtain \( \| x_n - q \| \leq M + \sum_{n=0}^{\infty} \| u_n \|, n \geq 0 \), which implies that \( \| x_n - q \| \leq D, n \geq 0 \). Using (2.1) and Lemma 1.2, we have

\[
\| x_{n+1} - q \|^2 = \| (1 - \alpha_n)(x_n - q) + \alpha_n(Ty_n - Tq) + u_n \|^2 \\
\leq \| (1 - \alpha_n)(x_n - q) + \alpha_n(Ty_n - Tq) \|^2 + 2M\| u_n \|. \tag{2.4}
\]

Let \( A_n = \| Ty_n - T(x_{n+1} - u_n) \| \). Then \( A_n \to 0 \) as \( n \to \infty \). Indeed, since \( T \) is uniformly continuous, we observe that \( \{ x_n \}_{n=0}^{\infty}, \{ T x_n \}_{n=0}^{\infty}, \) and \( \{ Ty_n \}_{n=0}^{\infty} \) are all bounded and \( \| y_n - (x_{n+1} - u_n) \| \to 0 \) as \( n \to \infty \), so that \( A_n \to 0 \) as \( n \to \infty \). Using Lemma 1.2, (2.1), and (2.2), we have

\[
\| x_{n+1} - u_n - q \|^2 \\
= \| (1 - \alpha_n)(x_n - q) + \alpha_n(Ty_n - Tq) \|^2 \\
\leq (1 - \alpha_n)^2 \| x_n - q \|^2 + 2\alpha_n \langle Ty_n - Tq, j(x_{n+1} - u_n - q) \rangle \\
\leq (1 - \alpha_n)^2 \| x_n - q \|^2 + 2\alpha_n \langle Ty_n - T(x_{n+1} - u_n), j(x_{n+1} - u_n - q) \rangle \\
+ 2\alpha_n \langle T(x_{n+1} - u_n) - Tq, j(x_{n+1} - u_n - q) \rangle \\
\leq (1 - \alpha_n)^2 \| x_n - q \|^2 + 2\alpha_n A_n \| x_{n+1} - u_n - q \| \\
+ 2\alpha_n \| x_{n+1} - u_n - q \|^2 - 2\alpha_n \Phi(\| x_{n+1} - u_n - q \|)\| x_{n+1} - u_n - q \| \\
\leq (1 - \alpha_n)^2 \| x_n - q \|^2 + 2\alpha_n A_n (1 - \alpha_n)\| x_n - q \| + 2\alpha_n^2 A_n D.
\]
\[+ 2\alpha_n \|x_{n+1} - u_n - q\|^2 - 2\alpha_n \Phi(||x_{n+1} - u_n - q||) \|x_{n+1} - u_n - q\|\]
\[\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + \alpha_n A_n (1 - \alpha_n) (1 + \|x_n - q\|^2) + 2\alpha_n^2 A_n D\]
\[+ 2\alpha_n \|x_{n+1} - u_n - q\|^2 - 2\alpha_n \Phi(||x_{n+1} - u_n - q||) \|x_{n+1} - u_n - q\|\]
\[\leq ((1 - \alpha_n)^2 + \alpha_n A_n)||x_n - q||^2 + \alpha_n A_n (1 + 2\alpha_n D)\]
\[+ 2\alpha_n \|x_{n+1} - u_n - q\|^2 - 2\alpha_n \Phi(||x_{n+1} - u_n - q||) \|x_{n+1} - u_n - q\|,\]
(2.5)

which implies that
\[\|x_{n+1} - u_n - q\|^2 \leq \frac{1 - \alpha_n}{1 - 2\alpha_n} \|x_n - q\|^2 + \frac{\alpha_n A_n (1 + 2\alpha_n D)}{1 - 2\alpha_n}\]
\[- \frac{2\alpha_n}{1 - 2\alpha_n} \Phi(||x_{n+1} - u_n - q||) \|x_{n+1} - u_n - q\|\]
\[\leq \|x_n - q\|^2 + \frac{2\alpha_n}{1 - 2\alpha_n} \left(\frac{D^2\alpha_n + D^2 A_n + A_n + 2\alpha_n A_n D}{2}\right)\]
\[- \Phi(||x_{n+1} - u_n - q||) \|x_{n+1} - u_n - q\| \right) + 2D||u_n||\]
(2.6)

Substituting (2.6) into (2.4) yields that
\[\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 + \frac{2\alpha_n}{1 - 2\alpha_n} \left(\frac{D^2\alpha_n + D^2 A_n + A_n + 2\alpha_n A_n D}{2}\right)\]
\[- \Phi(||x_{n+1} - u_n - q||) \|x_{n+1} - u_n - q\| \right) + 2D||u_n||\]
\[\leq \|x_n - q\|^2 + \frac{2\alpha_n}{1 - 2\alpha_n} \left(B_n - \Phi(||x_{n+1} - u_n - q||) \|x_{n+1} - u_n - q\| \right) + 2D||u_n||,\]
(2.7)

where \(B_n = D^2\alpha_n + D^2 A_n + A_n + 2\alpha_n A_n D/2\). Now we consider the following two possible cases.

