We study the multiplicativity factor and quadraticity factor for near quasinorm on certain sequence spaces of Maddox, namely, $l(p)$ and $l_\infty(p)$, where $p = (p_k)$ is a bounded sequence of positive real numbers.

1. Introduction

Let $X$ be an algebra over a field $F$ ($R$ or $C$). A quasinorm on $X$ is a function $|\cdot| : X \to \mathbb{R}$ such that

(i) $|0| = 0$,
(ii) $|x| \geq 0$, for all $x \in X$,
(iii) $|−x| = |x|$, for all $x \in X$,
(iv) $|x + y| \leq |x| + |y|$, for all $x, y \in X$,
(v) if $t_k \in F$, $|t_k − t| \to 0$, and $x_k, x \in X$, $|x_k − x| \to 0$, then $|t_kx_k − tx| \to 0$.

If $|\cdot|$ satisfies only properties (i) to (iv), then we call $|\cdot|$ a near quasinorm. If the quasinorm satisfies $|x| = 0$ if and only if $x = 0$, then it is said to be total.

A quasinormed linear space (QNLS) is a pair $(X, |\cdot|)$ where $|\cdot|$ is a quasinorm on $X$. If $(X, |\cdot|)$ is a quasinorm space, then the map $|\cdot| : X \to \mathbb{R}$ is continuous. For $p > 0$, a $p$-seminorm on $X$ is a function $\|\cdot\| : X \to \mathbb{R}$ satisfying

(i) $\|x\| \geq 0$, for all $x \in X$,
(ii) $\|tx\| = |t|^p\|x\|$, for all $t \in F$, for all $x \in X$,
(iii) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$.

A seminorm is called a norm if it satisfies the following condition:

(iv) $\|x\| = 0$ if and only if $x = 0$.

A $p$-seminormed linear space ($p$-semi-NLS) is a pair $(X, \|\cdot\|)$ where $\|\cdot\|$ is a seminorm on $X$. $p$-normed linear spaces ($p$-normed-LS) are defined similarly.

In [1, 2], multiplicativity factors (or $M$-factors) and quadraticity factors (or $Q$-factors) for seminorms on an algebra $X$ have been introduced and studied in detail. A number $\mu > 0$ is said to be a multiplicativity factor for a seminorm $S$ if and only if $S(xy) \leq \mu S(x)S(y)$, for all $x, y \in X$. Similarly, a number $\lambda > 0$ is said to be a quadraticity factor for $S$ if and
only if $S(x^2) \leq \lambda S(x)^2$, for all $x \in X$. The necessary and sufficient conditions for existence of $M$-factor and $Q_r$-factor for $S$ are answered in the following results.

**Theorem 1.1.** Let $X$ be an algebra and let $S \neq 0$ be a seminorm on $X$. Then

(a) $S$ has $M$-factors on $X$ if and only if $\ker S$ is an ideal in $X$ and

$$\mu_{\text{inf}} \equiv \sup \{ S(xy) : x, y \in X, S(x) = S(y) = 1 \} < +\infty,$$

(1.1)

(b) if $S$ has $M$-factors on $X$ and $\mu_{\text{inf}} > 0$, then $\mu_{\text{inf}}$ is the best (least) $M$-factor for $S$,

(c) if $S$ has $M$-factors on $X$ and $\mu_{\text{inf}} = 0$, then $\mu$ is an $M$-factor for $S$ if and only if $\mu > 0$.

**Theorem 1.2.** Let $X$ be an algebra and let $S \neq 0$ be a seminorm on $X$. Then

(a) $S$ has $Q_r$-factors on $X$ if and only if $\ker S$ is closed under squaring (i.e., $(\ker S)^2 \subset \ker S$) and

$$\lambda_{\text{inf}} \equiv \sup \{ S(x^2) : x \in X, S(x) = 1 \} < +\infty,$$

(1.2)

(b) if $S$ has $Q_r$-factors on $X$ and $\lambda_{\text{inf}} > 0$, then $\lambda_{\text{inf}}$ is the best (least) $Q_r$-factor for $S$,

(c) if $S$ has $Q_r$-factors on $X$ and $\lambda_{\text{inf}} = 0$, then $\lambda$ is a $Q_r$-factor for $S$ if and only if $\lambda > 0$.

