Let \( f(z) \) be an arbitrary entire function and \( M(f, r) = \max_{|z| = r} |f(z)| \). For a polynomial \( P(z) \) of degree \( n \), having no zeros in \( |z| < k, k \geq 1 \), Bidkham and Dewan (1992) proved \( \max_{|z| = r} |P'(z)| \leq (n(r + k)^{n-1}/(1 + k)^n) \max_{|z| = 1} |P(z)| \) for \( 1 \leq r \leq k \). In this paper, we generalize as well as improve upon the above inequality.

1. Introduction and statement of results

Let \( P(z) \) be a polynomial of degree \( n \) and \( M(P, r) = \max_{|z| = r} |P(z)| \), then according to Bernstein’s inequality

\[
\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| . \tag{1.1}
\]

The result is best possible and equality in (1.1) is obtained for \( P(z) = \alpha z^n, \alpha \neq 0 \).

If we restrict ourselves to the class of polynomials not vanishing in \( |z| < 1 \), then Erdős conjectured and Lax [4] proved

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| . \tag{1.2}
\]

Inequality (1.2) is best possible and the extremal polynomial is \( P(z) = \alpha + \beta z^n \) with \( |\alpha| = |\beta| \).

As an extension of (1.2), Malik [5] proved the following.

**Theorem 1.1.** If \( P(z) \) is a polynomial of degree \( n \) which does not vanish in \( |z| < k, k \geq 1 \), then

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k} \max_{|z|=1} |P(z)| . \tag{1.3}
\]

The result is best possible and equality holds for \( P(z) = (z + k)^n \).

Further, as a generalization of (1.3), Bidkham and Dewan [1] proved the following theorem.
Theorem 1.2. If \( P(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) having no zeros in \( |z| < k \), then for \( 1 \leq \rho \leq k \),
\[
\max_{|z|=\rho} |P'(z)| \leq \frac{n(\rho + k)^{n-1}}{(1 + k)^n} \max_{|z|=1} |P(z)|. \tag{1.4}
\]
The result is best possible and equality in (1.4) holds for \( P(z) = (z + k)^n \).

In this paper, we obtain the following result which is a generalization as well as an improvement of Theorem 1.2.

Theorem 1.3. If \( P(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) having no zeros in \( |z| < k \), then for \( 0 \leq r \leq \rho \leq k \),
\[
\max_{|z|=\rho} |P'(z)| \leq \frac{n(\rho + k)^{n-1}}{(k + r)^n} \left\{ 1 - \frac{k(k - \rho)(n|a_0| - k|a_1|)n}{(k^2 + \rho^2)n|a_0| + 2k^2 \rho|a_1|} \right\} M(P, r). \tag{1.5}
\]

Remark 1.4. Since it is well known that if \( P(z) = \sum_{v=0}^{n} a_v z^v \), \( P(z) \neq 0 \) in \( |z| < k \), \( k \geq 1 \), then \( |a_1|/|a_0| \leq n/k \), the above theorem with \( r = 1 \) gives a bound that is much better than obtainable from Theorem 1.2.

If we assume \( P'(0) = 0 \) in the above theorem, we get the following result.

Corollary 1.5. If \( P(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) having no zeros in \( |z| < k \), \( k \geq 1 \) and \( P'(0) = 0 \), then for \( 0 \leq r \leq \rho \leq k \),
\[
\max_{|z|=\rho} |P'(z)| \leq \frac{n(\rho + k)^{n-1}}{(k + r)^n} \left\{ 1 - \frac{k(k - \rho)(\rho - r)n}{(k^2 + \rho^2)(k + \rho)} \right\} M(P, r). \tag{1.6}
\]

2. Lemmas

We require the following lemmas for the proof of the theorem. The first lemma is due to Govil et al. \[2\].

Lemma 2.1. If \( P(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) having all its zeros in \( |z| \geq k \geq 1 \), then
\[
\max_{|z|=1} |P'(z)| \leq n\frac{|a_0| + k^2|a_1|}{(1 + k^2)n|a_0| + 2k^2|a_1|} \max_{|z|=1} |P(z)|. \tag{2.1}
\]

Lemma 2.2. If \( P(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) having no zeros in \( |z| < k \), \( k > 0 \), then for \( 0 \leq r \leq \rho \leq k \),
\[
M(P, r) \geq \left( \frac{r + k}{\rho + k} \right)^n M(P, \rho). \tag{2.2}
\]

There is equality in (2.2) for \( P(z) = (z + k)^n \).
The above lemma is due to Jain [3].