Case (i). \(\lim_{n \to \infty} \inf \|x_{n+1} - u_n - q\| = r > 0\). Since \(B_n \to 0, \alpha_n \to 0\) as \(n \to \infty\), then there exists a positive integer \(N\) such that \(B_n < 1/2\Phi(r)\alpha_n < 1/2\) for all \(n \geq N\). It follows from (2.7) that
\[\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 + \frac{\alpha_n}{1 - 2\alpha_n} \Phi(r) - \frac{2\alpha_n}{1 - 2\alpha_n} \Phi(r) + 2D||u_n||\]
\[\leq \|x_n - q\|^2 - \frac{\alpha_n}{1 - 2\alpha_n} \Phi(r) + 2D||u_n||\]
(2.8)

which implies that \(\Phi(r) r \sum_{n=N}^{\infty} \alpha_n/1 - 2\alpha_n \leq \|x_N - q\|^2 + 2D \sum_{n=N}^{\infty} ||u_n|| < \infty\). This contradicts the assumption that \(\sum_{n=0}^{\infty} \alpha_n = \infty\) and so the case (i) is impossible.
Thus we prove that

It is easily seen that under the assumptions of Corollary 2.3, the sequence for all $\{0,1\}$ satisfying the following conditions: (i) we have $K$ bounded sequences in $\mathbb{R}$. Let $\{K\} \subseteq \mathbb{R}$. Define iteratively the Ishikawa sequence $\{\gamma_n\} \subseteq \mathbb{R}$ as

\[\|x_{n+1} - q\|^2 = \|(1 - \alpha_n)(x_{n+1} - q) + \alpha_n(Ty_{n+1} - Tq) + u_{n+1}\|^2 \]

\[\leq \|x_{n+1} - u_{n+1} - q\|^2 + 2D\|u_{n+1}\| \]

\[\leq \varepsilon^2 + 2D\|u_{n+1}\|.

Case (ii-1). $\|x_{n+1} - u_{n+1} - q\| < \varepsilon$. Then using (2.7) yields that

\[\|x_{n+2} - q\|^2 \leq \varepsilon^2 + 2D(\|u_n\| + \|u_{n+1}\|).

(2.10)

For all $m \geq 1$, using induction, we have $\|x_{n+m} - q\|^2 \leq \varepsilon^2 + 2D\sum_{k=n_j}^{n_j+m-1}\|u_k\| < 2\varepsilon$. Thus we prove that $x_n \to q$ as $n \to \infty$. This completes the proof. □

Remark 2.2. The assumption $K + K \subseteq K$ only is used to guarantee that the iterative sequence $\{x_n\}_{n=0}^{\infty}$ is well defined. We can drop this assumption in Theorem 2.1 by using a revised iterative scheme.

Corollary 2.3. Let $E$ be a real Banach space, and let $K$ be a nonempty bounded and convex subset of $E$. Assume that $T : K \to K$ is a uniformly continuous $\Phi$-hemicontractive mapping. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$, $\{\hat{\alpha}_n\}_{n=0}^{\infty}$, $\{\hat{\beta}_n\}_{n=0}^{\infty}$, and $\{\hat{\gamma}_n\}_{n=0}^{\infty}$ be six real sequences in $[0,1]$ satisfying the following conditions: (i) $\beta_n \to 0, \hat{\beta}_n \to 0, \hat{\gamma}_n \to 0$ as $n \to \infty$; (ii) $\sum_{n=0}^{\infty}\beta_n = \sum_{n=0}^{\infty}\gamma_n < \infty$; (iii) $\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1$. Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two bounded sequences in $K$. Define iteratively the Ishikawa sequence $\{x_n\}_{n=0}^{\infty}$ with errors in $K$ as follows:

\[x_0 \in K,\]

\[y_n = \hat{\alpha}_n x_n + \hat{\beta}_n Ty_n + \hat{\gamma}_n v_n, \quad n \geq 0,\]

\[x_{n+1} = \alpha_n x_n + \beta_n Ty_n + \gamma_n u_n, \quad n \geq 0.