If $S$ is a norm, then $\ker S = \{0\}$. If in addition $X$ is finite-dimensional, then a simple compactness argument shows that $\mu_{\text{inf}}$ is finite. Therefore, by Theorem 1.1, norms on finite-dimensional algebras always have $M$-factors. If $S$ is a seminorm on a finite-dimensional algebra $X$, then $S$ has $M$-factors on $X$ if and only if $\ker S$ is a (two-sided) ideal in $X$. In [1, 2] several examples of seminorms having $M$-factors and $Q_r$-factors are given. In [3], scalar multiplicativity factors for near quasinorms on certain sequence spaces of Maddox are studied. Motivated by these results we define $M_r$-factors and $Q_{r'}$-factors for a near quasinorm $q$ on an algebra $X$ as follows.

A number $\mu > 0$ is an $M_r$-factor for $q$ if and only if $q(txy) \leq \mu |t|^r q(x)q(y)$, there exists $r > 0$, for all $t \in F$, for all $x, y \in X$.

A number $\lambda > 0$ is a $Q_{r'}$-factor for $q$ if and only if $q(tx^2) \leq \lambda |t|^r q(x)^2$, there exists $r > 0$, for all $t \in F$, for all $x \in X$.

Let

$$\mu_{\text{inf}} = \sup \left\{ \frac{q(txy)}{|t|^r q(x)q(y)} : t \in F - \{0\}, x, y \in X - \ker q \right\},$$

$$\lambda_{\text{inf}} = \sup \left\{ \frac{q(tx^2)}{|t|^r q(x)^2} : t \in F - \{0\}, x \in X - \ker q \right\}.$$

(1.3)

2. $M_r$-factors and $Q_{r'}$-factors for near quasinorms

In this section, we will prove the following theorems.

**Theorem 2.1.** Let $X$ be an algebra over a field $F$ ($F = C$ or $R$). Let $q$ be a near quasinorm on $X$. Then

(a) $q$ has $M_r$-factors on $X$ if and only if $\ker q$ is a (two-sided) ideal in $X$ and $\mu_{\text{inf}} < +\infty$,

(b) if $q$ has $M_r$-factors on $X$ and $\mu_{\text{inf}} > 0$, then $\mu_{\text{inf}}$ is the best (least) $M_r$-factor for $q$,

(c) if $q$ has $M_r$-factors on $X$ and $\mu_{\text{inf}} = 0$, then $\mu$ is an $M_r$-factor for $q$ if and only if $\mu > 0$. 
Theorem 2.2. Let $X$ be an algebra over a field $F$ ($F = C$ or $R$). Let $q$ be a near quasinorm on $X$. Then

(a) $q$ has $Q_r$-factors on $X$ if and only if $\text{Ker} q$ is closed under squaring (i.e., $x^2 \in \text{Ker} q$, for all $x \in \text{Ker} q$) and $\lambda_{\inf} < +\infty$,

(b) if $q$ has $Q_r$-factors on $X$ and $\lambda_{\inf} > 0$, then $\lambda_{\inf}$ is the best (least) $Q_r$-factors for $q$,

(c) if $q$ has $Q_r$-factors on $X$ and $\lambda_{\inf} = 0$, then $\lambda$ is a $Q_r$-factors for $q$ if and only if $\lambda > 0$.

Proof of Theorem 2.1. (a) Suppose that $q$ has an $M_r$-factor $\mu$ on $X$. Clearly, $\text{Ker} q$ is a subspace of $X$. Now take any $x \in \text{Ker} q$ and $y \in X$. Then $q(xy) = \mu q(x)q(y) = 0$ which implies that $xy \in \text{Ker} q$. Similarly, $yx \in \text{Ker} q$, so $\text{Ker} q$ is a (two-sided) ideal in $X$. Now for $t \in F - \{0\}$ and $x, y \in X - \text{Ker} q$, we have $q(txy) \leq \mu |t| t q(x)q(y)$ or $t q(x)q(y) / |t| q(x)q(y) \leq \mu$ which implies that $\mu_{\inf} \leq \mu < +\infty$. Conversely, suppose that $\text{Ker} q$ is a (two-sided) ideal in $X$ and $\mu_{\inf} < +\infty$. If $t = 0$, $x \in \text{Ker} q$, or $y \in \text{Ker} q$, then $t xy \in \text{Ker} q$, so $0 = q(txy) = \mu_{\inf} |t| t q(x)q(y)$. If $t \neq 0$ and $x, y \in \text{Ker} q$, then $t q(x)q(y) / |t| |t| q(x)q(y) \leq \mu_{\inf} / t$ or $t q(x)q(y) \leq \mu_{\inf} / |t| q(x)q(y)$. Therefore, $q(txy) \leq \mu_{\inf} |t| ^{\mu q(x)q(y)}$, for all $t \in F$ and for all $x, y \in X$ which implies that $q$ has $M_r$-factors on $X$.