**Lemma 2.3.** If \( P(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) having no zeros in \(|z| < k\), \( k \geq 1 \), then for \( 0 \leq r \leq \rho \leq k \),

\[
M(P, r) \geq \left( \frac{k + r}{k + \rho} \right)^n \left\{ 1 - \frac{k(k - \rho)(n|a_0| - k|a_1|)n|\rho - r|}{(k^2 + \rho^2)n|a_0| + 2k^2\rho|a_1|} \left( \frac{k + r}{k + \rho} \right)^{-1} \right\} \times M(P, \rho).
\] (2.3)

**Proof.** Since \( P(z) \) has no zeros in \(|z| < k\), \( k \geq 1 \), therefore, the polynomial \( T(z) = P(tz) \) where \( 0 \leq t \leq k \) has no zeros in \(|z| < k/t\), \( k/t \geq 1 \). Using Lemma 2.1 with the polynomial \( T(z) \), we get

\[
\max_{|z|=1} |T'(z)| \leq n \left\{ \frac{n|a_0| + k^2/t^2|a_1|}{(1 + k^2/t^2)n|a_0| + 2(k^2/t^2)|a_1|} \right\} \max_{|z|=1} |T(z)|, \quad (2.4)
\]

which implies

\[
\max_{|z|=t} |P'(z)| \leq n \left\{ \frac{n|a_0| t + k^2|a_1|}{(t^2 + k^2)n|a_0| + 2k^2|a_1|} \right\} \max_{|z|=t} |P(z)|. \quad (2.5)
\]

Now for \( 0 \leq r \leq \rho \leq k \) and \( 0 \leq \theta < 2\pi \), we have

\[
|P(\rho e^{i\theta}) - P(re^{i\theta})| \leq \int_{r}^{\rho} |P'(te^{i\theta})| \, dt \\
\leq \int_{r}^{\rho} n \left\{ \frac{n|a_0| t + k^2|a_1|}{(t^2 + k^2)n|a_0| + 2k^2|a_1|} \right\} \max_{|z|=t} |P(z)| \, dt \quad (by \ (2.5)),
\] (2.6)

which implies on using inequality (2.2) of Lemma 2.2,

\[
|P(\rho e^{i\theta}) - P(re^{i\theta})| \leq \int_{r}^{\rho} n \left\{ \frac{n|a_0| t + k^2|a_1|}{(t^2 + k^2)n|a_0| + 2k^2|a_1|} \right\} \left( \frac{k + t}{k + r} \right)^n M(P, r) \, dt \\
\leq nM(P, r) \int_{r}^{\rho} \left\{ \frac{n|a_0| t + k^2|a_1|}{(t^2 + k^2)n|a_0| + 2k^2|a_1|} \right\} (k + t)^n \, dt, \quad (2.7)
\]
which gives, for \(0 \leq r \leq \rho \leq k\),

\[
M(P, \rho) \leq \left[1 + \frac{n}{(k+r)^n} \int_r^\rho \left\{ \frac{n}{(t^2+k^2)n} \left| a_0 t + k^2 a_1 \right| \right\} (t+k)^n \, dt \right] M(P, r)
\]

\[
= \left[1 + \frac{n(k+\rho)}{(k+r)^n} \left\{ \frac{n}{(\rho^2+k^2)n} \left| a_0 \rho + k^2 a_1 \right| \right\} + \left\{ \frac{(k+\rho)(n|a_0| \rho + k^2 a_1)}{(\rho^2+k^2)n|a_0| + 2k^2 \rho a_1} \right\} \right] (k+\rho)^n M(P, r)
\]

\[
= \left[1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)}{(\rho^2+k^2)n|a_0| + 2k^2 \rho a_1} \right] \left(1 - \left(\frac{k+r}{k+\rho}\right)^n\right)^{-1} M(P, r)
\]

(3.2)

from which inequality (2.3) follows.

\[\square\]

3. Proof of theorem

Since the polynomial \(P(z) = \sum_{v=0}^n a_v z^v\) has no zero in \(|z| < k\), where \(k \geq 1\), therefore, it follows that \(F(z) = P(\rho z)\) has no zeros in \(|z| < k/\rho\) where \(k/\rho \geq 1\). Applying inequality (1.3) to the polynomial \(F(z)\), we get

\[
\max_{|z|=1} |F'(z)| \leq \frac{n \max_{|z|=1} |F(z)|}{1 + k/\rho}
\]

which gives

\[
\max_{|z|=1} |P'(z)| \leq \frac{n \max_{|z|=\rho} |F(z)|}{\rho + k}
\]

(3.1)

Now if \(0 \leq r \leq \rho \leq k\), then applying inequality (2.3) of Lemma 2.3 to (3.2), it follows that

\[
\max_{|z|=\rho} |P'(z)| \leq \frac{n(k+\rho)^n}{(k+r)^n} \left[1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)}{(\rho^2+k^2)n|a_0| + 2k^2 \rho a_1} \right] \left(\frac{\rho-r}{k+\rho}\right)^{n-1} \max_{|z|=r} |P(z)|
\]

(3.3)

which is (1.5) and the theorem is proved.
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