(2.11)

Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (2.11) converges strongly to the unique fixed point of $T$.

Proof. We observe that (2.11) can be rewritten as follows:

\[x_0 \in K,\]

\[y_n = (1 - \hat{\beta}_n)x_n + \hat{\beta}_n Ty_n + \hat{\gamma}_n(v_n - x_n), \quad n \geq 0,\]

\[x_{n+1} = (1 - \beta_n)x_n + \beta_n Ty_n + \gamma_n(u_n - x_n), \quad n \geq 0.

(2.12)

It is easily seen that under the assumptions of Corollary 2.3, the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded. Now the conclusion follows from Theorem 2.1. This completes the proof. □
THEOREM 2.4. Let $E$ be a real Banach space, and let $K$ be a nonempty closed convex subset of $E$ such that $K + K \subset K$. Assume that $T : K \to K$ is a uniformly continuous $\Phi$-hemicontractive mapping. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two real sequences in $[0, 1]$ satisfying the following conditions: (i) $\alpha_n, \beta_n \to 0$ as $n \to \infty$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are two sequences in $K$ satisfying $\|u_n\|, \|v_n\| \to 0$ as $n \to \infty$, where $\|u_n\| = o(\alpha_n)$. Define the Ishikawa iterative sequence $\{x_n\}_{n=0}^{\infty}$ with errors in $K$ by

\[
\begin{align*}
\text{(IS) } & x_0 \in K, \\
y_n &= (1 - \beta_n)x_n + \beta_nTx_n + v_n, \quad n \geq 0, \\
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n + u_n, \quad n \geq 0.
\end{align*}
\]  

(2.13)

If $\{Ty_n\}_{n=0}^{\infty}$ and $\{Tx_n\}_{n=0}^{\infty}$ are bounded, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of $T$.

Proof. Since $K + K \subset K$ and $K$ is convex, we see that the sequence $\{x_n\}_{n=0}^{\infty}$ is well defined. By the definition of $T$, $T$ has a unique fixed point in $K$. Let $q$ denote the unique fixed point. Now we shall show that $\{x_n\}_{n=0}^{\infty}$ is bounded. In fact, we may set $\|u_n\| = \varepsilon_n \alpha_n$, where $\varepsilon_n \to 0$ as $n \to \infty$. Set $D = \sup_{n \geq 0} \{\|Ty_n - q\| + \varepsilon_n + \|x_0 - q\|\}$, by induction, we can show that $\|x_n - q\| \leq D$ for all $n \geq 0$, so that $\{y_n\}$ is bounded. And we have

\[
\langle Tx - Tq, j(x - q) \rangle \leq \|x - q\|^2 - \Phi(\|x - q\|)\|x - q\|
\]  

(2.14)

for each $x \in K$. By using Lemma 1.2 and (2.7), we have

\[
\|x_{n+1} - q\|^2 \leq \|(1 - \alpha_n)(x_n - q) + \alpha_n(Ty_n - Tq)\|^2 + 2D\|u_n\|.
\]  

(2.15)

After repeating the usage of the proof of Theorem 2.1, we obtain

\[
\begin{align*}
\|(1 - \alpha_n)(x_n - q) + \alpha_n(Ty_n - Tq)\|^2 & \leq ((1 - \alpha_n)^2 + \alpha_n A_n)\|x_n - q\|^2 + \alpha_n A_n(1 + 2\alpha_n D) \\
& \quad + 2\alpha_n\|x_{n+1} - u_n - q\|^2 - 2\alpha_n \Phi(\|x_{n+1} - u_n - q\|)\|x_{n+1} - u_n - q\|
\end{align*}
\]  

(2.16)

Thus, we have

\[
\begin{align*}
\|x_{n+1} - q\|^2 & \leq \|x_n - q\|^2 + \frac{2\alpha_n}{1 - 2\alpha_n} \left( \frac{D^2 \alpha_n + D^2 A_n + A_n + 2\alpha_n A_n D}{2} \right. \\
& \quad \left. - \Phi(\|x_{n+1} - u_n - q\|)\|x_{n+1} - u_n - q\| \right) + 2D\|u_n\|
\end{align*}
\]  