(b) Let $\mu$ be an $M_r$-factor for $q$ on $X$ and $\mu_{\inf} > 0$. Then $q(txy) \leq \mu |t| t q(x)q(y)$ for all $t \in F$ and for all $x, y \in X$. Therefore, $q(txy) / |t| t q(x)q(y) \leq \mu$, for all $t \in F - \{0\}$ and for all $x, y \in \text{Ker} q$, so $\mu_{\inf} \leq \mu$. (c) This part follows directly from definition of $\mu_{\inf}$ and $M_r$-factors for $q$ on $X$.

Proof of Theorem 2.2. The proof of this theorem is a simple modification of the proof of Theorem 2.1 and will be omitted.

3. $M_r$-factors and $Q_r$-factors for near quasinorm on certain sequence spaces of Maddox

Let $p = (p_k)$ be a bounded sequence of positive real numbers. The sequence spaces of Maddox $l_\infty(p)$ and $l(p)$ are defined as follows:

$$l_\infty(p) = \left\{ (x_k) : x_k \in C, \sup_k |x_k|^{p_k} < \infty \right\},$$

$$l(p) = \left\{ (x_k) : x_k \in C, \sum_k |x_k|^{p_k} < \infty \right\}. \quad (3.1)$$

With the usual multiplication (i.e., $(x_k)(y_k) = (x_k y_k)$), both $l_\infty(p)$ and $l(p)$ are algebras over $C$. We define near quasinorms $q_1$ on $l_\infty(p)$ and $q_2$ on $l(p)$ as follows:

$$q_1((x_k)) = \sup_k |x_k|^{p_k / M}, \quad (x_k) \in l_\infty(p),$$

$$q_2((x_k)) = \left( \sum_k |x_k|^{p_k} \right)^{1 / M}, \quad (x_k) \in l(p), \quad (3.2)$$

where $M = \max\{1, \sup_k p_k\}$. We observe that $q_1$ and $q_2$ may or may not be quasinorms. For example, when $(p_k) = (1/k)$, then $q_1$ is a near quasinorm but not a quasinorm; if $(p_k) = (1 - 1/(k + 1))$, then $q_1$ is a quasinorm.
Similarly, we can show that $l$ and $k$ so that

3.1. Theorem

If $r < \sup_k p_k/M$, then $\mu_{\inf} = +\infty$ which is a contradiction. Therefore, $r \geq \sup_k p_k/M$. Similarly, we can show that $r \leq \inf_k p_k/M$ from which it follows that $r = \sup_k p_k/M = \inf_k p_k/M$ and the proof is complete.

(a)$\Rightarrow$(b) The same proof as (a)$\Rightarrow$(b).

(c)$\Rightarrow$(a) The same proof as (b)$\Rightarrow$(a).
(d) ⇒ (b) This is obvious.
(b) ⇒ (d) Assume that $q_1$ has $M_r$-factors. Then, by (a), $p_0 = p_k = p_{k+1}$ for all $k \geq 0$ where $p_0$ is a positive real number. Moreover, we have

$$q_1(txy) = \sup_k |t \cdot (x_k)(y_k)|^{|p_0/M|} \sup_k |x_k y_k|^{|p_0/M|} = |t|^{1/q_0} q_1(xy)$$

(3.7)

for all $x = (x_k), y = (y_k) \in l_{\infty}(p)$ and all $t \in F$. Putting $y = (1, 1, 1, \ldots)$ we see that

$$q_1(tx) = |t|^{1/q_0} q_1(x)$$

(3.8)

and the proof is complete.

$\square$

Proof of Theorem 3.2. The proof is almost the same as in Theorem 3.1 and will be omitted.

Remark 3.3. If the algebra $X$ has an identity element $x_0$ for multiplication and $q \neq 0$ is a near-quasinorm on $X$ which has an $M_r$-factor on $X$, then we obtain $q(x_0) > 0$, $\mu_{\inf} \geq 1/q(x_0)$ and

$$\frac{1}{q(x_0)\mu_{\inf}} |t|^r q(xy) \leq q(txy) \leq \mu_{\inf} |t|^r q(x)q(y)$$

(3.9)

for all $x, y \in X$ and all $t \in F$.

References


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