(2.17)

where $B_n = D^2 \alpha_n + D^2 A_n + A_n + 2\alpha_n A_n D/2 \to 0, C_n = 1 - 2\alpha_n/\alpha_n D\|u_n\| \to 0$ as $n \to \infty$. Then $\lim_{n \to \infty} \inf \|x_{n+1} - u_n - q\| = 0$. If it is not the case, then there exist $\delta > 0$ and
positive integer \(N\) such that \(B_n + C_n < 1/2\Phi(r)r, \alpha_n < 1/2\) for all \(n \geq N\). It follows that 
\[
\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n/1 - 2\alpha_n\Phi(r)r, \text{ which leads to } \|x_N - q\|^2 < \infty, \text{ a contradiction.}
\]
Hence, there exists a subsequence \(\{x_{n_j} + 1\}\) such that \(x_{n_j} + 1 \to q\) as \(j \to \infty\). At this point, we can choose a positive integer \(n_j\) such that \(\|x_{n_j} + 1 - q\| < \varepsilon\) and \(B_n + C_n < \Phi(\varepsilon/2)/4, \|u_n\| < \varepsilon/2\) for all \(n \geq n_j\). We show that \(\|x_{n_j+2} - q\| < \varepsilon\). If not, we assume that \(\|x_{n_j+2} - q\| \geq \varepsilon\), then \(\|x_{n_j+2} - u_{n_j+1} - q\| \geq \|x_{n_j+2} - q\| - \|u_{n_j+1}\| \geq \varepsilon/2\) so that \(\Phi(x_{n_j+2} - u_{n_j+1} - q) \geq \Phi(\varepsilon/2)\). Thus, using (2.17), we have
\[
\|x_{n_j+2} - q\|^2 \leq \|x_{n_j+1} - q\|^2 - \frac{\alpha_{n_j+1}}{1 - 2\alpha_{n_j+1}}\Phi\left(\frac{\varepsilon}{2}\right)\frac{\varepsilon}{2} < \varepsilon^2,
\]
this is a contradiction and so \(\|x_{n_j+2} - q\| < \varepsilon\). By induction, \(\|x_{n_j+m} - q\| < \varepsilon\) for all \(m \geq 1\).

**Corollary 2.5.** Let \(E\) be a real Banach space, and let \(K\) be a nonempty bounded and convex subset of \(E\). Assume that \(T : K \to K\) is a uniformly continuous \(\Phi\)-hemicontractive mapping. Let \(\{\alpha_n\}_{n=0}^\infty, \{\beta_n\}_{n=0}^\infty, \{\gamma_n\}_{n=0}^\infty, \{\hat{\alpha}_n\}_{n=0}^\infty, \{\hat{\beta}_n\}_{n=0}^\infty,\) and \(\{\hat{\gamma}_n\}_{n=0}^\infty\) be six real sequences in \([0,1]\) satisfying the following conditions: (i) \(\beta_n \to 0, \hat{\beta}_n \to 0, \hat{\gamma}_n \to 0\) as \(n \to \infty\); (ii) \(\sum_{n=0}^\infty \beta_n = \infty, \gamma_n = o(\beta_n); (iii) \alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1, n \geq 0\). Let \(\{u_n\}_{n=0}^\infty\) and \(\{v_n\}_{n=0}^\infty\) be two bounded sequences in \(K\). Define iteratively the Ishikawa sequence \(\{x_n\}_{n=0}^\infty\) with errors in \(K\) as follows:
\[
\begin{align*}
x_0 & \in K, \\
y_n &= \hat{\alpha}_nx_n + \hat{\beta}_nTx_n + \hat{\gamma}_nv_n, \quad n \geq 0, \\
x_{n+1} &= \alpha_nx_n + \beta_nTy_n + \gamma_nu_n, \quad n \geq 0.
\end{align*}
\]
Then the sequence \(\{x_n\}_{n=0}^\infty\) defined by (2.11) converges strongly to the unique fixed point of \(T\).

**Proof.** We observe that (2.11) can be rewritten as follows:
\[
\begin{align*}
x_0 & \in K, \\
y_n &= (1 - \hat{\beta}_n)x_n + \hat{\beta}_nTx_n + \hat{\gamma}_n(v_n - x_n), \quad n \geq 0, \\
x_{n+1} &= (1 - \beta_n)x_n + \beta_nTy_n + \gamma_n(u_n - x_n), \quad n \geq 0.
\end{align*}
\]
It is easily to obtain the conclusion from Theorem 2.4. This completes the proof.

**Remark 2.6.** Theorems 2.1 and 2.4 extend the results of [5] from real \(q\)-uniformly smooth Banach spaces to arbitrary real Banach spaces. It is also easy to see that our results are significant extensions of the results of [1, 2, 3, 4, 7] to arbitrary real Banach spaces and to the more general classes of mapping (\(\Phi\)-hemicontractive mapping) considered here. Moreover, our iteration schemes extend from the usual iterative sequences to the iterative sequences with errors.
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